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SOME RESULTS ON SUM AND PRODUCT OF RELATIVE GROWTH FACTORS OF COMPOSITE ENTIRE FUNCTIONS

SANJIB KUMAR DATTA*, BANANI DUTTA AND NITYAGOPAL BISWAS

Abstract. In this paper we study about the sum and product of relative $(p,q,t)L$-th type and relative $(p,q,t)L$-th lower type of an entire function with respect to another entire function in the light of a special type of non-decreasing, unbounded function Ψ.

1. Introduction, definitions and notations

We assume that reader is familiar with the fundamental results and the standard notations of the Nevanlinna's theory of meromorphic functions ([7]). Let $\mathbb{C}$ be the set of finite complex numbers and f be an entire function defined on $\mathbb{C}$. To characterize the growth of an entire function a special growth scale called maximum modulus function on $|z| = r$ is introduced as

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

Some times the symbol $M(r,f)$ can also be written as $M_f(r)$. It plays an important role in the theory of entire functions. If f is a non-constant entire function, the its maximum modulus $M(r,f)$ increases as r increases and is continuous therefore there exists its inverse function $M_f^{-1} : (|f(0)|,\infty) \to (0,\infty)$ with $\lim_{r\to\infty} M_f^{-1}(r) = \infty$. Again the sum of proximity function and counting function is denoted by $T(r,f)$, known as Nevanlinna's characteristic function. Further a non-constant entire function f is said to have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $\left[M_f(r)\right]^2 \leqslant M_f(r^\sigma)$ holds (see [2]).

Many authors like Datta et al. $(cf.$ [1], [6], [5] and [11]$)$ investigated about the growth of entire functions. This investigation can be made using slowly changing function also. A positive continuous function $L(r)$ is said to be slowly changing function if $L(ar) \sim L(r)$ as $r \to \infty$, i.e. $\lim_{r\to\infty} \frac{L(ar)}{L(r)} = 1$. Somasundaram and Tamizharasi [13] introduced the notion of L-order and L-lower order.

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* Corresponding author.

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DEFINITION 1. Let f be an entire function. The order ρ_f and lower order λ_f of an entire function f is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$, for $k = 1,2,3,\ldots$ and $\log^{[0]} x = x$.

Juneja, Kapoor and Bajpai [8, 9] introduced the idea of $(p,q)^{th}$ -order

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r},$$

where p,q are positive integers and $p \geqslant q$. Clearly, when $p = 2, q = 1$ it reduces to order ρ_f.

When the growth of an entire function is defined with respect to another entire function, this is known as relative growth. Ruiz et al. [10], introduced the concept of relative $(p,q)^{th}$ -order and relative $(p,q)^{th}$ -lower order of an entire function with respect to another entire function in the light of index pair.

Recently, Biswas [3] introduced the idea of relative $(p,q,t)^{th} - L$ order and relative $(p,q,t)^{th} - L$ lower order of an entire function as follows:

DEFINITION 2. Let f, g be two entire functions. Then the relative $(p,q,t)L^{th}$-order and relative $(p,q,t)L^{th}$-lower order of f with respect to the function g is defined as

$$\rho_g^{(p,q,t)L}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M(r,f)}{\log^{[q]} r + \exp^{[t]} L(r)}$$

and

$$\lambda_g^{(p,q,t)L}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M(r,f)}{\log^{[q]} r + \exp^{[t]} L(r)},$$

where, $p,q \in N$, $t \in N \cup \{-1,0\}$.

Chyzhykov et al. [4], introduced the definition of Ψ-order of a meromorphic function on single variable in the unit disc. For details about Ψ-order, one may see [4] and [12].

