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ON GEOMETRICAL PROPERTIES OF STARLIKE LOGHARMONIC MAPPINGS

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Abstract. In this paper, we find the radius of the disk Ω_r such that every starlike logharmonic mapping $f(z)$ of order α, is starlike in $|z| \leqslant r$ with respect to any point of Ω_r. We also establish a relation between the set of starlike logharmonic mappings and the set of starlike logharmonic mappings of order α. Moreover, the radius of starlikeness and univalence for the set of close to starlike logharmonic mappings of order α is determined.

1. Introduction

Let $H(U)$ be the linear space of all analytic functions defined in the unit disk $U = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$ and let B denote the set of functions $a \in H(U)$ satisfying $|a(z)| < 1$ in U. A logharmonic mapping defined on U is a solution of the nonlinear elliptic partial differential equation

$$\frac{\overline{f_{\bar{z}}}}{\overline{f}} = a\frac{f_z}{f}, \tag{1.1}$$

where the second dilatation function a belongs to the class B. Thus the Jacobian

$$J_f = |f_z|^2 (1-|a|^2)$$

is positive and hence, all non-constant logharmonic mappings are sense-preserving and open on U. If f is a non-constant logharmonic mapping of U and vanishes only at $z = 0$, then f admits the representation

$$f(z) = z^m |z|^{2\beta m} h(z)\overline{g(z)}, \tag{1.2}$$

where m is a nonnegative integer, $\mathrm{Re}(\beta) > -1/2$, and h and g are analytic functions in U satisfying $g(0) = 1$ and $h(0) \neq 0$ (see [1]). The exponent β in (1.2) depends only on $a(0)$ and can be expressed by

$$\beta = \overline{a(0)}\frac{1+a(0)}{1-|a(0)|^2}.$$

Note that $f(0) \neq 0$ if and only if $m = 0$, and that a univalent logharmonic mapping on U vanishes at the origin if and only if $m = 1$, that is, f has the form

$$f(z) = z|z|^{2\beta} h(z)\overline{g(z)},$$

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where $\text{Re}(\beta) > -1/2$ and $0 \notin (hg)(U)$. This class has been studied extensively in recent years, for instance in [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 21, 22, 26]. As further evidence of its importance, note that $F(\zeta) = \log f(e^{\zeta})$ is a univalent harmonic mapping of the half-plane $\{\zeta : \text{Re}(\zeta) < 0\}$. Studies on univalent harmonic mappings can be found in [10, 13, 14, 15, 16, 17, 18, 19, 20]. Such mappings are closely related to the theory of minimal surfaces (see [24, 25]). When f is a nonvanishing logharmonic mapping in U, it is known that f can be expressed as

$$f(z) = h(z)\overline{g(z)},$$

where h and g are nonvanishing analytic functions in U. Let $f = zh(z)\overline{g(z)}$ be a univalent logharmonic mapping. We say that f is a starlike logharmonic mapping of order α if

$$\frac{\partial \arg f(re^{i\theta})}{\partial \theta} = \Re \frac{zf_z - \overline{z}f_{\overline{z}}}{f} > \alpha, 0 \leqslant \alpha < 1$$

for all $z \in U$. Denote by $ST_{Lh}(\alpha)$ the set of all starlike logharmonic mappings of order α. If $\alpha = 0$, we get the class of starlike logharmonic mappings. We also denote $ST(\alpha) = \{f \in ST_{Lh}(\alpha) \text{ and } f \in H(U)\}$. A detailed study of the class $ST_{Lh}(\alpha)$ may be found in [4]. In particular, the following are representation theorem and distortion theorem for mappings in the set $ST_{Lh}(\alpha)$.

