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ON COEFFICIENT FUNCTIONALS ASSOCIATED WITH THE ZALCMAN CONJECTURE

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Abstract. For a function f which is analytic and univalent in the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ having the power series expansion of the normalized form $z + \sum_{n=2}^{\infty} a_n z^n$, Zalcman conjectured that $|a_n^2 - a_{2n-1}| \leqslant (n-1)^2$, $n = 2, 3, \ldots$. In this article, we obtain the sharp estimate for the classical Zalcman coefficient functional $a_n^2 - a_{2n-1}$ for the above class of functions with the restriction that the n-th coefficient, a_n, has certain integral representation associated with probability measure. Moreover, we also study a similar problem for the classes of functions of the above form whose coefficients satisfy certain inequalities.

1. Introduction

We denote by $\mathscr{A}$, the class of all analytic functions f in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

and by $\mathscr{S}$, the class of *univalent functions* in $\mathscr{A}$. Then $|a_2^2 - a_3| \leqslant 1$ holds for $f \in \mathscr{S}$, see [20, Theorem 1.5]. At the end of 1960's, Zalcman made a conjecture that each $f \in \mathscr{S}$ satisfies the inequality

$$|a_n^2 - a_{2n-1}| \leqslant (n-1)^2, \quad n \geqslant 2 \tag{2}$$

with equality for the Koebe function $k(z) = z/(1-z)^2$ and its rotations. One of the main aims of the Zalcman conjecture was to prove the Bieberbach conjecture: $|a_n| \leqslant n$, for $n \geqslant 2$, when $f \in \mathscr{S}$, using the famous Hayman Regularity Theorem (see [5, Theorem 5.6, pp. 163]). The Bieberbach conjecture was a challenging open problem for function theorists for several decades and was finally settled by de Branges [3] in 1985.

There are several approaches made to prove the Zalcman conjecture. One of the approaches is to prove the conjecture for some subclasses of $\mathscr{S}$. For example, in [4], Brown and Tsao proved that (2) holds for the class $\mathscr{T}$ of typically real functions and the class $\mathscr{S}^*$ of starlike functions. In [17], Ma proved the Zalcman conjecture for the class $\mathscr{K}$ of close-to-convex functions when $n \geqslant 4$. However, this conjecture was remained open for $n = 3$ and this has recently been settled in [14]. Readers can refer to, for instance, [1, 10, 11, 13, 14] and references therein for more information on this topic.

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A generalized version of Zalcman's inequality, in terms of the so-called *generalized coefficient functional* $\lambda a_n^2 - a_{2n-1}$, $\lambda > 0$, has been considered in [1, 4, 6, 13].

In [18], Ma proposed a generalized version of the Zalcman conjecture as follows: for $f \in \mathscr{S}$,

$$|a_n a_m - a_{n+m-1}| \leqslant (n-1)(m-1) \quad (n,m = 2,3,\ldots)$$

and proved that this holds for starlike functions and univalent functions with real coefficients. Recently, Efraimidis and Vukotić in [6] proved that the Zalcman conjecture is asymptotically true.

In this paper, we establish sharp estimates of the Zalcman conjecture in the form proposed by Ma in [18] for some classes of analytic functions of the form (1) such that the n-th coefficient a_n has the form

$$a_n = s(n) \int_0^{2\pi} e^{i(n-1)\theta} \mathrm{d}\mu(\theta),$$

where $s(n)$ is some non-negative function of n and $\mu(\theta)$ is a probability measure on $[0, 2\pi]$. We denote such class of functions by $\mathscr{F}$. Note that the class $\mathscr{F}$ no longer consists exclusively the univalent functions. For example, consider the function

$$f(z) = \frac{z}{1-z^3} = \sum_{n=0}^{\infty} z^{3n+1} = z + z^4 + z^7 + \ldots, \quad z \in \mathbb{D}.$$

Here $a_{3n+1} = 1$ whereas $a_n = 0$. Hence $f \in \mathscr{F}$. It can easily be seen that $f(z)$ is not univalent in $\mathbb{D}$. In addition, we consider the functions of the form (1) such that the n-th coefficient a_n satisfy the inequality

$$\sum_{n=2}^{\infty} r(n)|a_n| \leqslant 1, \quad r(n) > 0.$$

We denote such class of functions by $\mathscr{H}$ and obtain the sharp estimate for the Zalcman coefficient functional for the class $\mathscr{H}$.

