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A REMARK ON $\varphi-\left|\overline{N},q_n;\delta\right|_k$ SUMMABILITY

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Abstract. In a quite recent paper Özarslan [2] has tried to get necessary and sufficient conditions in order that every φ- $\left|\overline{N},p_n;\delta\right|_k$ summable series should be φ- $\left|\overline{N},q_n;\delta\right|_k$ summable, which extends a result of Bor [1], but does not finalize. In this paper we have solved this open problem.

1. Introduction

Let Σa_v be a given infinite series with partial sums (s_n) and (φ_n) be a sequence of positive real numbers. According to [2], the series Σa_v is summable φ- $\left|\overline{N},p_n;\delta\right|_k$, $k \geqslant 1$, if

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \right|^k < \infty, \tag{1}$$

where (p_n) is a sequence of positive numbers with $P_n = p_0 + p_1 + \ldots + p_n \to \infty$ as $n \to \infty$.

Also, in the special case $\varphi_n = P_n/p_n$, this method is reduced to the summability $\left|\overline{N},p_n;\delta\right|_k$ (see [1]). Bor [1] has recently proved the following theorem concerning the comparison of the methods $\left|\overline{N},p_n;\delta\right|_k$ and $\left|\overline{N},q_n;\delta\right|_k$.

THEOREM 1. *Let $k \geqslant 1$ and $0 \leqslant \delta < 1/k$, $(\varphi_n), (p_n)$ and (q_n) be sequences of positive numbers and let*

$$\sum_{n=v+1}^{\infty} \left(\frac{Q_n}{q_n}\right)^{\delta k-1} \frac{1}{Q_{n-1}} = O\left\{\left(\frac{Q_v}{q_v}\right)^{\delta k} \frac{1}{Q_v}\right\}. \tag{2}$$

In order that every $\left|\overline{N},p_n;\delta\right|_k$ summable series be $\left|\overline{N},q_n;\delta\right|_k$ summable it is necessary that

$$\frac{q_n P_n}{Q_n p_n} = O(1) \tag{3}$$

holds. If

$$\frac{Q_n p_n}{q_n P_n} = O(1). \tag{4}$$

holds, then (3) is also sufficient for the conclusion.

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More recently by following the paper of Bor [1], Özarslan [2] has extended Theorem 1 to the methods φ- $\left|\overline{N},p_n;\delta\right|_k$ and φ- $\left|\overline{N},q_n;\delta\right|_k$ as follows.

THEOREM 2. *Let $k \geqslant 1$ and $0 \leqslant \delta < 1/k$, $(\varphi_n), (p_n)$ and (q_n) be sequences of positive numbers, and let*

$$\sum_{n=v+1}^{\infty} \frac{\varphi_n^{\delta k+k-1} q_n^k}{Q_n^k Q_{n-1}} = O\left\{\frac{\varphi_v^{\delta k+k-1} q_v^{k-1}}{Q_v^k}\right\}. \tag{5}$$

In order that every φ- $\left|\overline{N},p_n;\delta\right|_k$ summable series be φ- $\left|\overline{N},q_n;\delta\right|_k$ summable it is necessary that (3) holds. If (4) holds, then (3) is also sufficient for the conlusion.

2. Main result

Here we note that the author tries to get necessary and sufficient conditions under the condition (5), but does not finalize. Now we solve this problem for arbitrary sequences (p_n) and (q_n) that needn't satisfy condition (5) as follows.

THEOREM 3. *Let $k \geqslant 1$ and $0 \leqslant \delta < 1/k$, (φ_n), (p_n) and (q_n) be sequences of positive numbers.*

(i) If $1 < k < \infty$, then, in order that every φ- $\left|\overline{N},p_n;\delta\right|_k$ summable series should be φ- $\left|\overline{N},q_n;\delta\right|_k$ summable it is necessary and sufficient that (3) and

$$\left\{\sum_{v=1}^{n-1} \varphi_v^{-\delta k^*-1} \left|\frac{P_v P_{v-1}}{p_v} \Delta\left(\frac{Q_{v-1}}{P_{v-1}}\right)\right|^{k^*}\right\}^{1/k^*} K_n^{(k)} = O(1) \tag{6}$$

are satisfied, where

$$K_n^{(k)} = \left\{\sum_{v=n+1}^{\infty} \varphi_v^{\delta k+k-1} \left(\frac{q_v}{Q_v Q_{v-1}}\right)^k\right\}^{1/k}.$$

(ii) If $k = 1$, then in order that every φ- $\left|\overline{N},p_n;\delta\right|$ summable series should be φ- $\left|\overline{N},q_n;\delta\right|$ summable, it is necessary and sufficient that (3) and

$$\left|\frac{P_n P_{n-1}}{\varphi_n^{\delta} p_n} \Delta\left(\frac{Q_{n-1}}{P_{n-1}}\right)\right| K_n^{(1)} = O(1) \tag{7}$$

are satisfied.