REMARK 1. ([12]) Throughout this paper, we assume that $\Psi : [0,\infty) \to (0,\infty)$ is a non-decreasing unbounded function and always satisfies the following two conditions:

(i)

$$\lim_{r\to\infty} \frac{\log^{[p]} r}{\log^{[q]} \Psi(r)} = 0$$

and

(ii)

$$\lim_{r\to\infty}\frac{\log^{[q]}\Psi(\alpha r)}{\log^{[q]}\Psi(r)}=1, \text{ for some } \alpha>1.$$

Now we give the following classical definitions of growth indicators of f with the help of the function Ψ :

DEFINITION 3. Let f, g be two entire functions. Then the relative Ψ-$(p,q)^{th}$-order and relative Ψ-$(p,q)^{th}$-lower order of f with respect to the function g is defined as

$$\rho_{g,\Psi}^{(p,q)}(f)=\limsup_{r\to\infty}\frac{\log^{[p]}M_g^{-1}M(r,f)}{\log^{[q]}\Psi(r)} \text{ and } \lambda_{g,\Psi}^{(p,q)}(f)=\liminf_{r\to\infty}\frac{\log^{[p]}M_g^{-1}M(r,f)}{\log^{[q]}\Psi(r)},$$

where p,q are positive integers with $p\geqslant q$.

Using the functions Ψ and $L(r)$, we introduce the relative $(p,q,t)^{th}-L$ order of an entire function.

DEFINITION 4. Let f, g be two entire functions. Then the relative $(p,q,t)L^{th}$-order and relative $(p,q,t)L^{th}$-lower order of f with respect to the function g is defined as

$$\rho_{g,\Psi}^{(p,q,t)L}(f)=\limsup_{r\to\infty}\frac{\log^{[p]}M_g^{-1}M(r,f)}{\log^{[q]}\Psi(r)+\exp^{[t]}L(r)}$$

and

$$\lambda_{g,\Psi}^{(p,q,t)L}(f)=\liminf_{r\to\infty}\frac{\log^{[p]}M_g^{-1}M(r,f)}{\log^{[q]}\Psi(r)+\exp^{[t]}L(r)},$$

where, $p,q\in N$, $t\in N\cup\{-1,0\}$.

DEFINITION 5. Let f, g be two entire functions. Then the relative $(p,q,t)L^{th}$-type and relative $(p,q,t)L^{th}$-lower type of f with respect to the function g is defined as

$$\sigma_{g,\Psi}^{(p,q,t)L}(f)=\limsup_{r\to\infty}\frac{\log^{[p-1]}M_g^{-1}M(r,f)}{[\log^{[q-1]}\Psi(r).\exp^{[t+1]}L(r)]^{\rho_{g,\Psi}^{(p,q,t)L}(f)}}$$

and

$$\overline{\sigma}_{g,\Psi}^{(p,q,t)L}(f)=\liminf_{r\to\infty}\frac{\log^{[p-1]}M_g^{-1}M(r,f)}{[\log^{[q-1]}\Psi(r).\exp^{[t+1]}L(r)]^{\rho_{g,\Psi}^{(p,q,t)L}(f)}},$$

where, $p,q\in N$, $t\in N\cup\{-1,0\}$.

DEFINITION 6. Let f, g be two entire functions. Then the relative $(p,q,t)L^{th}$-weak type and relative $(p,q,t)L^{th}$-growth indicator of f with respect to the function g is defined as

$$\tau_{g,\Psi}^{(p,q,t)L}(f) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M(r,f)}{[\log^{[q-1]} \Psi(r).\exp^{[t+1]} L(r)]^{\lambda_{g,\Psi}^{(p,q,t)L}(f)}}$$

and

$$\overline{\tau}_{g,\Psi}^{(p,q,t)L}(f) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M(r,f)}{[\log^{[q-1]} \Psi(r).\exp^{[t+1]} L(r)]^{\lambda_{g,\Psi}^{(p,q,t)L}(f)}},$$

where, $p,q \in N$, $t \in N \cup \{-1,0\}$.

In this paper we will discuss about some property of sum and product of different growth factors of an entire function with respect to another entire function such as relative (p,q,t)-L^{th} type, relative (p,q,t)-L^{th} lower type, relative (p,q,t)-L^{th} weak type etc.

2. Lemmas

Now we state some lemmas which will be needed in the proof of the theorems.

LEMMA 1. [2] *Suppose f be an entire function and α, β be such that $\alpha > 1$ and $0 < \beta < \alpha$. Then*

$$M_f(\alpha r) > \beta M_f(r).$$

LEMMA 2. [2] *Let f be an entire function satisfying the Property (A). Then for any positive integer n and for all sufficiently large r*

$$[M_f(r)]^n \leqslant M_f\left(r^\delta\right)$$

holds, where $\delta > 1$.

