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# BOUNDARY SCHWARZ INEQUALITIES ARISING FROM ROGOSINSKI'S LEMMA

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*Abstract.* We consider some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski.

## 1. Introduction

Denote by $\Delta \subset \mathbb{C}$ the open unit disk. *Schwarz's Lemma*, which is a consequence of the Maximum Principle, says that if $f : \Delta \to \Delta$ is analytic with $f(0) = 0$, then

$$|f(\lambda)| \leqslant |\lambda| \quad \forall \lambda \in \Delta, \qquad \text{and therefore} \qquad |f'(0)| \leqslant 1.$$

A sharpened version of this is *Rogosinski's Lemma* (e.g. [3, 4]), which says that

$$|\, f(\lambda) - c_1 \,| \leqslant r_1 \quad \forall \lambda \in \Delta,$$

where

$$c_1 = \frac{\lambda f'(0)(1-|\lambda|^2)}{1-|\lambda|^2|f'(0)|^2} \qquad \text{and} \qquad r_1 = \frac{|\lambda|^2(1-|f'(0)|^2)}{1-|\lambda|^2|f'(0)|^2}.$$

Consequently,

$$|f(\lambda)| \leqslant |c_1| + r_1 = |\lambda| \frac{|\lambda| + |f'(0)|}{1+|\lambda||f'(0)|} \qquad \forall \lambda \in \Delta. \tag{1}$$

Now let us suppose that $f : \Delta \to \Delta$ is analytic and extends continuously to $x \in \partial\Delta$, say along a radius. By pre-composing with a rotation (if necessary) we may assume that $x = 1$ and by post-composing with a rotation (if necessary) we may assume that $f(1) = 1$. Suppose also that the radial derivative of $f$ exists at $1 \in \partial\Delta$:

$$\lim_{r \nearrow 1} \frac{f(r) - f(1)}{r-1} = f'(1).$$

It is easily seen that if $f$ is not constant, then $|f'(1)| > 0$.

But if also $f(0) = 0$, Schwarz's Lemma implies the boundary Schwarz estimate $|f'(1)| \geqslant 1$.

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Osserman [8] showed that in this case, we have in fact

$$|f'(1)| \geqslant 1 + \frac{1-|f'(0)|}{1+|f'(0)|}. \qquad (2)$$

Dubinin [2] incorporated more information about $f$, likewise assuming that $f(0) = 0$, in obtaining

$$|f'(1)| \geqslant 1 + \frac{2(1-|f'(0)|)^2}{1-|f'(0)|^2+|f''(0)|/2}. \qquad (3)$$

A main ingredient in the proof of (2) is (1), though Rogosinski's Lemma is not explicitly mentioned there. The proof of (3) relies heavily on (2); it is a byproduct of a rather complicated investigation which involves zeros of $f$ other than just $\lambda = 0$.

Related to the Schwarz Lemma is *Carathéodory's Inequality* (e.g. [1]), which says that if $f : \Delta \to \mathbb{C}$ is analytic, with $f(0) = 0$ and $\mathrm{Re}(f) \leqslant A$. Then

$$|f(\lambda)| \leqslant \frac{2A|\lambda|}{1-|\lambda|} \quad \forall \lambda \in \Delta, \quad \text{and therefore} \quad |f'(0)| \leqslant 2A. \qquad (4)$$

Örnek [5] showed that if $f$ also extends continuously to $1 \in \partial\Delta$, $\mathrm{Re}(f(1)) = A$, and $f'(1)$ exists, then

$$|f'(1)| \geqslant \frac{A}{2}. \qquad (5)$$

It is our purpose here to: provide an elementary argument, using (1), which yields (3) without appealing to (2); to refine (4), again using (1); and to use (3) to improve (5) and a related result along with it.

