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A SHARP CARATHÉODORY'S INEQUALITY ON THE RIGHT HALF PLANE

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Abstract. In this paper, a boundary version of Carathéodory's inequality on the right half plane is investigated. Here, the function $Z(s)$, is given as $Z(s) = 1 + c_1(s-1) + c_2(s-1)^2 + \ldots$ be an analytic in the right half plane with $\Re Z(s) \leqslant A$ $(A > 1)$ for $\Re s \geqslant 0$. We derive inequalities for the modulus of $Z(s)$ function, $|Z'(0)|$, by assuming the $Z(s)$ function is also analytic at the boundary point $s = 0$ on the imaginary axis and finally, the sharpness of these inequalities is proved.

1. Introduction

The most classical version of the Schwarz Lemma examines the behavior of a bounded, analytic function mapping the origin to the origin in the unit disc $D = \{z : |z| < 1\}$. It is possible to see its effectiveness in the proofs of many important theorems. The Schwarz Lemma, which has broad applications and is the direct application of the maximum modulus principle, is given in the most basic form as follows:

Let D be the unit disc in the complex plane $\mathbb{C}$. Let $f : D \to D$ be an analytic function with $f(z) = c_1 z + c_2 z^2 \ldots\ldots$ Under these conditions, $|f(z)| \leqslant |z|$ for all $z \in D$ and $|f'(0)| \leqslant 1$. In addition, if the equality $|f(z)| = |z|$ holds for any $z \neq 0$, or $|f'(0)| = 1$, then f is a rotation; that is $f(z) = ze^{i\theta}$, θ real ([6], p.329). The Schwarz lemma is one of the most important results in the classical complex analysis, which has become a crucial theme in many branches of mathematical research for over a hundred years. On the other hand, in the book [7], Sharp Real-Parts Theorem's (in particular Carathéodory's inequalities), which are frequently used in the theory of entire functions and analytic function theory, have been studied. Also, a boundary version of the Carathéodory's inequality is considered in unit disc and novel results are obtained in [16, 17]. As being assumed that the value of 1 at $s = 1$ of the function, at first, as in Schwarz lemma, Carathéodory's inequality at right half plane will be presented. Mercer [13] prove a version of the Schwarz lemma where the images of two points are known. Also, he considers some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [14].

Let $Z(s) = 1 + c_1(s-1) + c_2(s-1)^2 + \ldots$ be an analytic in the right half plane with $\Re Z(s) \leqslant A$ $(A > 1)$ for $\Re s \geqslant 0$.

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Consider the function

$$f(z)=\frac{Z(s)-1}{Z(s)+1-2A},\ z=\frac{s-1}{s+1}.$$

Here, the function $f(z)$ is an analytic function in D, $f(0)=0$ and $|f(z)|<1$ for $z\in D$. Now, let us show that $|f(z)|<1$ for $|z|<1$. Since

$$\begin{aligned}\left|Z\left(\frac{1+z}{1-z}\right)-1\right|^2&=\left(Z\left(\frac{1+z}{1-z}\right)-1\right)\left(\overline{Z\left(\frac{1+z}{1-z}\right)}-1\right)\\&=\left|Z\left(\frac{1+z}{1-z}\right)\right|^2-Z\left(\frac{1+z}{1-z}\right)-\overline{Z\left(\frac{1+z}{1-z}\right)}+1\end{aligned}$$

and

$$\begin{aligned}\left|Z\left(\frac{1+z}{1-z}\right)+1-2A\right|^2&=\left(Z\left(\frac{1+z}{1-z}\right)+1-2A\right)\left(\overline{Z\left(\frac{1+z}{1-z}\right)}+1-2A\right)\\&=\left|Z\left(\frac{1+z}{1-z}\right)\right|^2+(1-2A)Z\left(\frac{1+z}{1-z}\right)\\&\quad+(1-2A)\overline{Z\left(\frac{1+z}{1-z}\right)}+(1-2A)^2,\end{aligned}$$

we obtain

$$\begin{aligned}&\left|Z\left(\frac{1+z}{1-z}\right)-1\right|^2-\left|Z\left(\frac{1+z}{1-z}\right)+1-2A\right|^2\\&=-2(1-A)\left(Z\left(\frac{1+z}{1-z}\right)+\overline{Z\left(\frac{1+z}{1-z}\right)}\right)+4A-4A^2\\&=-4(1-A)\Re Z\left(\frac{1+z}{1-z}\right)+4A-4A^2\\&\leqslant-4(1-A)A+4A-4A^2=0.\end{aligned}$$

Therefore, we have $|f(z)|<1$ for $|z|<1$.

