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ON ABSOLUTE MATRIX SUMMABILITY FACTORS OF INFINITE SERIES

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Abstract. In the present paper, a general theorem dealing with $|A,p_n;\delta|_k$ summability method of infinite series has been proved by using almost increasing sequences. Some results have also been given.

1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n=\sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad (n\to\infty), \quad (P_{-i}=p_{-i}=0, \quad i\geqslant 1). \tag{1}$$

Let $A=(a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s=(s_n)$ to $As=(A_n(s))$, where

$$A_n(s)=\sum_{v=0}^{n} a_{nv}s_v, \quad n=0,1,\ldots \tag{2}$$

The series $\sum a_n$ is said to be summable $|A,p_n;\delta|_k$, $k\geqslant 1$ and $\delta\geqslant 0$, if (see [10])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|\bar{\Delta}A_n(s)|^k<\infty, \tag{3}$$

where

$$\bar{\Delta}A_n(s)=A_n(s)-A_{n-1}(s).$$

If we take $\delta=0$, then $|A,p_n;\delta|_k$ summability reduces to $|A,p_n|_k$ summability (see [18]). If we take $\delta=0$, $a_{nv}=\frac{p_v}{P_n}$, then we get $|\bar{N},p_n|_k$ summability (see [2]). Furthermore, if we take $\delta=0$, $a_{nv}=\frac{p_v}{P_n}$ and $p_n=1$ for all values of n, then $|A,p_n;\delta|_k$ summability reduces to $|C,1|_k$ summability (see [7]).

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2. Known result

In [3], Bor has proved the following theorem for $\left|\bar{N},p_n\right|_k$ summability factors of infinite series using positive non-decreasing sequence.

THEOREM 1. *Let (X_n) be a positive non-decreasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$\mid \Delta\lambda_n \mid \leqslant \beta_n, \tag{4}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{5}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{6}$$

$$\mid \lambda_n \mid X_n = O(1). \tag{7}$$

If

$$\sum_{v=1}^{n} \frac{\mid s_v \mid^k}{v} = O(X_n) \quad as \quad n \to \infty \tag{8}$$

and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{9}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{10}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N},p_n \mid_k$, $k \geqslant 1$.

REMARK 1. It should be noted that, from the hypotheses of Theorem 1, (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [3]).

3. Main result

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants K and L such that $Kc_n \leqslant b_n \leqslant Lc_n$ (see [1]). Many works on almost increasing sequences have been done (see [4]–[6], [11]–[17]). The purpose of this paper is to generalize Theorem 1 for $|A,p_n;\delta|_k$ summability. Before giving the main theorem, we must first introduce some further notations.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n,v = 0,1,\ldots \tag{11}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1,2,\ldots \tag{12}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{i=0}^{n} \bar{a}_{ni}a_i \tag{13}$$

and

$$\bar{\Delta} A_n(s) = \sum_{i=0}^{n} \hat{a}_{ni}a_i. \tag{14}$$

Now, we shall prove the following theorem.

THEOREM 2. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1,\ n = 0,1,\ldots, \tag{15}$$

$$a_{n-1,v} \geqslant a_{nv},\ for\ n \geqslant v+1, \tag{16}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{17}$$

$$|\ \hat{a}_{n,v+1}\ | = O(v\ |\ \Delta_v(\hat{a}_{nv})\ |), \tag{18}$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v \hat{a}_{nv}| = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{p_v}{P_v}\right) \quad as \quad m \to \infty, \tag{19}$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}| = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\right) \quad as \quad m \to \infty. \tag{20}$$

Let (X_n) be an almost increasing sequence. If conditions (4)–(7) and (9)–(10) of Theorem 1 and

$$\sum_{v=1}^{n}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|s_v|^k}{v} = O(X_n) \quad as \quad n \to \infty, \tag{21}$$

are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ is summable $|A,p_n;\delta|_k$, $k \geqslant 1$ and $0 \leqslant \delta < 1/k$.

We should give the following lemmas for the proof of Theorem 2.

