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FOUR DIMENSIONAL LOGARITHMIC TRANSFORMATION INTO $\mathscr{L}_u$

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Abstract. Let $t=(t_m)$ and $\bar{t}=(\bar{t}_n)$ be two null sequences in the interval $(0,1)$ and define the four dimensional logarithmic matrix $L_{t,\bar{t}}=(a^{t,\bar{t}}_{mnkl})$ by

$$a^{t,\bar{t}}_{mnkl}=\frac{1}{\log(1-t_m)\log(1-\bar{t}_n)}\frac{1}{(k+1)(l+1)}t_m^{k+1}(\bar{t}_n)^{l+1}.$$

The matrix $L_{t,\bar{t}}$ determines a sequence -to-sequence variant of classicial logarithmic summability method. The aim of this paper is to study these transformations to be $\mathscr{L}_u-\mathscr{L}_u$ summability methods.

1. Introduction

The most well-known notion of convergence for double sequences is the convergence in the sense of Pringsheim. Recall that a double sequence $x=\{x_{kl}\}$ of complex (or real) numbers is called convergent to a scalar ℓ in Pringsheim's sense (denoted by P-$\lim x=\ell$) if for every $\varepsilon>0$ there exists an $N\in\mathbb{N}$ such that $|x_{kl}-\ell|<\varepsilon$ whenever $k,l>N$. Such an x is described more briefly as "P-convergent". It is easy to verify that $x=\{x_{kl}\}$ convergences in Pringsheim's sense if and only if for every $\varepsilon>0$ there exists an integer $N=N(\varepsilon)$ such that $\left|x_{ij}-x_{kl}\right|<\varepsilon$ whenever $\min\{i,j,k,l\}\geqslant N$. A double sequence $x=\{x_{kl}\}$ is bounded if there exists a positive number M such that $|x_{kl}|\leqslant M$ for all k and l, that is, if $\sup_{k,l}|x_{kl}|<\infty$.
A double sequence $x=\{x_{kl}\}$ is said to convergence regularly if it converges in Pringsheim's sense and, in addition, the following finite limits exist:

$$\lim_{k\to\infty}x_{kl}=\ell_l \quad (l=1,2,...),$$

$$\lim_{l\to\infty}x_{kl}=j_k \quad (k=1,2,...).$$

Note that the main drawback of the Pringsheim's convergence is that a convergent sequence fails in general to be bounded. The notion of regular convergence lacks this disadvantage.
Let $A=(a_{mnkl})$ denote a four dimensional summability method that maps the complex

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double sequence x into the double sequence $Ax=\{(Ax)_{mn}\}$ where $(Ax)_{mn}$ is defined as follows:

$$(Ax)_{mn}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}a_{mnkl}x_{kl}.$$

In [17] Robison presented the following notion of regularity for four-dimensional matrix transformation and a Silverman-Toeplitz type characterization of such notion.

DEFINITION 1. The four-dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

The assumption of bounded was added because a double sequence which is P-convergent is not necessarily bounded. Along these same lines, Robison and Hamilton presented a Silverman-Toeplitz type multidimensional characterization of regularity in [6] and [17].

THEOREM 1. *The four-dimensional matrix A is RH-regular if and only if*

RH_1: $P\text{-}\lim_{m,n} a_{mnkl}=0$ *for each k and l;*
RH_2: $P\text{-}\lim_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{mnkl}=1$*;*
RH_3: $P\text{-}\lim_{m,n}\sum_{k=0}^{\infty}|a_{mnkl}|=0$ *for each l;*
RH_4: $P\text{-}\lim_{m,n}\sum_{l=0}^{\infty}|a_{mnkl}|=0$ *for each k;*
RH_5: $\sum_{k,l=0,0}^{\infty,\infty}|a_{mnkl}|$ *is P-convergent;*
RH_6: *there exist finite positive integers Δ and Γ such that* $\sum_{k,l>\Gamma}|a_{mnkl}|<\Delta$.