Proof. First we recall (see [3]) that if (a_n), (b_n), (A_n) and (B_n) be sequences of positive numbers connected with

$$X_n^* = A_n \sum_{v=1}^{n} a_{v-1} x_v \quad \text{and} \quad Y_n^* = B_n \sum_{v=1}^{n} b_{v-1} \varepsilon_v x_v \tag{8}$$

then $\Sigma|Y_n^*|^s<\infty$ whenever $\Sigma|X_n^*|^k<\infty$ for $1<k\leqslant s<\infty$ if and only if

$$\frac{b_{n-1}B_n}{a_{n-1}A_n}\varepsilon_n=O(1) \tag{9}$$

$$\left(\sum_{v=1}^{n}\left|\frac{1}{A_v}\Delta\left(\frac{b_{v-1}}{a_{v-1}}\varepsilon_v\right)\right|^{k^*}\right)^{1/k^*}\left(\sum_{v=n+1}^{\infty}B_n^s\right)^{1/s}=O(1) \tag{10}$$

are satisfied, where (ε_n) is any sequence of complex numbers and k^* is the conjugate of k. Also, $\Sigma|Y_n^*|^k<\infty$ whenever $\Sigma|X_n^*|<\infty$ for $1\leqslant k<\infty$ if and only if (9) and

$$\left|\frac{1}{A_v}\Delta\left(\frac{b_{v-1}}{a_{v-1}}\varepsilon_v\right)\right|\left(\sum_{v=n+1}^{\infty}B_n^k\right)^{1/k}=O(1) \tag{11}$$

are satisfied. If we take $k=s,\ \varepsilon_n=1,\ A_n=\varphi_n^{\delta+1/k^*}p_n/P_nP_{n-1},\ a_n=P_n,\ B_n=\varphi_n^{\delta+1/k^*}q_n/Q_nQ_{n-1}$ and $b_n=Q_n$, then, it follows from (8) that the series Σa_n is summable φ- $\left|\overline{N},p_n;\delta\right|_k$ and φ- $\left|\overline{N},q_n;\delta\right|_k$ if and only if $\Sigma|X_n^*|^k<\infty$ and $\Sigma|Y_n^*|^k<\infty$, respectively. Also, the conditions (9) and (10) are reduced to the conditions (3) and (6), respectively, which completes the proof.

On the other hand, if we take $k=\varepsilon_n=1,\ A_n=(P_n/p_n)^{\delta+1/k^*}p_n/P_nP_{n-1},\ a_n=P_n$, $B_n=(Q_n/q_n)^{\delta+1/k^*}q_n/Q_nQ_{n-1}$ and $b_n=Q_n$, then we get a modification of Theorem 1. □

THEOREM 4. *Let $k\geqslant 1$ and $0\leqslant\delta<1/k$, (p_n) and (q_n) be sequences of positive numbers.*

(i) If $1<k<\infty$, then, in order that every $\left|\overline{N},p_n;\delta\right|_k$ summable series should be $\left|\overline{N},q_n;\delta\right|_k$ summable it is necessary and sufficient that (3) and

$$\left\{\sum_{v=1}^{n-1}\left|P_{v-1}\left(\frac{p_v}{P_v}\right)^{\delta-1/k}\Delta\left(\frac{Q_{v-1}}{P_{v-1}}\right)\right|^{k^*}\right\}^{1/k^*}H_n^{(k)}=O(1)$$

are satisfied, where

$$H_n^{(k)}=\left\{\sum_{v=n+1}^{\infty}\left(\frac{Q_v}{q_v}\right)^{\delta k-1}\frac{1}{Q_v^k}\right\}^{1/k}.$$

(ii) If $k=1$, then in order that every $\left|\overline{N},p_n;\delta\right|$ summable series should be $\left|\overline{N},q_n;\delta\right|$ summable, it is necessary and sufficient that (3) and

$$\left|P_{n-1}\Delta\left(\frac{p_n}{P_n}\right)^{\delta-1}\Delta\left(\frac{Q_{v-1}}{P_{v-1}}\right)\right|H_n^{(1)}=O(1)$$

are satisfied.

REFERENCES

[1] H. BOR, *Some equivalence theorems on absolute summability methods*, Acta Math. Hungar **149**, (2016), 208–214.

[2] H. S. ÖZARSLAN, *Some equivalence theorems on absolute summability methods*, Inter. J. Anal. Appl. **13**, (2017), 93–97.

[3] M. A. SARIGÖL, *On the local properties of factored Fourier series*, Appl. Math. Comput. **216**, (2010), 3386–3390.

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