LEMMA 1. ([8]) *If (X_n) is an almost increasing sequence, then under the conditions (5)–(6), we have*

$$nX_n\beta_n = O(1) \quad as \quad n \to \infty, \tag{22}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{23}$$

LEMMA 2. ([9]) *If the conditions (9) and (10) are satisfied, then we have*

$$\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right). \tag{24}$$

4. Proof of Theorem 2

Let (M_n) denotes the A-transform of the series $\sum \frac{a_n\lambda_n P_n}{np_n}$. Then, we have

$$\bar{\Delta} M_n = \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v \lambda_v P_v}{v p_v}$$

by (13) and (14). By applying Abel's transformation, we get

$$\begin{aligned}
\bar{\Delta} M_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv}\lambda_v P_v}{v p_v} \right) \sum_{r=1}^{v} a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n p_n} \sum_{v=1}^{n} a_v \\
&= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv}\lambda_v P_v}{v p_v} \right) s_v + \frac{a_{nn} P_n \lambda_n}{n p_n} s_n \\
&= \sum_{v=1}^{n-1} \frac{P_v \lambda_v \Delta_v(\hat{a}_{nv})}{v p_v} s_v + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} \Delta\lambda_v P_{v+1}}{(v+1) p_{v+1}} s_v \\
&\quad + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_v \Delta \left(\frac{P_v}{v p_v} \right) s_v + \frac{a_{nn} P_n \lambda_n}{n p_n} s_n \\
&= M_{n,1} + M_{n,2} + M_{n,3} + M_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 2, by Minkowski's inequality, it is enough to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \mid M_{n,r} \mid^k < \infty, \quad \text{for} \quad r = 1,2,3,4.$$

First, by applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |M_{n,1}|^k &\leqslant \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \left\{ \sum_{v=1}^{n-1} \left(\frac{P_v}{v p_v} \right) |\Delta_v(\hat{a}_{nv})| \, |\lambda_v| |s_v| \right\}^k \\
&\leqslant \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \left\{ \sum_{v=1}^{n-1} \left(\frac{P_v}{v p_v} \right)^k |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |s_v|^k \right\} \\
&\quad \times \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right\}^{k-1}.
\end{aligned}$$

By (11) and (12), we have

$$\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1} = \bar{a}_{nv} - \bar{a}_{n-1,v} - \bar{a}_{n,v+1} + \bar{a}_{n-1,v+1} = a_{nv} - a_{n-1,v}.$$

Thus using (11), (15) and (16)

$$\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1} (a_{n-1,v} - a_{nv}) \leqslant a_{nn}.$$

Hence,

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,1}|^k &\leqslant \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k \frac{1}{v^k}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|s_v|^k\right\}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k \frac{1}{v^k}|\lambda_v|^k|s_v|^k \sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\left(\frac{P_v}{p_v}\right)^{k-1}\frac{1}{v^k}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}|\lambda_v|\frac{|s_v|^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{i=1}^{v}\left(\frac{P_i}{p_i}\right)^{\delta k}\frac{|s_i|^k}{i}+O(1)|\lambda_m|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|s_v|^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\
&= O(1)\sum_{v=1}^{m-1}\beta_v X_v+O(1)|\lambda_m|X_m\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2 and Lemma 1.

By using (9) and Hölder's inequality, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||s_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||s_v|^k\right\}\\
&\quad\times\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v\,|s_v|^k\right\}\\
&= O(1)\sum_{v=1}^{m}\beta_v\,|s_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k} v\beta_v\frac{|s_v|^k}{v}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{i=1}^{v}\left(\frac{P_i}{p_i}\right)^{\delta k}\frac{|s_i|^k}{i}+O(1)m\beta_m\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|s_v|^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v+O(1)m\beta_m X_m\\
&= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v+O(1)\sum_{v=1}^{m-1}\beta_v X_v+O(1)m\beta_m X_m\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2 and Lemma 1.

Since $\Delta\left(\frac{P_v}{vp_v}\right)=O\left(\frac{1}{v}\right)$ by (24), as in $M_{n,1}$, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}||\lambda_v||s_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}||\lambda_v|^k|s_v|^k\\
&\quad\times\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}||\lambda_v|^k|s_v|^k\\
&= O(1)\sum_{v=1}^{m}\frac{1}{v}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}|\lambda_v|\frac{|s_v|^k}{v}\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypothese of Theorem 2 and Lemma 1.

Finally, as in $M_{n,1}$, we have that

$$
\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|M_{n,4}|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{n^k}\left(\frac{P_n}{p_n}\right)^k|\lambda_n|^{k-1}|\lambda_n||s_n|^k\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}|\lambda_n|\frac{|s_n|^k}{n}\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2 and Lemma 1.

5. Conclusions

If we take (X_n) as a positive non-decreasing sequence, $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 2, then we get Theorem 1. In this case, the condition (21) reduces to the condition (8). Also, the conditions (15)–(20) are automatically satisfied. Also, if we take $\delta = 0$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result for $|C,1|_k$ summability.

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