Consider the product

$$B(z)=\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}.$$

The function $B(z)$ is called a finite Blaschke product, where $z_1,z_2,...,z_n\in D$.

Let

$$g(z)=\frac{f(z)}{\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}},\ z_i=\frac{s_i-1}{s_i+1}.$$

Here, $s_1, s_2, \ldots, s_n$ are points in the right half of the s-plane with $Z(s_i) = 1$ and $z_1, z_2, \ldots, z_n$ are zeros $f(z)$. In addition, $g(z)$ is an analytic function in D, $g(0) = 0$ and $|g(z)| < 1$ for $z \in D$. Therefore, $g(z)$ satisfy the conditions of the Schwarz lemma. Thus, from the Schwarz lemma, we obtain

$$\begin{aligned} g(z) &= \frac{Z\left(\frac{1+z}{1-z}\right) - 1}{Z\left(\frac{1+z}{1-z}\right) + 1 - 2A} \frac{1}{\prod\limits_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}} \\ &= \frac{c_1 \frac{2z}{1-z} + c_2 \frac{4z^2}{(1-z)^2} + \ldots}{2(1-A) + c_1 \frac{2z}{1-z} + c_2 \frac{4z^2}{(1-z)^2} + \ldots} \frac{1}{\prod\limits_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}}, \end{aligned}$$

$$\frac{g(z)}{z} = \frac{c_1 \frac{2}{1-z} + c_2 \frac{4z}{(1-z)^2} + \ldots}{2(1-A) + c_1 \frac{2z}{1-z} + c_2 \frac{4z^2}{(1-z)^2} + \ldots} \frac{1}{\prod\limits_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}}$$

and

$$\left|g'(0)\right| = \frac{|c_1|}{A-1} \frac{1}{\prod\limits_{i=1}^{n} |z_i|} \leqslant 1.$$

Since $|c_1| = |Z'(1)|$ and $|z_i| = \left|\frac{s_i-1}{s_i+1}\right|$, we get

$$\left|Z'(1)\right| \leqslant (A-1) \prod_{i=1}^{n} \left|\frac{s_i - 1}{s_i + 1}\right|.$$

This result is sharp and the extremal function is

$$Z(s) = \frac{1 + (1-2A) \frac{s-1}{s+1} \prod\limits_{i=1}^{n} \frac{\frac{s-1}{s+1} - \frac{s_i-1}{s_i+1}}{1 - \overline{\frac{s_i-1}{s_i+1}} \frac{s-1}{s+1}}}{1 - \frac{s-1}{s+1} \prod\limits_{i=1}^{n} \frac{\frac{s-1}{s+1} - \frac{s_i-1}{s_i+1}}{1 - \overline{\frac{s_i-1}{s_i+1}} \frac{s-1}{s+1}}},$$

where $s_1, s_2, \ldots, s_n$ are positive real numbers.

We thus obtain the following lemma.

LEMMA 1. *Let* $Z(s) = 1 + c_1(s-1) + c_2(s-1)^2 + \ldots$ *be an analytic in the right half plane with* $\Re Z(s) \leqslant A$ $(A > 1)$ *for* $\Re s \geqslant 0$. *Assume that* $s_1, s_2, \ldots, s_n$ *are points in the right half of the* s*-plane with* $Z(s_i) = 1$, $i = 1, 2, \ldots, n$. *Then we have the inequality*

$$\left|Z'(1)\right| \leqslant (A-1) \prod_{i=1}^{n} \left|\frac{s_i - 1}{s_i + 1}\right|. \tag{1.1}$$