LEMMA 3. *([7], p. 18) Let f be an entire function. Then for all sufficiently large values of r*

$$T_f(r) \leqslant \log M_f(r) \leqslant 3T_f(2r).$$

3. Main results

Next we will establish some results using sum and product of type and lower type of entire functions.

THEOREM 1. (Main) *Let f_1, f_2, g_1 and g_2 be entire functions such that $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)$, $\rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$, $\rho_{g_2,\Psi}^{(p,q,t)L}(f_2)$ and $\rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$ are non zero finite.*

(i) *If* $\rho_{g_1,\Psi}^{(p,q,t)L}(f_i) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_j)$, *for* $(i,j) = (1,2)$ *or* $(2,1)$, *then*

$$\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1 \pm f_2) = \sigma_{g_1,\Psi}^{(p,q,t)L}(f_i) \text{ and } \overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1 \pm f_2) = \overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_i).$$

(ii) *If* $\rho_{g_i,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_j,\Psi}^{(p,q,t)L}(f_1)$, *for* $(i,j) = (1,2)$ *or* $(2,1)$, f_1 *is of regular relative growth with respect to* g_j, *then*

$$\sigma_{g_1 \pm g_2,\Psi}^{(p,q,t)L}(f_1) = \sigma_{g_i,\Psi}^{(p,q,t)L}(f_1) \text{ and } \overline{\sigma}_{g_1 \pm g_2,\Psi}^{(p,q,t)L}(f_1) = \overline{\sigma}_{g_i,\Psi}^{(p,q,t)L}(f_1).$$

(iii) *If*

(a) $\rho_{g_i,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_j,\Psi}^{(p,q,t)L}(f_1)$, f_1 *is of regular relative growth with respect to* g_j,

(b) $\rho_{g_i,\Psi}^{(p,q,t)L}(f_2) < \rho_{g_j,\Psi}^{(p,q,t)L}(f_2)$, f_2 *is of regular relative growth with respect to* g_j,

(c) $\rho_{g_1,\Psi}^{(p,q,t)L}(f_i) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_j)$, $\rho_{g_2,\Psi}^{(p,q,t)L}(f_i) > \rho_{g_2,\Psi}^{(p,q,t)L}(f_j)$ *holds simultaneously,*

(d)

$$\rho_{g_m,\Psi}^{(p,q,t)L}(f_l) = \max\left\{\begin{array}{c} \min\left\{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1), \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)\right\}, \\ \min\left\{\rho_{g_1,\Psi}^{(p,q,t)L}(f_2), \rho_{g_2,\Psi}^{(p,q,t)L}(f_2)\right\} \end{array}\right\}, l,m = 1,2,$$

then

$$\sigma_{g_1 \pm g_2,\Psi}^{(p,q,t)L}(f_1 \pm f_2) = \sigma_{g_m,\Psi}^{(p,q,t)L}(f_l),$$
$$\overline{\sigma}_{g_1 \pm g_2,\Psi}^{(p,q,t)L}(f_1 \pm f_2) = \overline{\sigma}_{g_m,\Psi}^{(p,q,t)L}(f_l).$$

Proof. From the Definition of $\sigma_{g_l,\Psi}^{(p,q,t)L}(f_k)$ and $\overline{\sigma}_{g_l,\Psi}^{(p,q,t)L}(f_k)$ we have, for all sufficiently large values of r,

$$M(r,f_k) \leqslant M_{g_l}\left[\exp^{[p-1]}\left[\begin{array}{c} \left(\sigma_{g_l,\Psi}^{(p,q,t)L}(f_k) + \varepsilon\right) \\ \cdot\left(\log^{[q-1]}\Psi(r) \cdot \exp^{[t+1]} L(r)\right)^{\rho_{g_l,\Psi}^{(p,q,t)L}(f_k)} \end{array}\right]\right], \tag{3.1}$$

$$M(r,f_k) \geqslant M_{g_l}\left[\exp^{[p-1]}\left[\begin{array}{c} \left(\overline{\sigma}_{g_l,\Psi}^{(p,q,t)L}(f_k) - \varepsilon\right) \\ \cdot\left(\log^{[q-1]}\Psi(r) \cdot \exp^{[t+1]} L(r)\right)^{\rho_{g_l,\Psi}^{(p,q,t)L}(f_k)} \end{array}\right]\right] \tag{3.2}$$