## 2. Direct Proof of (3)

If $|f'(0)| = 1$, then (3) holds. Otherwise, as is customary, set

$$g(\lambda) = \frac{f(\lambda)}{\lambda} \quad (\text{with } g(0) := f'(0)), \quad \text{and} \quad h(\lambda) = \frac{f'(0) - g(\lambda)}{1-\overline{f'(0)}g(\lambda)}.$$

Then $h$ is analytic on $\Delta$ with $h(0) = 0$ and by Schwarz's Lemma $h : \Delta \to \Delta$, with

$$h'(0) = \frac{-f''(0)}{2(1-|f'(0)|^2)}.$$

Applying estimate (1) to $h$ and then isolating $f$ (cf. [3, 4]) gives

$$|f(\lambda) - c_2| \leqslant r_2,$$

where

$$c_2 = \frac{\lambda|f'(0)|(1-a^2)}{1-a^2|f'(0)|^2}, \quad r_2 = \frac{a|\lambda|(1-|f'(0)|^2)}{1-a^2|f'(0)|^2}, \quad \text{and} \quad a = |\lambda|\frac{|\lambda|+|h'(0)|}{1+|\lambda||h'(0)|}.$$

Therefore

$$\left|\frac{f(\lambda)-1}{\lambda-1}\right| \geqslant \frac{1-|c_2|-r_2}{|\lambda-1|} = \frac{1-a^2|f'(0)|^2-|\lambda||f'(0)|(1-a^2)-a|\lambda|(1-|f'(0)|^2)}{|\lambda-1|(1-a^2|f'(0)|^2)}$$
$$= \frac{1}{1+a|f'(0)|}\frac{1+a|f'(0)|-a|\lambda|-|\lambda||f'(0)|}{|\lambda-1|}.$$

Now having $\lambda = r \to 1$ radially, so that $|\lambda-1| = 1-r$, an application of L'Hospital's Rule (while noticing that $a = a(r) \to 1$ as $r \to 1$) gives

$$|f'(1)| \geqslant \frac{3+|h'(0)|+|f'(0)||h'(0)|-|f'(0)|}{(1+|f'(0)|)(1+|h'(0)|)}.$$

Finally, some more manipulations lead to

$$|f'(1)| \geqslant 1+\frac{1}{1+|h'(0)|}\frac{2(1-|f'(0)|)^2}{(1-|f'(0)|^2)} = 1+\frac{4(1-|f'(0)|)^2}{2(1-|f'(0)|^2)+|f''(0)|},$$

which is the same as (3). □

REMARK 2.1. Schwarz's Lemma applied to $h$ gives $|f''(0)|/2 \leqslant 1-|f'(0)|^2$, from which we see that Dubinin's estimate (3) improves (and implies) Osserman's (2).

## 3. Refinement of (4)

LEMMA 3.1. *Let $f:\Delta \to \mathbb{C}$ be analytic, with $f(0)=0$ and $\mathrm{Re}(f) \leqslant A$. Then*

$$|f(\lambda)| \leqslant |\lambda|\frac{2A|\lambda|+|f'(0)|}{1-|\lambda|^2} \quad \forall \lambda \in \Delta.$$

*Proof.* Wherever Schwarz's Lemma is used in the proof of (4), we instead use (1); we omit some of the details (c.f. [1]). Set

$$g(\lambda) = \frac{f(\lambda)}{2A-f(\lambda)}.$$

Then $g:\Delta \to \Delta$ is analytic, with $g(0)=0$ and $g'(0) = f'(0)/(2A)$. (And consequently $|f'(0)| \leqslant 2A$, by Schwarz's Lemma.) Applying (1), then the reverse triangle inequality, then (1) again, we get

$$|f(\lambda)| = \frac{2A|g(\lambda)|}{|1+g(\lambda)|} \leqslant \frac{2A}{|1+g(\lambda)|}|\lambda|\frac{|\lambda|+|f'(0)|/(2A)}{1+|\lambda||f'(0)|/(2A)}$$
$$\leqslant 2A|\lambda|\frac{|\lambda|+|f'(0)|/(2A)}{1+|\lambda||f'(0)|/(2A)}\frac{1}{1-|\lambda|\frac{|\lambda|+|f'(0)|/(2A)}{1+|\lambda||f'(0)|/(2A)}}$$
$$= |\lambda|\frac{2A|\lambda|+|f'(0)|}{1-|\lambda|^2}. \quad \square$$

REMARK 3.2. Lemma 3.1 is sharp – consider $f(\lambda) = \frac{2A\lambda}{1+\lambda}$.