THEOREM A. (Representation Theorem) *Let $f(z) = zh(z)\overline{g(z)}$ be a logharmonic mapping on U, $0 \notin hg(U)$. Then $f \in ST_{Lh}(\alpha)$ if and only if $\varphi(z) = zh(z)/g(z) \in ST(\alpha)$ and it follows that*

$$f(z) = \varphi(z) \exp 2\Re \int_0^z \frac{a(s)\varphi'(s)}{\varphi(s)(1-a(s))} ds.$$

THEOREM B. (Distorsion Theorem) *Let $f(z) = zh(z)\overline{g(z)} \in ST_{Lh}(\alpha)$ with $a(0) = 0$. Then for $z \in U$ we have*

$$\frac{|z|}{(1+|z|)^{2\alpha}} \exp\left((1-\alpha)\frac{-4|z|}{1+|z|}\right) \leqslant |f(z)| \leqslant \frac{|z|}{(1-|z|)^{2\alpha}} \exp\left((1-\alpha)\frac{4|z|}{1-|z|}\right).$$

The equalities occur if and only if $f(z) = \overline{\zeta} f_0(\zeta z)$, where $|\zeta| = 1$ and

$$f_0(z) = \frac{z(1-\overline{z})}{(1-z)} \frac{1}{(1-\overline{z})^{2\alpha}} \exp(1-\alpha)\Re \frac{4z}{1-z}.$$

Denote by P_{Lh} the set of all logharmonic mappings R defined on the unit disk U which are of the form $R = H\overline{G}$, where H and G are in $H(U)$, $H(0) = G(0) = 1$ and such that $Re(R(z)) > 0$ for all $z \in U$. In particular, the set P of all analytic functions $p(z)$ in U with $p(0) = 1$ and $Re(p(z)) > 0$ in U is a subset of P_{Lh} (for more details see[2]).

In Section 2, we consider a relation between $ST_{Lh}(\alpha)$ and $ST_{Lh}(\beta)$ and obtain the radius of the disk Ω_r such that every starlike logharmonic mapping $f(z)$ of order α, is starlike in $|z| < r$ with respect to any point of Ω_r. In section 3, the radius of univalence and starlikeness is determined for the set of close to starlike logharmonic mappings of order α.

2. Geometrical properties of the class $ST_{Lh}(\alpha)$

In the following proposition we establish a relationship between the classes $ST_{Lh}(\alpha)$ and $ST_{Lh}(\beta)$.

PROPOSITION 1. *Let $f(z)=zh(z)\overline{g(z)}$ be a logharmonic mapping with respect to $a\in B$ and*

$$K(z)=z\exp 2\Re\int_0^z \frac{a(z)}{1-a(z)}\frac{dz}{z}.$$

If $f\in ST_{Lh}(\alpha)$ then $F(z)=f(z)^{\delta}K(z)^{\gamma}\in ST_{Lh}(\beta)$, with $\beta=\delta\alpha+\gamma\geqslant 0$, and $\delta+\gamma=1$.

Proof. $K(z)=z\exp 2\Re\int_0^z \frac{a(z)}{1-a(z)}\frac{dz}{z}$ is starlike logharmonic univalent with respect to a.

Direct calculations yield

$$\frac{\overline{F_{\overline{z}}}}{\overline{F}}=\delta\frac{\overline{f_{\overline{z}}}}{\overline{f}}+\gamma\frac{\overline{K_{\overline{z}}}}{\overline{K}}=\delta a\frac{f_z}{f}+\gamma a\frac{K_z}{K}=a\frac{F_z}{F}.$$

Hence F is logharmonic with respect to the same a. Moreover,

$$\Re\frac{zK_z-\overline{z}K_{\overline{z}}}{K}=\Re\left(1+\frac{a(z)}{1-a(z)}-\overline{\frac{a(z)}{1-a(z)}}\right)=1.$$

It follows,

$$\Re\frac{zF_z-\overline{z}F_{\overline{z}}}{F}=\delta\,\Re\frac{zf_z-\overline{z}f_{\overline{z}}}{f}+\gamma\Re\frac{zK_z-\overline{z}K_{\overline{z}}}{K}>\delta\alpha+\gamma=\beta.$$