and for a sequence of values of r tending to infinity

$$M(r,f_k) \geqslant M_{g_l}\left[\exp^{[p-1]}\left[\begin{array}{c} \left(\sigma_{g_l,\Psi}^{(p,q,t)L}(f_k) - \varepsilon\right) \\ \cdot\left(\log^{[q-1]}\Psi(r) \cdot \exp^{[t+1]} L(r)\right)^{\rho_{g_l,\Psi}^{(p,q,t)L}(f_k)} \end{array}\right]\right], \tag{3.3}$$

$$M(r,f_k) \leqslant M_{g_l}\left[\exp^{[p-1]}\left[\begin{array}{c}\left(\overline{\sigma}_{g_l,\Psi}^{(p,q,t)L}(f_k)+\varepsilon\right)\\ \cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_l,\Psi}^{(p,q,t)L}(f_k)}\end{array}\right]\right], \tag{3.4}$$

for $k,l=1,2$.

Case 1: Let $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$. For arbitrary $\varepsilon>0$, and from equation (3.1), for sufficiently large values of r,

$$\begin{aligned}M_{f_1\pm f_2} &\leqslant M_{f_1}+M_{f_2}\\ &\leqslant M_{g_1}\left[\exp^{[p-1]}\left[\begin{array}{c}\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\varepsilon\right)\\ \cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\end{array}\right]\right]\\ &\quad +M_{g_1}\left[\exp^{[p-1]}\left[\begin{array}{c}\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_2)+\varepsilon\right)\\ \cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_2)}\end{array}\right]\right]\\ &= M_{g_1}\left[\exp^{[p-1]}\left[\begin{array}{c}\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\varepsilon\right)\\ \cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\end{array}\right]\right]\cdot[1+\omega],\end{aligned}$$

where

$$\omega=\frac{M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_2)+\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_2)}\right]\right]}{M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]}.$$

As $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$ and for all large values of r, ω is very small. Let $\alpha=1+\omega$, then $\alpha\to 1+$, i.e.,

$$M_{f_1\pm f_2}\leqslant M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_l,\Psi}^{(p,q,t)L}(f_1)+\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]\cdot\alpha.$$

Then we get that

$$\limsup_{r\to\infty}\frac{\log^{[p-1]}M_{g1}^{-1}M_{f_1\pm f_2}(r)}{\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}}\leqslant\sigma_{g_l,\Psi}^{(p,q,t)L}(f_1).$$

Therefore

$$\sigma_{g_1}^{(p,q,t)L}(f_1\pm f_2)\leqslant\sigma_{g_1}^{(p,q,t)L}(f_1).$$

Now let $f = f_1 \pm f_2$, as $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$, then

$$\sigma_{g_1}^{(p,q,t)L}(f) = \sigma_{g_1}^{(p,q,t)L}(f_1 \pm f_2) \leqslant \sigma_{g_1}^{(p,q,t)L}(f_1).$$

As $f_1 = f \pm f_2$, then

$$\rho_{g_1,\Psi}^{(p,q,t)L}(f) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_2),$$

then by above result

$$\sigma_{g_1}^{(p,q,t)L}(f_1) \leqslant \sigma_{g_1}^{(p,q,t)L}(f) = \sigma_{g_1}^{(p,q,t)L}(f_1 \pm f_2).$$

Hence,

$$\sigma_{g_1}^{(p,q,t)L}(f) = \sigma_{g_1}^{(p,q,t)L}(f_1) \Rightarrow \sigma_{g_1}^{(p,q,t)L}(f_1) = \sigma_{g_1}^{(p,q,t)L}(f_1 \pm f_2).$$

Thus case 1 is established. Similarly the result can be proved when $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$.