We conclude this section with some basic definitions. A function $f \in \mathscr{A}$ is said to be *starlike of order* β $(0 \leqslant \beta < 1)$ if $\operatorname{Re}\{zf'(z)/f(z)\} > \beta$ and denote the class of starlike functions of order β by $\mathscr{S}^*(\beta)$. Similarly, a function $f \in \mathscr{A}$ is said to be *convex of order* β $(0 \leqslant \beta < 1)$ if $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > \beta$ and denote the class of convex functions of order β by $\mathscr{C}(\beta)$. Clearly, functions in the classes $\mathscr{S}^*(\beta)$ and $\mathscr{C}(\beta)$ are univalent in $\mathbb{D}$. Moreover $\mathscr{S}^*(0) = \mathscr{S}^*$ and $\mathscr{C}(0) = \mathscr{C}$.

A function f is said to be *uniformly starlike* in $\mathbb{D}$ if f is starlike and has the property that for every circular arc γ contained in $\mathbb{D}$, with center $\zeta \in \mathbb{D}$, the arc $f(\gamma)$ is starlike with respect to $f(\zeta)$. We denote by $\mathscr{U}\mathscr{S}\mathscr{T}$, the class of all uniformly starlike functions. Similarly, we say that a convex function f in $\mathbb{D}$ is *uniformly convex* if for each circular arc γ in $\mathbb{D}$ with center η in $\mathbb{D}$, the image arc $f(\gamma)$ is convex. Denote the class of all uniformly convex functions by $\mathscr{U}\mathscr{C}\mathscr{V}$, see [7, 8]. We call a function $f \in \mathscr{A}$ is ν-*spiral-like of order* $\beta, 0 \leqslant \beta < 1$, if there is a real number ν $(-\pi/2 < \nu < \pi/2)$ such that $\operatorname{Re}[e^{i\nu}\{zf'(z)/f(z)\}] > \beta \cos\nu$ for $z \in \mathbb{D}$. We denote by $\mathscr{S}_p^{\nu}(\beta)$, the class of ν-spiral-like functions of order β, see [12]. More literature on spiral-like functions can be found in [2, 16, 19]. Recent investigation on spiral-like functions in connection with Yamashita conjecture, and integral means may be found from [21].

2. Main results

This section is devoted to our main results. The following lemma follows from the work of Ma [18] and other recent results (see also [22, Lemma 1.4]).

LEMMA A. *Let* $\mu(\theta)$ *be a probability measure on* $[0,2\pi]$. *Then for* $\lambda \in \mathbb{C}$,

$$\left|\lambda \int_0^{2\pi} e^{i(n-1)\theta} \mathrm{d}\mu(\theta) \int_0^{2\pi} e^{i(m-1)\theta} \mathrm{d}\mu(\theta) - \int_0^{2\pi} e^{i(n+m-2)\theta} \mathrm{d}\mu(\theta)\right| \leqslant \max\{|\lambda - 1|, 1\}$$

for $n, m = 2, 3, \ldots$.

Now we state the first main result of this paper.

THEOREM 1. *Let* $f \in \mathscr{F}$. *Then for* $\lambda \in \mathbb{C}$ *and* $n, m = 2, 3, \ldots$,

$$|\lambda a_n a_m - a_{n+m-1}| \leqslant \max\{|\lambda s(n)s(m) - s(n+m-1)|, s(n+m-1)\}.$$

The inequality is sharp.