Thus, $F\in ST_{Lh}(\beta)$. □

REMARK 1. Above proposition gives us in particular the following 2 special cases:

- If $f(z)=zh(z)\overline{g(z)}\in ST_{Lh}(\alpha)$ with respect to $a\in B$ then $F(z)=f(z)^{\frac{1}{1-\alpha}}K(z)^{\frac{-\alpha}{1-\alpha}}\in ST_{Lh}(0)$.
- If $f(z)=zh(z)\overline{g(z)}\in ST_{Lh}(0)$ with respect to $a\in B$ then $F(z)=f(z)^{1-\alpha}K(z)^{\alpha}\in ST_{Lh}(\alpha)$.

In what follows next, our objective is to find the region Ω_r in the $w-$plane such that every $f\in ST_{Lh}(\alpha)$ is starlike with respect to any point of Ω_r. Since $ST_{Lh}(\alpha)$ is compact (see[4]), it follows that Ω_r is a closed set. Therefore, Ω_r is a closed disk with center at $w=0$ and the determination of Ω_r is equivalent to the determination of the radius of the disk Ω_r.

Our main result is the following theorem

THEOREM 1. *Let* $f \in ST_{Lh}(\alpha)$, *then the radius of the disk* Ω_r *such that* f *is starlike with respect to any point of* Ω_r *is given by*

$$\lambda_\alpha(r_0) = \frac{r_0}{(1+r_0)^{2\alpha}} \exp\left(\frac{-(1-\alpha)4r_0}{(1+r_0)}\right) \frac{\left(\alpha + (1-\alpha)\dfrac{1-r_0}{1+r_0}\right)}{\left(\alpha + (1-\alpha)\left(\dfrac{1+r_0}{1-r_0}\right)\right)\left(\dfrac{1+r_0}{1-r_0}\right)},$$

where $r_0 \in (0,1)$ *and* r_0 *is the smallest positive root of the equation*

$$8r^5\alpha^3 - 12r^5\alpha^2 + 6r^5\alpha - r^5 - 16r^4\alpha^3 + 12r^4\alpha^2 + 4r^4\alpha - 3r^4 + 8r^3\alpha^3 - 36r^3\alpha^2 + 32r^3\alpha - 8r^3 + 4r^2\alpha^2 - 4r^2\alpha + 4r^2 - 6r\alpha + 9r - 1 = 0.$$

Proof. Let $U_r(f) = f(|z| \leqslant r < 1)$, $w = f(z) \in ST_{Lh}(\alpha)$. $U_r(f)$ is starlike with respect to w_0 if and only if

$$\frac{\partial \arg\left(f(re^{i\theta}) - w_0\right)}{\partial\theta} = \Re\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z) - w_0} > 0 \text{ for } |z| \leqslant r < 1.$$

This is equivalent to

$$\Re\frac{f(z) - w_0}{zf_z(z) - \overline{z}f_{\overline{z}}(z)} > 0 \text{ for } |z| \leqslant r < 1,$$

or

$$\Re\frac{f(z)}{zf_z(z) - \overline{z}f_{\overline{z}}(z)} > \Re\frac{w_0}{zf_z(z) - \overline{z}f_{\overline{z}}(z)} \text{ for } |z| \leqslant r < 1. \tag{2.1}$$

It follows from (2.1) that

$$|f(z)|^2\Re\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)} > |w_0|^2\Re\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{w_0} \text{ for } |z| \leqslant r < 1. \tag{2.2}$$

Now if $f(z) \in ST_{Lh}(\alpha)$, we have $e^{i\theta}f(e^{-i\theta}z) \in ST_{Lh}(\alpha)$. It follows that if $w_0 \in \Omega_r$ then $\rho e^{i\theta} \in \Omega_r$, with $\rho = |w_0|$ and $-\pi < \theta \leqslant \pi$. Therefore, if $w_0 \in \Omega_r$, (2.2) must holds for all points $w = |w_0|e^{i\theta}$, $-\pi < \theta \leqslant \pi$ and so

$$\left|f(z)^2\right|\Re\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)} \geqslant |w_0|\,|zf_z(z) - \overline{z}f_{\overline{z}}(z)| \text{ for } |z| \leqslant r < 1.$$