Case 2: Let $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$. From equations (3.1) and (3.4) we get for a sequence of values of r tending to infinity

$$M_{f_1 \pm f_2} \leqslant M_{g_1}\left[\exp^{[p-1]}\left[\begin{array}{c}\left(\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1) + \varepsilon\right) \\ \cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\end{array}\right]\right]\cdot[1+\omega_1],$$

where

$$\omega_1 = \frac{M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_2) + \varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_2)}\right]\right]}{M_{g_1}\left[\exp^{[p-1]}\left[\left(\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1) + \varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]}.$$

As $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$ and for all large values of r, ω_1 is very small. Let $\alpha = 1 + \omega_1$, then $\alpha \to 1+$, i.e.,

$$M_{f_1 \pm f_2} \leqslant M_{g_1}\left[\exp^{[p-1]}\left[\begin{array}{c}\left(\overline{\sigma}_{g_l,\Psi}^{(p,q,t)L}(f_1) + \varepsilon\right) \\ \cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\end{array}\right]\right]\cdot\alpha.$$

In a similar way, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}M_{g_1}^{-1}M_{f_1 \pm f_2}(r)}{\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}} \leqslant \overline{\sigma}_{g_l,\Psi}^{(p,q,t)L}(f_1).$$

Hence we obtain that

$$\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1 \pm f_2) \leqslant \overline{\sigma}_{g_1}^{(p,q,t)L}(f_1).$$

For the equality part proceeding as case 1, we get that

$$\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \overline{\sigma}_{g_1}^{(p,q,t)L}(f_1).$$

So the proof of first part of the theorem is done.

Case 3: Let $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$ and f_1 is of regular relative growth with respect to g_2. Let

$$\omega_2 = \frac{M_{g_2}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) - \varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]}{M_{g_2}\left[\exp^{[p-1]}\left[\left(\sigma_{g_2,\Psi}^{(p,q,t)L}(f_1) - \varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_2,\Psi}^{(p,q,t)L}(f_1)}\right]\right]}.$$

Now

$$\begin{aligned}
&M_{g_1\pm g_2}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) - \varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]\\
&\leqslant M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) - \varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]\\
&\quad + M_{g_2}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) - \varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right].
\end{aligned}$$

We can make ω_2 very small for sufficiently large values of r. we take $1+\omega_2 = \alpha$, then $\alpha \to 1+$. Now by using Lemma 3 for a sequence of values of r tending to infinity

$$\begin{aligned}
&M_{g_1\pm g_2}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) - \varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]\\
&\leqslant \alpha \cdot M_{f_1}(r),
\end{aligned}$$

i.e.,

$$\limsup_{r\to\infty}\frac{\log^{[p-1]} M_{g_1\pm g_2}^{-1} M_{f_1}(r)}{\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}} \geqslant \sigma_{g_1,\Psi}^{(p,q,t)L}(f_1).$$

Therefore we obtain

$$\sigma_{g_{1\pm g_2},\Psi}^{(p,q,t)L}(f_1) \geqslant \sigma_{g_1,\Psi}^{(p,q,t)L}(f_1).$$

For the equality part proceeding as case 1, we get that $\sigma_{g_1\pm g_2,\Psi}^{(p,q,t)L}(f_1) = \sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)$. Similarly, the result can be proved if we consider $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) > \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$.

Case 4: Let $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$ and f_1 is of regular relative growth with respect to g_2. Let

$$\omega_3 = \frac{M_{g_2}\left[\exp^{[p-1]}\left[\left(\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]}{M_{g_2}\left[\exp^{[p-1]}\left[\left(\overline{\sigma}_{g_2,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_2,\Psi}^{(p,q,t)L}(f_1)}\right]\right]}.$$

Now

$$\begin{aligned}
&M_{g_1\pm g_2}\left[\exp^{[p-1]}\left[\left(\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]\\
&\leqslant M_{g_1}\left[\exp^{[p-1]}\left[\left(\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]\\
&\quad+M_{g_2}\left[\exp^{[p-1]}\left[\left(\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right].
\end{aligned}$$

We can make ω_3 very small, by choosing sufficiently large values of r. We take $1+\omega_3=\alpha$, then $\alpha\to 1+$. Now by using Lemma 2, for all sufficiently large values of r we have

$$\begin{aligned}
&M_{g_1\pm g_2}\left[\exp^{[p-1]}\left[\left(\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]\\
&\leqslant \alpha\cdot M_{f_1}(r),
\end{aligned}$$

i.e.,

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}M_{g_1\pm g_2}^{-1}M_{f_1}(r)}{\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}}\geqslant \overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1).$$

Therefore we get

$$\overline{\sigma}_{g_{1\pm g_2},\Psi}^{(p,q,t)L}(f_1)\geqslant \overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1).$$

For the equality part proceeding as case 1, we get $\overline{\sigma}_{g_1\pm g_2,\Psi}^{(p,q,t)L}(f_1)=\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1)$. Similarly, the result can be proved if we consider $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) > \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$. Thus the second part of the theorem is proved. We can easily prove the third part by using part A, part B. So the proof is omitted.