Proof. Using the integral representations of the coefficients a_n, a_m and a_{n+m-1} in the expression $\lambda a_n a_m - a_{n+m-1}$, we rewrite

$$\begin{aligned}
&|\lambda a_n a_m - a_{n+m-1}| \\
&= \left|\lambda s(n)s(m) \int_0^{2\pi} e^{i(n-1)\theta} \mathrm{d}\mu(\theta) \int_0^{2\pi} e^{i(m-1)\theta} \mathrm{d}\mu(\theta)\right. \\
&\quad \left. - s(n+m-1) \int_0^{2\pi} e^{i(n+m-2)\theta} \mathrm{d}\mu(\theta)\right| \\
&= s(n+m-1)\left|\lambda \frac{s(n)s(m)}{s(n+m-1)} \int_0^{2\pi} e^{i(n-1)\theta} \mathrm{d}\mu(\theta) \int_0^{2\pi} e^{i(m-1)\theta} \mathrm{d}\mu(\theta)\right. \\
&\quad \left. - \int_0^{2\pi} e^{i(n+m-2)\theta} \mathrm{d}\mu(\theta)\right|.
\end{aligned}$$

Now, by applying Lemma A, we have

$$|\lambda a_n a_m - a_{n+m-1}| \leqslant \max\{|\lambda s(n)s(m) - s(n+m-1)|, s(n+m-1)\}.$$

The sharpness of the inequality can be verified from the following example:

$$F(z) = \begin{cases} z + \sum_{n=2}^{\infty} s(n) z^n \, (=: f_0(z)), & |\lambda s(n)s(m) - s(n+m-1)| \geqslant s(n+m-1), \\ \dfrac{1}{n+m-2} \sum_{k=1}^{n+m-2} e^{-i\phi_k} f_0(e^{i\phi_k} z), & |\lambda s(n)s(m) - s(n+m-1)| < s(n+m-1), \end{cases}$$

where $\phi_k = \dfrac{2k\pi}{n+m-2}$. The proof of our theorem is complete. □

The case $n = m$ in Theorem 1 gives

COROLLARY 1. *Let $f \in \mathscr{F}$. Then for $\lambda \in \mathbb{C}$*

$$|\lambda a_n^2 - a_{2n-1}| \leqslant \max\{|\lambda s(n)^2 - s(2n-1)|, s(2n-1)\}, \ \ n \geqslant 2.$$

The inequality is sharp and can be verified from the following example:

$$F(z) = \begin{cases} f_0(z), & |\lambda s(n)^2 - s(2n-1)| \geqslant s(2n-1), \\ \sum_{k=1}^{2n-2} m_k e^{-i\theta_k} f_0(e^{i\theta_k} z), & |\lambda s(n)^2 - s(2n-1)| < s(2n-1). \end{cases}$$

Here $0 \leqslant m_k \leqslant 1$, $\theta_k = \dfrac{(2k+1)\pi}{2n-2}$, *and* $\sum_{k=1}^{n-1} m_{2k} = \sum_{k=1}^{n-1} m_{2k-1} = \dfrac{1}{2}$. *Note that* m_k *can attain the value* $1/(2n-2)$.

REMARK 1. Theorem 1 and Corollary 1 help us to estimate the generalized Zalcman coefficient functional $\lambda a_n a_m - a_{n+m-1}$ and $\lambda a_n^2 - a_{2n-1}$ for several subclasses of functions in $\mathscr{F}$. For instance, the results stated below are consequences of Theorem 1 and Corollary 1.

Let us denote by $\overline{co(\mathscr{S}^*(\alpha))}$, $\alpha < 1$, the closed convex hull of $\mathscr{S}^*(\alpha)$. Then, for all f in $\overline{co(\mathscr{S}^*(\alpha))}$, the n-th coefficients of the series expansion of f can be written in the form that

$$a_n = \frac{1}{(n-1)!} \prod_{j=0}^{n-2} (2(1-\alpha)+j) \int_0^{2\pi} e^{i(n-1)\theta} \mathrm{d}\mu(\theta), \quad n \geqslant 2.$$

Hence, for $f \in \overline{co(\mathscr{S}^*(\alpha))}$, $s(n) = \dfrac{1}{(n-1)!} \prod_{j=0}^{n-2} (2(1-\alpha)+j) = A_n(say)$.