Hence

$$\left|\frac{f(z)^2}{zf_z(z) - \overline{z}f_{\overline{z}}(z)}\right|\Re\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)} \geqslant |w_0| \text{ for } |z| \leqslant r < 1.$$

We next consider the function

$$\Psi(f,z) = \left|\frac{f(z)^2}{zf_z(z) - \overline{z}f_{\overline{z}}(z)}\right|\Re\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)}, \tag{2.3}$$

where $f \in ST_{Lh}(\alpha)$ and z is fixed, $|z| = r$. Clearly, $\min_{f\in ST_{Lh}(\alpha)} \Psi(f,z)$ is independent of the choice of $z = re^{i\theta}$, $-\pi < \theta \leqslant \pi$. Let $|z| = r$, $r > 0$. Then $\lambda_\alpha(r) = \min_{f\in ST_{Lh}(\alpha)} \Psi(f,z)$ is the radius of Ω_r. Since $f \in ST_{Lh}(\alpha)$ by Theorem A, there exists a $\varphi \in ST(\alpha)$ such that

$$\Re\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)} = \Re\frac{z\varphi'(z)}{\varphi(z)} = \Re((1-\alpha)p(z)+\alpha), \tag{2.4}$$

where $p \in P$. From Theorem B, it follows that if $f \in ST_{Lh}(\alpha)$ then

$$|f(z)| \geqslant \frac{r}{(1+r)^{2\alpha}} \exp\left(\frac{-(1-\alpha)4r}{(1+r)}\right). \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.3), we get

$$\Psi(f,z) = \left|\frac{f(z)}{\frac{zf_z(z)-\overline{z}f_{\overline{z}}(z)}{f(z)}}\right| \Re\frac{zf_z(z)-\overline{z}f_{\overline{z}}(z)}{f(z)}$$

$$\geqslant \frac{r}{(1+r)^{2\alpha}} \exp\left(\frac{-(1-\alpha)4r}{(1+r)}\right) \frac{\Re\left[\alpha+(1-\alpha)p(z)\right]}{\left|\frac{1}{1-a}\left[\alpha+(1-\alpha)p(z)\right] - \frac{\overline{a}}{1-\overline{a}}\left[\alpha+(1-\alpha)\overline{p(z)}\right]\right|}$$

$$\geqslant \frac{r}{(1+r)^{2\alpha}} \exp\left(\frac{-(1-\alpha)4r}{(1+r)}\right) \frac{\left[\alpha+(1-\alpha)\frac{1-r}{1+r}\right]}{\left|\frac{1}{|1-a|}\left[\alpha+(1-\alpha)|p(z)|\right] + \frac{|a|}{|1-a|}\left[\alpha+(1-\alpha)|p(z)|\right]\right|}$$

$$\geqslant \frac{r}{(1+r)^{2\alpha}} \exp\left(\frac{-(1-\alpha)4r}{(1+r)}\right) \frac{\left(\alpha+(1-\alpha)\frac{1-r}{1+r}\right)}{\left(\alpha+(1-\alpha)\left(\frac{1+r}{1-r}\right)\right)\left(\frac{1+r}{1-r}\right)}.$$

We set

$$\lambda_\alpha(r) = \frac{r}{(1+r)^{2\alpha}} \exp\left(\frac{-(1-\alpha)4r}{(1+r)}\right) \frac{\left(\alpha+(1-\alpha)\frac{1-r}{1+r}\right)}{\left(\alpha+(1-\alpha)\left(\frac{1+r}{1-r}\right)\right)\left(\frac{1+r}{1-r}\right)}.$$