A double series $\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}x_{kl}$ converges to a sum ℓ if
(a) the sequence of "rectangular" partial sums S_{mn} converges:

$$\ell=P-\lim_{m,n\to\infty}\sum_{k=1}^{m}\sum_{l=0}^{n}x_{kl};$$

(b) every "row series" $\sum_{l=0}^{\infty}x_{kl}$ converges;
(c) every "column series" $\sum_{k=0}^{\infty}x_{kl}$ converges.
A double series $\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}x_{kl}$ is called absolutely convergent if the series

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}|x_{kl}|$$

converges. The space of all absolutely convergent double sequences will be denoted $\mathscr{L}_u$, that is

$$\mathscr{L}_u:=\{x=\{x_{kl}\}:\qquad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|x_{kl}|<\infty\}.$$

Observe that every absolutely convergent double series is convergent. The reader may refer to the textbooks [2] and [12], and recent paper [18] on the spaces of double sequences, four dimensional matrices and related topics.

In [14], Patterson proved that the matrix $A=(a_{mnkl})$ determines an $\mathcal{L}_u-\mathcal{L}_u$ method if and only if

$$\sup_{k,l}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|a_{mnkl}|<\infty. \tag{1}$$

The aim of this paper is study four dimensional Abel matrices as $\mathcal{L}_u-\mathcal{L}_u$ matrices.

2. Four dimensional logarithmic $\mathcal{L}_u-\mathcal{L}_u$ method

The logarithmic power series method of summability, denoted by $\mathcal{L}_u$, is following sequences-to-function transformation if

$$\lim_{r_1\to 1^-,r_2\to 1^-}\left\{\frac{1}{\log(1-r_1)\log(1-r_2)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{(k+1)(l+1)}x_{kl}r_1^{k+1}r_2^{l+1}\right\}=a,$$

then $x=(x_{kl})$ is $\mathcal{L}_u$-summable to a. This can be modified into a sequence-to-sequence transformation by replacing the continuous parameters r_1 and r_2 with the sequences (t_m) and $(\bar{t}_n)$ such that $0<t_m<1$ for all m, $0<\bar{t}_n<1$ for all n, $\lim_m t_m=1$ and $\lim_n \bar{t}_n=1$. Thus the sequence $x=\{x_{kl}\}$ is transformed into the sequence $L_{t,\bar{t}}x$ whose mn*th* term is given by

$$(L_{t,\bar{t}}x)_{mn}=\frac{1}{\log(1-t_m)\log(1-\bar{t}_n)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{(k+1)(l+1)}x_{k,l}t_m^{k+1}(\bar{t}_n)^{l+1}.$$

This transformation is represented by the matrix $L_{t,\bar{t}}=(a_{mnkl}^{t,\bar{t}})$ given by

$$a_{mnkl}^{t,\bar{t}}=\frac{1}{\log(1-t_m)\log(1-\bar{t}_n)}\frac{1}{(k+1)(l+1)}x_{k,l}t_m^{k+1}(\bar{t}_n)^{l+1}.$$

The matrix $L_{t,\bar{t}}$ is called a four dimensional logarithmic matrix. It is clear that $A_{t,\bar{t}}$ is RH-regular matrix.

THEOREM 2. *The four dimensional logarithmic matrix $L_{t,\bar{t}}$ is an $\mathcal{L}_u-\mathcal{L}_u$ matrix if and only if $\frac{1}{\log(1-t_m)\log(1-\bar{t}_n)}\in\mathcal{L}_u$.*

Proof. Since $0<t_m<1$ for all m and $0<\bar{t}_n<1$ for all n, we have

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|a_{mnkl}^{t,\bar{t}}|&=\frac{1}{(k+1)(l+1)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\log(1-t_m)\log(1-\bar{t}_n)}t_m^{k+1}(\bar{t}_n)^{l+1}\\&\leqslant\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\log(1-t_m)\log(1-\bar{t}_n)},\end{aligned}$$

for every k and l. Thus if $(t_m\bar{t}_n)\in\mathcal{L}_u$, Theorem 1 in [14] guarantees that $L_{t,\bar{t}}$ is an $\mathcal{L}_u-\mathcal{L}_u$ matrix. Now suppose that $\frac{1}{\log(1-t_m)\log(1-\bar{t}_n)}\notin\mathcal{L}_u$, then we consider the sum of the $(a^{t,\bar{t}}_{mn00})$ elements of $L_{t,\bar{t}}$:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|a^{t,\bar{t}}_{mn00}| = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{t_m\bar{t}_n}{\log(1-t_m)\log(1-\bar{t}_n)} = \infty,$$

which shows that $L_{t,\bar{t}}$ is not an $\mathscr{L}_u - \mathscr{L}_u$ matrix. □