This result is sharp and the extremal function is

$$Z(s)=\frac{1+(1-2A)\frac{s-1}{s+1}\prod_{i=1}^{n}\frac{\frac{s-1}{s+1}-\frac{s_i-1}{s_i+1}}{1-\overline{\frac{s_i-1}{s_i+1}}\frac{s-1}{s+1}}}{1-\frac{s-1}{s+1}\prod_{i=1}^{n}\frac{\frac{s-1}{s+1}-\frac{s_i-1}{s_i+1}}{1-\overline{\frac{s_i-1}{s_i+1}}\frac{s-1}{s+1}}},$$

where $s_1,s_2,...,s_n$ *are positive real numbers.*

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point c with $|c|=1$, and if $|f(c)|=1$ and $f'(c)$ exists, then $|f'(c)|\geqslant 1$, which is known as the Schwarz lemma on the boundary. In [15], R. Osserman proposed the boundary refinement of the classical Schwarz lemma as follows:

Let $f:D\rightarrow D$ be an analytic function with $f(z)=c_1z+c_2z^2....,$. Assume that there is a $c\in\partial D$ so that f extends continuously to c, $|f(c)|=1$ and $f'(c)$ exists. Then

$$\left|f'(c)\right|\geqslant\frac{2}{1+|f'(0)|}. \tag{1.2}$$

The equality in (1.2) holds if and only if f is of the form $f(z)=z\frac{k-z}{1-\overline{k}z}$, for some constant $k\in(-1,0]$. Inequality (1.2) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1, 2, 3, 4, 5, 8, 9, 10, 11, 12, 14, 15, 18, 19, 20].

In this paper, we studied “a boundary Carathéodory’s inequalities” on the right half plane as analog the Schwarz lemma at the boundary [15]. We present an analytic to understand the behaviour of the derivative of $Z(s)$ function at the zero on the right half plane. In the resulting theorems of the analysis, assuming that $\Re Z(0)=A$, a lower boundary for modulus of the derivative of the $Z(s)$ function at the zero on right half plane, $|Z'(0)|$, are obtained.

Also, we target to find the answer of the question: ”What can be said about $Z'(s)$ when it is considered at the boundary? The answer of the question relies on the boundary analysis of the Carathéodory’s inequality, that is, analysis of $Z(s)$ function at $s=0$. As a result, in our study, we give a bounded version of Caratheodory inequality on the right half-plane. Moreover, by assuming $Z(s)$ is also analytic at the boundary point $s=0$ on the imaginer axis, we shall give an estimate for $|Z'(0)|$ from below using Taylor expansion coefficients. The sharpness of this inequality is also proved.

2. Main results

In this section, a boundary version of Carathéodory’s inequality on the right half plane is investigated. Here, the function $Z(s)$, is given as $Z(s)=1+c_1(s-1)+c_2(s-1)^2+...$ be an analytic in the right half plane with $\Re Z(s)\leqslant A$ $(A>1)$ for $\Re s\geqslant 0$. We derive inequalities for the modulus of $Z(s)$ function, $|Z'(0)|$, by assuming the $Z(s)$ function is also analytic at the boundary point $s=0$ on the imaginary axis and finally, the sharpness of these inequalities is proved. We have following results, which can be offered as the boundary refinement of Carathéodory's inequality on the right half plane. Also, in the following inequalities, the value of 1 at point $s=1$ and the Taylor coefficient that, is different from the first zero, are used.

THEOREM 1. *Let $Z(s)=1+c_1(s-1)+c_2(s-1)^2+\ldots$ be an analytic in the right half plane with $\Re Z(s)\leqslant A$ for $\Re s\geqslant 0$. Suppose that $Z(s)$ is analytic at the point $s=0$ of the imaginary axis with $\Re Z(0)=A$. Assume that $1,s_1,s_2,\ldots,s_n$ are points in the right half of the s-plane with $Z(s_i)=1$, $i=1,2,\ldots,n$. Then we have the inequality*

$$|Z'(0)|\geqslant (A-1)\left(1+\sum_{i=1}^{n}\frac{\Re s_i}{|s_i|^2}\right. \tag{2.1}$$

$$\left.+\frac{2\left(\left((A-1)\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\right)-|c_1|\right)^2}{\left((A-1)\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\right)^2-|c_1|^2+(A-1)\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\left|2c_2+c_1\left(1+\sum_{i=1}^{n}\frac{4\Re s_i}{|s_i|^2+2i\Im s_i-1}\right)\right|}\right).$$