Then $\lambda_\alpha(r_0)$ is the radius of Ω_r, where $r_0 \in (0,1)$ and r_0 is the smallest positive root of the equation $8r^5\alpha^3 - 12r^5\alpha^2 + 6r^5\alpha - r^5 - 16r^4\alpha^3 + 12r^4\alpha^2 + 4r^4\alpha - 3r^4 + 8r^3\alpha^3 - 36r^3\alpha^2 + 32r^3\alpha - 8r^3 + 4r^2\alpha^2 - 4r^2\alpha + 4r^2 - 6r\alpha + 9r - 1 = 0$. We note that $\min_{f\in ST_{Lh}(\alpha)} \Psi(f,z)$ is attained in $ST_{Lh}(\alpha)$ by a function of the form $f(z) = \overline{\eta} f_0(\eta z)$, $|\eta| = 1$ and where

$$f_0(z) = \frac{z(1-\overline{z})}{(1-z)} \frac{1}{(1-\overline{z})^{2\alpha}} \exp(1-\alpha)\Re\frac{4z}{1-z}. \quad \square \tag{2.6}$$

For the particular case, where $f \in ST_{Lh}(0)$ we establish the following corollary.

COROLLARY 1. *Let $f \in ST_{Lh}(0)$, then f is starlike with respect to any point of Ω_r, where Ω_r is a disk $\{w : |w| < \lambda_\alpha(r_0)\}$ with $\lambda_\alpha(r_0) = 8.7462 \times 10^{-2}$.*

Proof. Let $U_r(f) = f(|z| \leqslant r < 1)$, $w = f(z) \in ST_{Lh}(0)$. Proceeding in a similar fashion as in the above proof, we show that $U_r(f)$ is starlike with respect to w_0 if and only if $|w_0| \leqslant \Psi(f,z)$, where

$$\Psi(f,z) = \left| \frac{f(z)}{\frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)}} \right| \Re \frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)}.$$

In particular $|w_0| \leqslant \lambda_\alpha(r) = \min\limits_{f \in ST_{Lh}(0)} \Psi(f,z)$, with z fixed, $|z| = r$. Since $f \in ST_{Lh}(0)$, we have

$$\lambda_\alpha(r) \geqslant r\left(\frac{1-r}{1+r}\right)^3 \exp\left(\frac{-4r}{(1+r)}\right).$$

We can minimize $\lambda_\alpha(r)$ by taking the smallest positive root of the equation

$$r^3 + 3r^2 + 9r - 1 = 0$$

which is $r_0 = 0.10715$. Hence $\lambda_\alpha(r_0) = 8.7462 \times 10^{-2}$. We note that $\min\limits_{f \in ST_{Lh}(\alpha)} \Psi(f,z)$ is attained in $ST_{Lh}(0)$ by a function of the form $f(z) = \overline{\eta} f_0(\eta z)$, $|\eta| = 1$ and where

$$f_0(z) = \frac{z(1-\overline{z})}{(1-z)} \exp \Re \frac{4z}{1-z}. \qquad \square \tag{2.7}$$

3. Close to starlike logharmonic mappings of order α

In this section, we consider the set of all logharmonic mappings $F(z)$ which can be factorized as the product of a logharmonic mapping $f(z) \in ST_{Lh}(\alpha)$ with respect to $a \in B$ and a logharmonic mapping $R(z) \in P_{Lh}$ with respect to the same a.

DEFINITION 1. We say $F(z)$ is close to starlike of order α, if $F(z) = f(z)R(z)$, where $f \in ST_{Lh}(\alpha)$ with respect to $a \in B$ and $R \in P_{Lh}$ with respect to the same a. We denote by $CST_{Lh}(\alpha)$ the set of all close to starlike logharmonic mappings of order α.

Note that, if $\alpha = 0$, we get the class of close to starlike logharmonic mappings and if $R(z) = 1$ then $F \in ST_{Lh}(\alpha)$.