An immediate corollary to Theorem 1 for the class $\overline{co(\mathscr{S}^*(\alpha))}$ is the following result.

COROLLARY 2. [22, Theorem 2.1] *If* $f \in \overline{co(\mathscr{S}^*(\alpha))}$ $(\alpha < 1)$, *then*

$$|\lambda a_n a_m - a_{n+m-1}| \leqslant \max\{|\lambda A_n A_m - A_{n+m-1}|, A_{n+m-1}\}.$$

Equality occur for the functions given by

$$F(z) = \begin{cases} k_\alpha(z) = z/(1-z)^{2(1-\alpha)} = z + \sum_{n=2}^{\infty} s(n) z^n, & |\lambda A_n A_m - A_{n+m-1}| \geqslant A_{n+m-1}, \\ k_\alpha^{n,m}(z) = \dfrac{1}{n+m-2} \sum_{k=1}^{n+m-2} e^{-i\phi_k} k_\alpha(e^{i\phi_k} z), & |\lambda A_n A_m - A_{n+m-1}| < A_{n+m-1}. \end{cases}$$

Here m_k, θ_k *and* ϕ_k *are the same quantities as defined in Theorem 1 and Corollary 1.*

REMARK 2. Here we have pointed out several consequences of Theorem 1 and Corollary 1 for the classes $\overline{co(\mathscr{S}^*(\alpha))}$ and $\overline{co(\mathscr{S}^*)}$.

- The case $m = n$ in Corollary 2 gives

$$|\lambda a_n^2 - a_{2n-1}| \leqslant \max\{|\lambda A_n^2 - A_{2n+1}|, A_{2n-1}\}$$

for $n = 2, 3, \ldots$. Equality occurs for the functions $k_\alpha(z)$ and

$$k_\alpha^n(z) = \sum_{k=1}^{2n-2} m_k \frac{z}{(1 - e^{i\theta_k} z)^{2(1-\alpha)}}.$$

Here $k_\alpha(z)$ is the function as defined in Corollary 2. This result is also pointed out in [22, Corollary 2.2].

- The case $\alpha = 0$ in Corollary 2 gives: if $f \in \overline{co(\mathcal{S}^*)}$, then

$$|\lambda a_n a_m - a_{n+m-1}| \leqslant \max\{|\lambda nm - n - m + 1)|, n + m - 1\}.$$

Equality occurs for the functions $k_0(z) = \sum_{n=1}^{\infty} nz^n = z/(1-z)^2$ and its rotations when $|\lambda nm - n - m + 1)| \geqslant n + m - 1$ and for the function

$$k_0^{n,m}(z) = \frac{1}{n+m-2} \sum_{k=1}^{n+m-2} \frac{z}{(1 - e^{i\phi_k} z)^2} = \sum_{r=0}^{\infty} (r(n+m-2)+1) z^{r(n+m-2)+1},$$

when $|\lambda nm - n - m + 1)| \leqslant n + m - 1$. This is proved in [6, Theorem 3.5].

- The restrictions on λ, $\frac{2(n+m-1)}{nm} \leqslant \lambda \in \mathbb{R}$, in Theorem 1 obtains the result proved by Ma [18, Theorem 2.2].

- If $f \in \overline{co(\mathcal{S}^*)}$, then for $m = n$, we have

$$|\lambda a_n^2 - a_{2n-1}| \leqslant \max\{|\lambda n^2 - 2n + 1|, 2n - 1\}$$

for $n = 2, 3, \ldots$. Equality occurs for the functions $k_0(z)$ and

$$k_0^n(z) = \sum_{k=1}^{2n-2} m_k \frac{z}{(1 - e^{i\theta_k} z)^2}.$$

This result is a consequence of Corollary 2.