Moreover, the equality in (2.1) occurs for the function

$$Z(s)=\frac{1+(1-2A)\left(\frac{s-1}{s+1}\right)^2\prod_{i=1}^{n}\frac{\frac{s-1}{s+1}-\frac{s_i-1}{s_i+1}}{1-\overline{\frac{s_i-1}{s_i+1}}\frac{s-1}{s+1}}}{1-\left(\frac{s-1}{s+1}\right)^2\prod_{i=1}^{n}\frac{\frac{s-1}{s+1}-\frac{s_i-1}{s_i+1}}{1-\overline{\frac{s_i-1}{s_i+1}}\frac{s-1}{s+1}}},$$

where $s_1,s_2,\ldots,s_n$ are positive real numbers.

Proof. Let

$$f(z)=\frac{Z\left(\frac{1+z}{1-z}\right)-1}{Z\left(\frac{1+z}{1-z}\right)+1-2A},\, z=\frac{s-1}{s+1}$$

and $z_1,z_2,\ldots,z_n$ be the zeros of the function $f(z)$ in D that are different from zero. The function

$$\Theta(z)=z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}$$

is analytic in D, $|\Theta(z)|<1$ for $z\in D$. From the maximum principle, for each $z\in D$, we have $|f(z)|\leqslant|\Theta(z)|$. The composite function

$$\varphi(z)=\frac{f(z)}{\Theta(z)}$$

is analytic in the unit disc D and $|\varphi(z)|<1$ for $z\in D$. In particular,

$$\begin{aligned}\varphi(z) &= \frac{Z\left(\frac{1+z}{1-z}\right)-1}{Z\left(\frac{1+z}{1-z}\right)+1-2A}\frac{1}{z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}} \\ &= \frac{c_1\frac{2z}{1-z}+c_2\frac{4z^2}{(1-z)^2}+\ldots}{2(1-A)+c_1\frac{2z}{1-z}+c_2\frac{4z^2}{(1-z)^2}+\ldots}\frac{1}{z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}} \\ &= \frac{c_1\frac{2}{1-z}+c_2\frac{4z}{(1-z)^2}+\ldots}{2(1-A)+c_1\frac{2z}{1-z}+c_2\frac{4z^2}{(1-z)^2}+\ldots}\frac{1}{\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}},\end{aligned}$$

$$|\varphi(0)|=\frac{|c_1|}{(A-1)\prod_{i=1}^{n}|z_i|}$$

and

$$\left|\varphi'(0)\right|=\frac{\left|2c_2+c_1\left(1+\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right)\right|}{(A-1)\prod_{i=1}^{n}|z_i|}.$$

Moreover, it can be easliy seen that

$$\frac{cf'(c)}{f(c)}=\left|f'(c)\right|\geqslant\left|\Theta'(c)\right|=\frac{c\Theta'(c)}{\Theta(c)},\ \ c\in\partial D.$$

The auxiliary function

$$\phi(z)=\frac{\varphi(z)-\varphi(0)}{1-\overline{\varphi(0)}\varphi(z)}$$

is an analytic function in D, $|\phi(z)|<1$ for $|z|<1$, $\phi(0)=0$ and $|\phi(c)|=1$ for $-1=c\in\partial D$. From (1.2), we obtain

$$\begin{aligned}\frac{2}{1+|\phi'(0)|}\leqslant\left|\phi'(-1)\right| &= \frac{1-|\varphi(0)|^2}{\left|1-\overline{\varphi(0)}\varphi(-1)\right|^2}\left|\varphi'(-1)\right| \\ &\leqslant\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\left\{\left|f'(-1)\right|-\left|\Theta'(-1)\right|\right\}.\end{aligned}$$