Close to starlike logharmonic mappings have the following geometrical property: Under the mapping $F(z)$, the radius vector of the image of $|z| = r < 1$, never turns back by an amount more than π. Observe that F is not necessarily univalent starlike on U. For example, take $F(z) = z(1-z)$, where $z \in ST(\alpha)$ and $1 - z \in P$.

In the next result we determine the radius of univalence and starlikeness for these mappings $F \in CST_{Lh}(\alpha)$.

THEOREM 2. *Let $F \in CST_{Lh}(\alpha)$. Then F maps the disk $|z| < \rho$ onto a starlike domain, where* $\rho \leqslant \dfrac{2-\alpha-\sqrt{\alpha^2-2\alpha+3}}{1-2\alpha}$ *for $\alpha \neq \frac{1}{2}$, and $\rho \leqslant \frac{1}{3}$ for $\alpha = \frac{1}{2}$. The upper bound is best possible for all $a \in B$.*

Proof. Let $F(z) = f(z)R(z) \in CST_{Lh}(\alpha)$, where $f = zh\overline{g} \in ST_{Lh}(\alpha)$ with respect to $a \in B$ and $R = H\overline{G} \in P_{Lh}$ with respect to the same a. $F(z)$ is logharmonic with respect to the same a and we have

$$\Re\frac{zF_z(z)-\overline{z}F_{\overline{z}}(z)}{F(z)} = \Re\frac{zf_z(z)-\overline{z}f_{\overline{z}}(z)}{f(z)} + \Re\frac{zR_z(z)-\overline{z}R_{\overline{z}}(z)}{R(z)}. \tag{3.1}$$

From Theorem A, we have

$$f(z) = \varphi(z)\exp 2\Re\int_0^z \frac{a(s)\varphi'(s)}{(1-a(s))\varphi(s)}ds, \tag{3.2}$$

where $\varphi(z) = \dfrac{zh}{g} \in ST(\alpha)$. Moreover, from [2] it follows that

$$R(z) = p(z)\exp 2\Re\int_0^z \frac{a(s)p'(s)}{(1-a(s))p(s)}ds, \tag{3.3}$$

where $p = \dfrac{H}{G} \in P$.

Substituting (3.2), (3.3) into (3.1), simple calculations lead to

$$\begin{aligned}\Re\frac{zF_z(z)-\overline{z}F_{\overline{z}}(z)}{F(z)} &= \Re\frac{zf_z(z)-\overline{z}f_{\overline{z}}(z)}{f(z)} + \Re\frac{zR_z(z)-\overline{z}R_{\overline{z}}(z)}{R(z)}\\ &= \Re\left(\frac{z\varphi'(z)}{\varphi(s)}\right) + \Re\left(\frac{zp'(z)}{p(z)}\right).\end{aligned}$$

But since

$$\Re\left(\frac{zp'(z)}{p(z)}\right) \geqslant \frac{-2|z|}{1-|z|^2}, \text{ and } \Re\left(\frac{z\varphi'(z)}{\varphi(s)}\right) > (1-\alpha)\frac{1-|z|}{1+|z|}+\alpha,$$

we get

$$\Re\frac{zF_z(z)-\overline{z}F_{\overline{z}}(z)}{F(z)} \geqslant (1-\alpha)\frac{1-|z|}{1+|z|}+\alpha-\frac{2|z|}{1-|z|^2} = \frac{(1-2\alpha)|z|^2+(2\alpha-4)|z|+1}{1-|z|^2}.$$

Hence, $\Re\dfrac{zF_z(z)-\overline{z}F_{\overline{z}}(z)}{F(z)} > 0$ if

$$(1-2\alpha)|z|^2+(2\alpha-4)|z|+1 > 0.$$

In the case $\alpha = \frac{1}{2}$, the above is satisfied for $|z| < \frac{1}{3}$, so the radius of starlikeness is $\rho = \frac{1}{3}$. For $\alpha \neq \frac{1}{2}$, the radius of starlikeness ρ is the smallest positive root(less than 1) of