THEOREM 3. *If $L_{t,\bar{t}}$ is an $\mathscr{L}_u - \mathscr{L}_u$ matrix and the series $\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}x_{kl}$ has bounded partial sums, then $x \in \mathscr{L}_{L_{t,\bar{t}}}$.*

Proof. Define $s_{kl} = \sum_{i=0}^{k}\sum_{j=0}^{l}x_{kl}$, $s_{00} = 0$, $s_{0l} = 0$, $s_{k0} = 0$ and $w_m^k = \frac{1}{k+1}t_m^{k+1}$, $v_n^l = \frac{1}{l+1}(\bar{t}_n)^{l+1}$. Then

$$\begin{aligned}
&\left|\sum_{k=1}^{m}\sum_{l=1}^{n}\frac{1}{(k+1)(l+1)}t_m^{k+1}(\bar{t}_n)^{l+1}x_{k,l}\right| \\
&= \left|\sum_{k=1}^{m}\sum_{l=1}^{n}w_m^k v_n^l x_{kl}\right| \\
&= \left|\sum_{k=1}^{m}\sum_{l=1}^{n}w_m^k v_n^l (s_{kl} - s_{k,l-1} - s_{k-1,l} + s_{k-1,l-1})\right| \\
&= \left|\sum_{k=1}^{m}\sum_{l=1}^{n}s_{kl}[w_m^k v_n^l - w_m^{k+1}v_n^l - w_m^k v_n^{l+1} + w_m^{k+1}v_n^{l+1}\right| \\
&\leqslant 4\log(1-t_m)\log(1-\bar{t}_n)\sup_{k\leqslant m, l\leqslant n}|s_{kl}| \\
&< M\log(1-t_m)\log(1-\bar{t}_n).
\end{aligned}$$

This yields that

$$\left|\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{(k+1)(l+1)}t_m^{k+1}(\bar{t}_n)^{l+1}x_{kl}\right| < M\log(1-t_m)\log(1-\bar{t}_n).$$

Hence,

$$|(L_{t,\bar{t}}x)_{mn}| < M$$

thus $L_{t,\bar{t}}$ is an $\mathscr{L}_u - \mathscr{L}_u$ matrix, so $x \in \mathscr{L}_{L_{t,\bar{t}}}$. □

3. A Tauberian theorem

We now prove an $\mathscr{L}_u - \mathscr{L}_u$ Tauberian theorem for the four dimensional logarithmic matrices.

THEOREM 4. *Let $L_{t,\bar{t}}$ be an $\mathscr{L}_u - \mathscr{L}_u$ logarithmic matrix; if x is a double sequence such that $L_{t,\bar{t}}x$ is in $\mathscr{L}_u$, and*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|\Delta_{10}x_{ij}|ij < \infty \tag{2}$$

and

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|\Delta_{01}x_{ij}|ij < \infty. \tag{3}$$

then x in $\mathscr{L}_u$ where $\Delta_{10}x_{ij} = x_{ij} - x_{i+1,j}$ and $\Delta_{01}x_{ij} = x_{ij} - x_{i,j+1}$.

Proof. In order to show that $L_{t,\bar{t}}x - x$ is in $\mathscr{L}_u$ we write

$$(L_{t,\bar{t}}x)_{mn} - x_{mn} = \frac{1}{\log(1-t_m)\log(1-\bar{t}_n)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{(k+1)(l+1)}t_m^{k+1}(\bar{t}_n)^{l+1}(x_{kl} - x_{mn}).$$

Letting

$$a_{mnkl}^{t,\bar{t}} = \frac{1}{\log(1-t_m)\log(1-\bar{t}_n)}\frac{1}{(k+1)(l+1)}t_m^{k+1}(\bar{t}_n)^{l+1},$$

we shall prove that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}a_{mnkl}^{t,\bar{t}}|x_{kl} - x_{mn}| < \infty.$$