Since

$$\phi'(z)=\frac{1-|\varphi(0)|^2}{\left(1-\overline{\varphi(0)}\varphi(z)\right)^2}\varphi'(z),$$

$$\left|\phi'(0)\right| = \frac{|\varphi'(0)|}{1-|\varphi(0)|^2} = \frac{\dfrac{\left|2c_2+c_1\left(1+\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right)\right|}{(A-1)\prod\limits_{i=1}^{n}|z_i|}}{1-\left(\dfrac{|c_1|}{(A-1)\prod\limits_{i=1}^{n}|z_i|}\right)^2},$$

$$\left|\phi'(0)\right| = (A-1)\prod_{i=1}^{n}|z_i|\,\frac{\left|2c_2+c_1\left(1+\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right)\right|}{\left((A-1)\prod\limits_{i=1}^{n}|z_i|\right)^2-|c_1|^2}$$

and

$$\left|\Theta'(-1)\right| = 1+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|1+z_i|^2},$$

we obtain

$$\frac{2}{1+(A-1)\prod\limits_{i=1}^{n}|z_i|\,\dfrac{\left|2c_2+c_1\left(1+\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right)\right|}{\left((A-1)\prod\limits_{i=1}^{n}|z_i|\right)^2-|c_1|^2}}$$

$$\leqslant \frac{1+\dfrac{|c_1|}{(A-1)\prod\limits_{i=1}^{n}|z_i|}}{1-\dfrac{|c_1|}{(A-1)\prod\limits_{i=1}^{n}|z_i|}}\left\{\frac{|Z'(0)|}{A-1}-\left(1+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|1+z_i|^2}\right)\right\}$$

$$= \frac{(A-1)\prod\limits_{i=1}^{n}|z_i|+|c_1|}{(A-1)\prod\limits_{i=1}^{n}|z_i|-|c_1|}\left\{\frac{|Z'(0)|}{A-1}-\left(1+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|1+z_i|^2}\right)\right\},$$

$$\frac{2\left(\left((A-1)\prod\limits_{i=1}^{n}|z_i|\right)^2-|c_1|^2\right)}{\left((A-1)\prod\limits_{i=1}^{n}|z_i|\right)^2-|c_1|^2+(A-1)\prod\limits_{i=1}^{n}|z_i|\left|2c_2+c_1\left(1+\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right)\right|}$$

$$\leqslant \frac{(A-1)\prod\limits_{i=1}^{n}|z_i|+|c_1|}{(A-1)\prod\limits_{i=1}^{n}|z_i|-|c_1|}\left\{\frac{|Z'(0)|}{A-1}-\left(1+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|1+z_i|^2}\right)\right\}$$

and

$$\frac{2\left(\left((A-1)\prod_{i=1}^{n}|z_i|\right)-|c_1|\right)^2}{\left((A-1)\prod_{i=1}^{n}|z_i|\right)^2-|c_1|^2+(A-1)\prod_{i=1}^{n}|z_i|\left|2c_2+c_1\left(1+\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right)\right|}$$
$$\leqslant\frac{|Z'(0)|}{A-1}-\left(1+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|1+z_i|^2}\right).$$

Also, for $z_i=\frac{s_i-1}{s_i+1}$, we have

$$1-|z_i|^2=1-\left|\frac{s_i-1}{s_i+1}\right|^2=\frac{4\Re s_i}{|s_i+1|^2},$$

$$|1+z_i|^2=\left|1+\frac{s_i-1}{s_i+1}\right|^2=\frac{4|s_i|^2}{|s_i+1|^2},$$

$$\frac{1-|z_i|^2}{z_i}=\frac{\frac{4\Re s_i}{|s_i+1|^2}}{\frac{s_i-1}{s_i+1}}=\frac{4\Re s_i}{|s_i+1|^2}\frac{s_i+1}{s_i-1}=\frac{4\Re s_i}{|s_i|^2+2i\Im s_i-1}$$

and

$$\frac{1-|z_i|^2}{|1+z_i|^2}=\frac{\Re s_i}{|s_i|^2}.$$

Therefore, we obtain

$$\frac{2\left(\left((A-1)\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\right)-|c_1|\right)^2}{\left((A-1)\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\right)^2-|c_1|^2+(A-1)\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\left|2c_2+c_1\left(1+\sum_{i=1}^{n}\frac{4\Re s_i}{|s_i|^2+2i\Im s_i-1}\right)\right|}$$
$$\leqslant\frac{|Z'(0)|}{A-1}-\left(1+\sum_{i=1}^{n}\frac{\Re s_i}{|s_i|^2}\right).$$