$(1-2\alpha)\rho^2+(2\alpha-4)\rho+1=0$ which is $\dfrac{2-\alpha-\sqrt{\alpha^2-2\alpha+3}}{1-2\alpha}$. Therefore, F is univalent on $|z|<\dfrac{2-\alpha-\sqrt{\alpha^2-2\alpha+3}}{1-2\alpha}$ and maps $\left\{z:|z|<\dfrac{2-\alpha-\sqrt{\alpha^2-2\alpha+3}}{1-2\alpha}\right\}$ onto a starlike domain. We consider the analytic function $F(z)=\dfrac{z}{(1-z)^{2-2\alpha}}\dfrac{1+z}{1-z}$, where $f(z)=\dfrac{z}{(1-z)^{2-2\alpha}}\in ST(\alpha)\subset ST_{Lh}(\alpha)$ and $p(z)=\dfrac{1+z}{1-z}\in P\subset P_{Lh}$. We have $F'\left(\dfrac{2-\alpha-\sqrt{\alpha^2-2\alpha+3}}{1-2\alpha}\right)=0$ for $\alpha\neq\frac{1}{2}$ and $F'(\frac{-1}{3})=0$ for $\alpha=\frac{1}{2}$. Hence, the upper bound is best possible for the class $ST_{Lh}(\alpha)$ and P_{Lh}. Since $f(z)=zh\overline{g}\in ST_{Lh}(\alpha)$ if and only if $\varphi(z)=zh/g\in ST(\alpha)$ and $R(z)=H\overline{G}\in P_{Lh}$ if and only if $p=H/G\in P$ (see[2]). The same bound is best possible for all $a\in B$. $\square$

COROLLARY 2. *Let $F\in CST_{Lh}(\alpha)$, then $F\in ST_{Lh}(\alpha)$ in $|z|<\rho$, for*

$$\rho\leqslant\frac{2-\alpha-\sqrt{-2\alpha+3}}{1-\alpha}.$$

Proof. $F\in ST_{Lh}(\alpha)$ if $\Re\dfrac{zF_z(z)-\overline{z}F_{\overline{z}}(z)}{F(z)}>\alpha$. Using the proof of the previous theorem, this will be satisfied for

$$\frac{(1-2\alpha)|z|^2+(2\alpha-4)|z|+1}{1-|z|^2}>\alpha,$$

that is for $(1-\alpha)|z|^2+(2\alpha-4)|z|+1-\alpha>0$. The radius of starlikeness ρ is the smallest positive root (less than 1) of $(1-\alpha)|z|^2+(2\alpha-4)|z|+1-\alpha=0$ which is $\dfrac{2-\alpha-\sqrt{-2\alpha+3}}{1-\alpha}$. $\square$

THEOREM 3. *Let $F\in CST_{Lh}(\alpha)$ with respect to $a\in B$. Let $f^*\in ST_{Lh}(\alpha)$ with respect to the same a. Then $Q(z)=F(z)^{\lambda}f^*(z)^{1-\lambda}$, $0<\lambda<1$, is univalent and starlike in*

$$|z|<\frac{1+\lambda-\alpha-\sqrt{\alpha^2-2\lambda\alpha+\lambda^2+2\lambda}}{1-2\alpha}$$

for $\alpha\neq\frac{1}{2}$, and in

$$|z|<\frac{1}{2\lambda+1},$$

for $\alpha=\frac{1}{2}$. The bound is best possible for all $a\in B$.