Let us write

$$\mathscr{S} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}a_{mnkl}^{t,\bar{t}}|x_{kl} - x_{mn}|.$$

Let $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$, where

$$\mathscr{S}_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}a_{mnkl}^{t,\bar{t}}|x_{kl} - x_{mn}|$$

and

$$\mathscr{S}_2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=m}^{\infty}\sum_{l=n}^{\infty}a_{mnkl}^{t,\bar{t}}|x_{kl} - x_{mn}|.$$

Since

$$|x_{kl} - x_{mn}| = |x_{mn} - x_{kl}| = \left|\sum_{i=m}^{k-1}\Delta_{10}x_{ij} + \sum_{j=n}^{l-1}\Delta_{01}x_{ij}\right|,$$

this leads to

$$
\begin{aligned}
\mathscr{S}_1 &\leqslant \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{m-1}\sum_{l=0}^{n-1} a_{mnkl}^{t,\bar{t}}\left(\sum_{i=m}^{k-1}|\Delta_{10}x_{ij}|+\sum_{j=n}^{l-1}|\Delta_{01}x_{ij}|\right)\\
&\leqslant \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|\Delta_{10}x_{ij}|\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j} a_{mnkl}^{t,\bar{t}}\\
&\quad+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|\Delta_{01}x_{ij}|\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j} a_{mnkl}^{t,\bar{t}}\\
&= \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|\Delta_{10}x_{ij}|+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|\Delta_{01}x_{ij}|\right)\zeta_{ij}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathscr{S}_2 &\leqslant \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=m+1}^{\infty}\sum_{l=n+1}^{\infty} a_{mnkl}^{t,\bar{t}}\left(\sum_{i=m}^{k-1}|\Delta_{10}x_{ij}|+\sum_{i=n}^{l-1}|\Delta_{01}x_{ki}|\right)\\
&\leqslant \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|\Delta_{10}x_{ij}|\sum_{m=0}^{i}\sum_{n=0}^{j}\sum_{k=i+1}^{\infty}\sum_{l=j+1}^{\infty} a_{mnkl}^{t,\bar{t}}\\
&\quad+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|\Delta_{01}x_{ij}|\sum_{m=0}^{i}\sum_{n=0}^{j}\sum_{k=i+1}^{\infty}\sum_{l=j+1}^{\infty} a_{mnkl}^{t,\bar{t}}\\
&= \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|\Delta_{10}x_{ij}|+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|\Delta_{01}x_{ij}|\right)\varsigma_{ij},
\end{aligned}
$$

where

$$
\zeta_{ij}=\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j} a_{mnkl}^{t,\bar{t}} \quad\text{and}\quad \varsigma_{ij}=\sum_{m=0}^{i}\sum_{n=0}^{j}\sum_{k=i+1}^{\infty}\sum_{l=j+1}^{\infty} a_{mnkl}^{t,\bar{t}}
$$

By showing that $\zeta_{ij}=O(ij)$ and $\varsigma_{ij}=O(ij)$, we will prove that $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|\Delta_{10}x_{ij}|ij<\infty$ and $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|\Delta_{01}x_{ij}|ij<\infty$ implies that $L_{t,\bar{t}}x-x$ is in $\mathscr{L}_u$. These $O(ij)$ assertions are very easily verified since $L_{t,\bar{t}}$ is $\mathscr{L}_u-\mathscr{L}_u$ we have

$$
\zeta_{ij}=\sum_{k=0}^{i}\sum_{l=0}^{j}\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty} a_{mnkl}^{t,\bar{t}} \leqslant (i+1)(j+1)\sup_{k,l}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{mnkl}^{t,\bar{t}}|=O(ij)
$$

and since $L_{t,\bar{t}}$ is RH-regular we have

$$
\varsigma_{i,j}=\sum_{m=0}^{i}\sum_{n=0}^{j}\sum_{k=i+1}^{\infty}\sum_{l=j+1}^{\infty} a_{mnkl}^{t,\bar{t}} \leqslant (i+1)(j+1)\sup_{m,n}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}|a_{mnkl}^{t,\bar{t}}|=O(ij).
$$

Thus, the proof is completed. $\square$

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