- Another consequence of Corollary 2 obtains a result of Brown and Tsao [4, p. 474]. That is, If $\lambda \in \mathbb{R}$ and $f \in \overline{co(\mathcal{S}^*)}$, then

$$|\lambda a_n^2 - a_{2n-1}| \leqslant \begin{cases} 2n-1, & 0 \leqslant \lambda \leqslant \dfrac{2(2n-1)}{n^2}, \\ \lambda n^2 - 2n + 1, & \lambda > \dfrac{2(2n-1)}{n^2}, \end{cases}$$

for $n = 2, 3, \ldots$.

Let us denote by $\overline{co(\mathscr{C}(\alpha))}$, $\alpha < 1$, the closed convex hull of $\mathscr{C}(\alpha)$. Then, for all f in $\overline{co(\mathscr{C}(\alpha))}$, the n-th coefficients of the series expansion of f can be written in the form that

$$a_n = \frac{1}{n!}\prod_{j=0}^{n-2}(2(1-\alpha)+j)\int_0^{2\pi} e^{i(n-1)\theta}\mathrm{d}\mu(\theta), \quad n \geqslant 2.$$

Hence, for $f \in \overline{co(\mathscr{C}(\alpha))}$, $s(n) = \frac{1}{n!}\prod_{j=0}^{n-2}(2(1-\alpha)+j) = \frac{A_n}{n} = B_n(say)$.

The function defined by

$$l_\alpha(z) = \begin{cases} \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1}, & \text{for } \alpha \neq 1/2, \\ -\log(1-z), & \text{for } \alpha = 1/2, \end{cases}$$

is often extremal in the class $\mathscr{C}(\alpha)$. The coefficients a_n of $l_\alpha(z)$ are B_n.

As a consequence of Theorem 1, we have the following result for the class $\overline{co(\mathscr{C}(\alpha))}$ which is also proved in [22, Corollary 2.3] and [6, Theorem 3.4].

COROLLARY 3. *If $f \in \overline{co(\mathscr{C}(\alpha))}$ $(\alpha < 1)$, then*

$$|\lambda a_n a_m - a_{n+m-1}| \leqslant \max\{|\lambda B_n B_m - B_{n+m-1}|, B_{n+m-1}\}.$$

Equality occurs for the functions $l_\alpha(z)$ and its rotations when $|\lambda B_n B_m - B_{n+m-1}| \geqslant B_{n+m-1}$ and for the function

$$l_\alpha^{n,m}(z) = \frac{1}{n+m-2}\sum_{k=1}^{n+m-2} e^{-i\phi_k} l_\alpha(e^{i\phi_k}z),$$

when $|\lambda B_n B_m - B_{n+m-1}| \leqslant B_{n+m-1}$. Here ϕ_k is the same quantity as defined in Theorem 1.

COROLLARY 4. [22, Corollary 2.4] *If $f \in \overline{co(\mathscr{C}(\alpha))}$, then Corollary 1 gives*

$$|\lambda a_n^2 - a_{2n-1}| \leqslant \max\{|\lambda B_n^2 - B_{2n-1}|, B_{2n-1}\}$$

for $n = 2,3,\ldots$. Equality occurs for the functions $l_\alpha(z)$ and

$$l_\alpha^n(z) = \sum_{k=1}^{2n-2} m_k e^{-i\theta_k} l_\alpha(e^{i\theta_k}z),$$

where m_k and θ_k are defined as in Corollary 1.

It can easily be checked that for $\alpha = -1/2$, $B_n = (n+1)/2$. As a consequence of Corollary 3, we have the following result for the class $\overline{co(\mathscr{C}(-1/2))}$. Note that in [13, Theorem 3.3], the authors have proved the following result by considering three cases for $n \geqslant 3$. Here all the three cases are covered in two cases.