Now, we shall show that the inequality (2.1). Let

$$Z\left(\frac{1+z}{1-z}\right)=\frac{1+(1-2A)z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z}_iz}}{1-z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z}_iz}}=2A-1+\frac{2(1-A)}{1-z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z}_iz}}. \tag{2.2}$$

Then we obtain

$$\frac{2}{(1-z)^2}Z'\left(\frac{1+z}{1-z}\right)=2(1-A)\frac{\left(2z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z}_iz}+z^2\sum_{i=1}^{n}\frac{1-|z_i|^2}{(1-\overline{z}_iz)(z-z_i)}\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z}_iz}\right)}{\left(1-z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z}_iz}\right)^2}.$$

For $z=-1$, we get

$$\frac{Z'(0)}{2}=2(1-A)\frac{\left(2(-1)\prod_{i=1}^{n}\frac{-1-z_i}{1+\overline{z_i}}+(-1)^2\sum_{i=1}^{n}\frac{1-|z_i|^2}{(1+\overline{z_i})(-1-z_i)}\prod_{i=1}^{n}\frac{-1-z_i}{1+\overline{z_i}}\right)}{\left(1-(-1)^2\prod_{i=1}^{n}\frac{-1-z_i}{1+\overline{z_i}}\right)^2}$$

$$Z'(0)=4(1-A)\frac{\left(2\prod_{i=1}^{n}\frac{1+z_i}{1+\overline{z_i}}+\sum_{i=1}^{n}\frac{1-|z_i|^2}{(1+\overline{z_i})(1+z_i)}\prod_{i=1}^{n}\frac{1+z_i}{1+\overline{z_i}}\right)}{\left(1+\prod_{i=1}^{n}\frac{1+z_i}{1+\overline{z_i}}\right)^2}.$$

Since $z_1,z_2,\dots,z_n$ are positive real numbers, we obtain

$$Z'(0)=4(1-A)\frac{\left(2+\sum_{i=1}^{n}\frac{1-(z_i)^2}{(1+z_i)^2}\right)}{4}$$

and

$$\left|Z'(0)\right|=(A-1)\left(2+\sum_{i=1}^{n}\frac{1-z_i}{1+z_i}\right).$$

Also, for $z_i=\frac{s_i-1}{s_i+1}$, we take

$$\begin{aligned}\left|Z'(0)\right|&=(A-1)\left(2+\sum_{i=1}^{n}\frac{1-\frac{s_i-1}{s_i+1}}{1+\frac{s_i-1}{s_i+1}}\right)\\&=(A-1)\left(2+\sum_{i=1}^{n}\frac{1}{s_i}\right).\end{aligned}$$

Moreover, from (2.2), we have $|c_1|=0$ and $|c_2|=\frac{A-1}{2}\prod_{i=1}^{n}|z_i|=\frac{A-1}{2}\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|$.

Therefore, we obtain

$$\begin{aligned}&(A-1)\left(1+\sum_{i=1}^{n}\frac{\Re s_i}{|s_i|^2}\right.\\&\left.+\frac{2\left(\left((A-1)\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\right)-|c_1|\right)^2}{\left((A-1)\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\right)^2-|c_1|^2+(A-1)\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\left|2c_2+c_1\left(1+\sum_{i=1}^{n}\frac{4\Re s_i}{|s_i|^2+2i\Im s_i-1}\right)\right|}\right)\\&=(A-1)\left(1+\sum_{i=1}^{n}\frac{\Re s_i}{|s_i|^2}+\frac{2\left((A-1)\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\right)^2}{\left((A-1)\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\right)^2+\left((A-1)\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\right)^2}\right)\end{aligned}$$

$$= (A-1)\left(2+\sum_{i=1}^{n}\frac{\Re s_i}{|s_i|^2}\right) = (A-1)\left(2+\sum_{i=1}^{n}\frac{1}{s_i}\right). \quad \square$$

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