Proof. Let $Q(z)=F(z)^{\lambda}\ f^*(z)^{1-\lambda}$, $0<\lambda<1$, where $F(z)=f(z)R(z)$, $f\in ST_{Lh}(\alpha)$, $R\in P_{Lh}$, and $f^*\in ST_{Lh}(\alpha)$. $Q(z)$ is logharmonic with respect to the same

$a \in B$. Moreover, we have

$$\begin{aligned}
&\Re \frac{zQ_z(z) - \overline{z}Q_{\overline{z}}(z)}{Q(z)} \\
&= \lambda \Re \frac{zf_z(z) - \overline{z}f_{\overline{z}}(z)}{f(z)} + \lambda \Re \frac{zR_z(z) - \overline{z}R_{\overline{z}}(z)}{R(z)} + (1-\lambda)\Re \frac{zf^*_z(z) - \overline{z}f^*_{\overline{z}}(z)}{f^*(z)} \\
&\geqslant \lambda \frac{(1-2\alpha)|z|^2 + (2\alpha-4)|z| + 1}{1-|z|^2} + (1-\lambda)\left((1-\alpha)\frac{1-|z|}{1+|z|} + \alpha\right) \\
&= \frac{(1-2\alpha)|z|^2 + 2(\alpha-\lambda-1)|z| + 1}{1-|z|^2}.
\end{aligned}$$

Hence, $\Re \frac{zQ_z(z) - \overline{z}Q_{\overline{z}}(z)}{Q(z)} > 0$ if

$$(1-2\alpha)|z|^2 + 2(\alpha-\lambda-1)|z| + 1 > 0.$$

For $\alpha = \frac{1}{2}$, the last inequality is satisfied for $|z| < \frac{1}{1+2\lambda}$. Hence, $Q(z)$ is univalent in $|z| < \frac{1}{1+2\lambda}$ and maps that circle onto a starlike domain. For $\alpha \neq \frac{1}{2}$, the last inequality is satisfied for $|z| < \frac{1+\lambda-\alpha-\sqrt{\alpha^2-2\lambda\alpha+\lambda^2+2\lambda}}{1-2\alpha}$. Hence, $Q(z)$ is univalent in $|z| < \frac{1+\lambda-\alpha-\sqrt{\alpha^2-2\lambda\alpha+\lambda^2+2\lambda}}{1-2\alpha}$ and maps that circle onto a starlike domain. We consider the function

$$Q(z) = F_0(z)^{\lambda}\ f_0^*(z)^{1-\lambda},$$

where

$$F_0(z) = \frac{z}{(1-z)^{2-2\alpha}} \frac{1-z}{1+z}$$

and

$$f_0^*(z) = \frac{z}{(1+z)^{2-2\alpha}}.$$

$Q(z)$ satisfies the hypothesis of the theorem since $F_0(z)$ is a product of an analytic function which is starlike of order α and an analytic function with real part positive. Also, $f_0^*(z)$ is starlike analytic function of order α and therefore, it belongs to the set $ST_{Lh}(\alpha)$. Moreover, $Q'\left(\frac{1+\lambda-\alpha-\sqrt{\alpha^2-2\lambda\alpha+\lambda^2+2\lambda}}{1-2\alpha}\right) = 0$ for $\alpha \neq \frac{1}{2}$ and $Q'(\frac{1}{1+2\lambda}) = 0$ for $\alpha = \frac{1}{2}$. From Theorem A, it follows that the same bound is best possible for all $a \in B$. □

For the particular case where $f^* \in ST_{Lh}(0)$, we have the following theorem:

THEOREM 4. *Let $F \in CST_{Lh}(\alpha)$ with respect to $a \in B$ and let $f^* \in ST_{Lh}(0)$ with respect to the same a. Then $Q(z) = F(z)^{\lambda} f^*(z)^{1-\lambda}$, $0 < \lambda < 1$, is univalent and starlike in $|z| < \dfrac{1+\lambda-\lambda\alpha-\sqrt{\lambda^2\alpha^2-2\lambda^2\alpha+\lambda^2+2\lambda}}{1-2\lambda\alpha}$ for $\alpha \neq \frac{1}{2\lambda}$, and in $|z| < \frac{1}{2\lambda+1}$, for $\alpha = \frac{1}{2\lambda}$. The bound is best possible for all $a \in B$.*

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