COROLLARY 5. *If* $f \in \overline{co(\mathscr{C}(-1/2)}$, *then*

$$|\lambda a_n^2 - a_{2n-1}| \leqslant \begin{cases} n, & 0 \leqslant \lambda \leqslant \dfrac{8n}{(n+1)^2}, \\ \left|\dfrac{(n+1)^2}{4}\lambda - n\right|, & elsewhere, \end{cases}$$

for $n = 2,3,\ldots$. *The sharpness of the second inequality can be verified by the function* $l_{-1/2}(z) = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n = \frac{z - z^2/2}{(1-z)^2}$ *and its rotations; and sharpness of the first inequality can be verified by the function*

$$l_{-1/2}^{n}(z) = \sum_{k=1}^{2n-2} m_k e^{-i\theta_k} l_{-1/2}(e^{i\theta_k} z).$$

Here m_k *and* θ_k *are the same as defined in Corollary 1.*

Moreover, as a consequence of Theorem 1, Corollary 5 can be generalized in the following way:

COROLLARY 6. *If* $f \in \overline{co(\mathscr{C}(-1/2)}$, *then*

$$|\lambda a_n a_m - a_{n+m-1}| \leqslant \begin{cases} \dfrac{n+m}{2}, & 0 \leqslant \lambda \leqslant \dfrac{4(n+m)}{(n+1)(m+1)}, \\ \left|\dfrac{(n+1)(m+1)}{4}\lambda - \dfrac{n+m}{2}\right|, & elsewhere, \end{cases}$$

for $n = 2,3,\ldots$. *The second inequality is sharp for the function* $l_{-1/2}(z)$ *whereas the first inequality is sharp for the function*

$$l_{-1/2}^{n,m}(z) = \frac{1}{n+m-2} \sum_{k=1}^{n+m-2} e^{-i\phi_k} l_{-1/2}(e^{i\phi_k} z).$$

Observe that

- For $\alpha = 0$, Corollary 3 reduces to [6, Theorem 3.3].
- For the case $\alpha = 0$ and $\lambda \in \mathbb{R}$, Corollary 4 reduces to the result obtained by Li et al. in [15, Theorem 1].
- For $-1/2 \leqslant \alpha < 0$, Corollary 4 coincides with [15, Theorem 2].
- For $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$, Corollary 4 coincides with [15, Theorem 3].
- For $\alpha = 1/2$, Corollary 4 coincides with [15, Theorem 4].

We now consider the functions in the normalized class

$$\mathscr{R}(\beta) := \{f \in \mathscr{A} : \operatorname{Re} f'(z) > \beta\}$$

where $\beta \in [0,1)$. Denote $\mathscr{R} = \mathscr{R}(0)$.

By the Herglotz representation theorem for functions with positive real part [5, 1.9], there is a unique probability measure μ on $[0,2\pi]$ such that

$$\frac{f'(z)-\beta}{1-\beta} = \int_0^{2\pi} \frac{1+e^{i\theta}z}{1-e^{i\theta}z} \mathrm{d}\mu(\theta)$$

or, equivalently,

$$1+\sum_{n=2}^{\infty} n a_n z^{n-1} = 1+(1-\beta)\sum_{n=2}^{\infty} 2\int_0^{2\pi} e^{in\theta} \mathrm{d}\mu(\theta) z^n.$$

Comparing the coefficients, we obtain

$$a_n = \frac{2(1-\beta)}{n}\int_0^{2\pi} e^{i(n-1)\theta} \mathrm{d}\mu(\theta), \quad n \geqslant 2.$$

Hence for $f \in \mathscr{R}(\beta)$, $\beta \in [0,1)$, $s(n) = 2(1-\beta)/n$.

Now, as consequence of Theorem 1 we have the following result which is also pointed out in [22, Theorem 3.1].