## 4. Improvement of (5) and a related result

LEMMA 4.1. *Let $f:\Delta\to\mathbb{C}$ be analytic, with $f(0)=0$ and* $\operatorname{Re}(f)\leqslant A$. *Suppose also that $f$ extends continuously to $1\in\partial\Delta$, say along a radius, that* $\operatorname{Re}(f(1))=A$, *and that the radial derivative $f'(1)$ exists. Then*

$$|f'(1)|\geqslant\frac{A}{2}+\frac{A(2A-|f'(0)|)^2}{4A^2-|f'(0)|^2+|Af''(0)+(f'(0))^2|}.$$

*Proof.* We rework the argument in [5], which uses (2), but instead we use (3). The function $g$ in the proof of Lemma 3.1 satisfies

$$g''(0)=\frac{Af''(0)+(f'(0))^2}{2A^2}.$$

Then from (3) applied to $g$ we get

$$\frac{2A|f'(1)|}{|f(1)-2A|^2}=|g'(1)|\geqslant 1+\frac{2(2A-|f'(0)|)^2}{4A^2-|f'(0)|^2+|Af''(0)+(f'(0))^2|}.$$

Now $|f(1)|\geqslant A$, and so $2|f'(1)|/A\geqslant\frac{2A|f'(1)|}{|f(1)-2A|^2}$ . Therefore,

$$|f'(1)|\geqslant\frac{A}{2}+\frac{A(2A-|f'(0)|)^2}{4A^2-|f'(0)|^2+|Af''(0)+(f'(0))^2|},$$

as desired. □

REMARK 4.2. Lemma 4.1 (which is sharp – again consider $f(\lambda)=\frac{2A\lambda}{1+\lambda}$ ) may be regarded as a companion to Theorem 1 of [6], wherein the the same hypotheses are in play, but also $f'(0)=0$.

In a very similar way one can use (3) to obtain the following result which, as Theorem 2.3 of [7], was obtained quite differently. We merely sketch the idea.

LEMMA 4.3. *Let $f:\Delta\to\mathbb{C}$ be analytic, with $f(0)=0$, and* $|\operatorname{Re}(f)|\leqslant 1$. *Suppose also that $f$ extends continuously to $1\in\partial\Delta$, say along a radius, that* $\operatorname{Re}(f(1))=1$, *and that the radial derivative $f'(1)$ exists. Then*

$$|f'(1)|\geqslant\frac{2}{\pi}\left(1+\frac{2(4-\pi|f'(0)|)^2}{16-\pi^2|f'(0)|^2+2\pi|f''(0)|}\right).$$

*Proof.* As in [7], set

$$\phi(\lambda)=\frac{e^{\frac{i\pi}{2}f(\lambda)}-1}{e^{\frac{i\pi}{2}f(\lambda)}+1}.$$

It is easily verified that $\phi:\Delta\to\Delta$ is analytic, $\phi(0)=0$, $|\phi(1)|=1$, $\phi'(0)=\frac{i\pi}{4}f'(0)$, $\phi''(0)=\frac{i\pi}{4}f''(0)$, and $|\phi'(1)|=\frac{\pi}{2}|f'(1)|$. Then applying (3) to $\phi$ and writing the result in terms of $f$, the inequality follows. □

REMARK 4.4. This result is sharp – consider $f(\lambda)=\frac{2}{i\pi}\ln\frac{1+\lambda}{1-\lambda}$. By Remark 2.1, Theorems 2.1 and 2.2 of [7] are simple corollaries of Theorem 2.3 there.

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