COROLLARY 7. *If $f \in \mathscr{R}(\beta)$, then*

$$|\lambda a_n a_m - a_{n+m-1}| \leqslant \max\left\{\left|\frac{4\lambda(1-\beta)^2}{nm} - \frac{2(1-\beta)}{n+m-1}\right|, \frac{2(1-\beta)}{n+m-1}\right\},$$

for $n = 2,3,\ldots$. The sharpness of the first inequality can easily be verified by using the function $m_\beta(z) = -2(1-\beta)\ln(1-z) - z(1-2\beta)$ and its rotations whereas the sharpness of the second inequality can be verified by using the function

$$m_\beta^{n,m}(z) = \frac{1}{n+m-2}\sum_{k=1}^{n+m-2} e^{-i\phi_k} m_\beta(e^{i\phi_k} z).$$

Here ϕ_k is the same quantity as defined in Theorem 1.

For $\beta = 0$, Corollary 7 reduces to [6, Theorem 3.2]. In particular, when $m = n$, Corollary 7 leads to

COROLLARY 8. *If $f \in \mathscr{R}(\beta)$, then*

$$|\lambda a_n^2 - a_{2n-1}| \leqslant \max\left\{\left|\frac{4\lambda(1-\beta)^2}{n^2} - \frac{2(1-\beta)}{2n-1}\right|, \frac{2(1-\beta)}{2n-1}\right\},$$

for $n = 2,3,\ldots$. The sharpness of the first inequality can easily be verified using the function $m_\beta(z)$ and its rotations. Sharpness of the second inequality can be verified for the function

$$m_\beta^n(z) = \sum_{k=1}^{2n-2} m_k e^{-i\theta_k} m_\beta(e^{i\theta_k} z).$$

Here m_k and θ_k are the same as defined in Corollary 1.

2.1. The class $\mathcal{H}$

Recall that

$$\mathcal{H} = \left\{ f \in \mathcal{A} : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } \sum_{n=2}^{\infty} r(n)|a_n| \leqslant 1, r(n) > 0 \text{ for } n \geqslant 2 \right\}.$$

Here is a partial list of restrictions on $r(n)$ such that $\mathcal{H}$ is a subclass of $\mathcal{S}$. For example,

- If $r(n) = (n-\beta)/(1-\beta)$, then $\mathcal{H} \subset \mathcal{S}^*(\beta) \subset \mathcal{S}$ [23]. In particular, for $\beta = 0$ we have $\mathcal{H} = H$, the Hurwitz class.
- If $r(n) = n(n-\beta)/(1-\beta)$, then $\mathcal{H} \subset \mathcal{C}(\beta) \subset \mathcal{S}$ [23].
- If $r(n) = 3n-2$, then $\mathcal{H} \subset \mathcal{U}\mathcal{S}\mathcal{T} \subset \mathcal{S}$ [9].
- If $r(n) = n(2n-1)$, then $\mathcal{H} \subset \mathcal{U}\mathcal{C}\mathcal{V} \subset \mathcal{S}$ [9].
- If $r(n) = n/(1-\beta)$, then $\mathcal{H} \subset \mathcal{R}(\beta) \subset \mathcal{S}$.
- If $r(n) = 1 + [(n-1)/(1-\beta)]\sec \nu$, then $\mathcal{H} \subset \mathcal{S}_p^{\nu}(\beta) \subset \mathcal{S}$ [12].

In all these classes $\beta \in [0,1)$. We now state our main result for the class $\mathcal{H}$.

THEOREM 2. *(a) Let $\lambda \in \mathbb{C}$ and $n = 2,3,\ldots$. For $f \in \mathcal{H}$, we have*

$$|\lambda a_n^2 - a_{2n-1}| \leqslant \max\left\{ \frac{|\lambda|}{r(n)^2}, \frac{1}{r(2n-1)} \right\}.$$

Equality holds if and only if

$$f(z) = \begin{cases} z + \dfrac{\alpha}{r(2n-1)} z^{2n-1} & \text{for } |\lambda| \leqslant \dfrac{r(n)^2}{r(2n-1)}, \\ z + \dfrac{\alpha}{r(n)} z^n & \text{for } |\lambda| \geqslant \dfrac{r(n)^2}{r(2n-1)}, \end{cases}$$

where α is a complex number such that $|\alpha| = 1$.

(b) If $f \in \mathcal{H}$ and $\lambda \in \mathbb{C}$ then for two distinct values $m,n \geqslant 2$ we have

$$|\lambda a_n a_m - a_{n+m-1}| \leqslant \max\left\{ \frac{|\lambda|}{4r(n)r(m)}, \frac{1}{r(n+m-1)} \right\}.$$

Equality holds if and only if

$$f(z) = \begin{cases} z + \dfrac{\alpha}{r(n+m-1)} z^{n+m-1} & \text{for } |\lambda| \leqslant \dfrac{4r(n)r(m)}{r(n+m-1)}, \\ z + \dfrac{\alpha}{2r(n)} z^n + \dfrac{\beta}{2r(m)} z^m & \text{for } |\lambda| \geqslant \dfrac{4r(n)r(m)}{r(n+m-1)}, \end{cases}$$

where α and β are complex numbers such that $|\alpha| = |\beta| = 1$.

We remark that for the choice $r(n) = n$, Theorem 2 turns into [6, Theorem 3.1(a)]. We here adopt the proof technique of [6, Theorem 3.1(a)]. To prove the generalized Zalcman problem for $\mathscr{H}$, we need the following lemma.

LEMMA 1. [6, Lemma 2.1] *Let $a,b \in \mathbb{C}$ be arbitrary and let $C,M > 0$. Then*

$$|a+\lambda b| \leqslant \max\{C, |\lambda|\}, \quad \textit{for all } \lambda \in \mathbb{C} \tag{3}$$

if and only if

$$|a| + |b|C \leqslant MC. \tag{4}$$

Assuming that $a,b \neq 0$, equality holds in (3) for some $\lambda \neq 0$ if and only if it holds in (4) and also $|\lambda| = C$ and $\arg\lambda = \arg a - \arg b$ (taking the values of the argument function modulus 2π).

Proof of Theorem 2. (a) By the definition of the class $\mathscr{H}$, $r(n)|a_n| \leqslant 1$ and $r(n)|a_n| + r(2n-1)|a_{2n-1}| \leqslant 1$. Therefore,

$$r(n)^2|a_n|^2 + r(2n-1)|a_{2n-1}| \leqslant r(n)|a_n| + r(2n-1)|a_{2n-1}| \leqslant 1.$$

Substituting the values $M = 1/r(n)^2$ and $C = r(n)^2/r(2n-1)$ in Lemma 1, we obtain the desired result.

(b) From the definition, it is clear that $r(n)|a_n| + r(m)|a_m| \leqslant 1$. Therefore,

$$4nm|a_n a_m| \leqslant (r(n)|a_n| + r(m)|a_m|)^2 \leqslant r(n)|a_n| + r(m)|a_m|.$$

Hence,

$$4r(n)r(m)|a_n a_m| + r(n+m-1)a_{n+m-1} \leqslant r(n)|a_n| + r(m)|a_m| + r(n+m-1)a_{n+m-1} \leqslant 1.$$

The conclusion now follows by taking $M = 1/4r(n)r(m)$ and $C = 4r(n)r(m)/r(n+m-1)$ in Lemma 1. $\square$

3. Concluding remarks

In the earlier version of this article (arXiv:1604.05494) we had posed the open problems on the generalized Zalcman conjecture in the form proposed by Ma in [18] for the classes $\overline{co(\mathbb{C})}$ and $\mathscr{R}(\beta)$ when $0 < \lambda < 2$ and $0 < \lambda < nm/(1-\beta)(n+m-1)$ respectively. These problems are recently settled by Ravichandran and Verma in [22] (see, [22, Corollary 2.4, Theorem 3.1]).

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