This proves the theorem. □

THEOREM 2. (Main) *Let f_1, f_2, g_1, g_2 be entire functions such that $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)$, $\rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$, $\rho_{g_2,\Psi}^{(p,q,t)L}(f_2)$ and $\rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$ are non zero finite. Also, let $p,q\in N$, $t\in N\cup\{-1,0\}$.*

(i) If $\lambda_{g_1,\Psi}^{(p,q,t)L}(f_i) > \lambda_{g_1,\Psi}^{(p,q,t)L}(f_j)$ *and* f_j *is of regular relative growth with respect to* g_1, *for* $(i,j) = (1,2)$ *or* $(2,1)$, *then*

$$\tau_{g_1,\Psi}^{(p,q,t)L}(f_1 \pm f_2) = \tau_{g_1,\Psi}^{(p,q,t)L}(f_i) \text{ and } \overline{\tau}_{g_1,\Psi}^{(p,q,t)L}(f_1 \pm f_2) = \overline{\tau}_{g_1,\Psi}^{(p,q,t)L}(f_i).$$

(ii) If $\lambda_{g_i,\Psi}^{(p,q,t)L}(f_1) < \lambda_{g_j,\Psi}^{(p,q,t)L}(f_1)$, *for* $(i,j) = (1,2)$ *or* $(2,1)$, *then*

$$\tau_{g_1 \pm g_2,\Psi}^{(p,q,t)L}(f_1) = \tau_{g_i,\Psi}^{(p,q,t)L}(f_1) \text{ and } \overline{\tau}_{g_1 \pm g_2,\Psi}^{(p,q,t)L}(f_1) = \overline{\tau}_{g_i,\Psi}^{(p,q,t)L}(f_1).$$

(iii) If

(a) $\lambda_{g_1,\Psi}^{(p,q,t)L}(f_i) > \lambda_{g_1,\Psi}^{(p,q,t)L}(f_j)$, f_j *is of regular relative growth with respect to* g_1,

(b) $\lambda_{g_2,\Psi}^{(p,q,t)L}(f_i) > \lambda_{g_2,\Psi}^{(p,q,t)L}(f_j)$, f_j *is of regular relative growth with respect to* g_2,

(c) $\lambda_{g_i,\Psi}^{(p,q,t)L}(f_1) < \lambda_{g_j,\Psi}^{(p,q,t)L}(f_1)$, $\lambda_{g_i,\Psi}^{(p,q,t)L}(f_2) < \lambda_{g_j,\Psi}^{(p,q,t)L}(f_2)$ *holds simultaneously,*

(d)

$$\lambda_{g_m,\Psi}^{(p,q,t)L}(f_l) = \min\{\max\{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1), \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)\}, \max\{\rho_{g_2,\Psi}^{(p,q,t)L}(f_1), \rho_{g_2,\Psi}^{(p,q,t)L}(f_2)\}\},$$

for $l,m = 1,2$, *then*

$$\tau_{g_1 \pm g_2,\Psi}^{(p,q,t)L}(f_1 \pm f_2) = \tau_{g_m,\Psi}^{(p,q,t)L}(f_l),$$
$$\overline{\tau}_{g_1 \pm g_2,\Psi}^{(p,q,t)L}(f_1 \pm f_2) = \overline{\tau}_{g_m,\Psi}^{(p,q,t)L}(f_l).$$

Proof. We omit the proof as its proof runs parallel to the proof of Theorem 1. □

THEOREM 3. (Main) *Let* f_1, f_2, g_1 *and* g_2 *be entire functions and* $p,q \in N, t \in N \cup \{-1,0\}$.

(i) If either $\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) \neq \sigma_{g_1,\Psi}^{(p,q,t)L}(f_2)$ *or* $\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_2)$ *and* $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$, *then*

$$\rho_{g_1,\Psi}^{(p,q,t)L}(f_1 \pm f_2) = \rho_{g_1,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_1,\Psi}^{(p,q,t)L}(f_2).$$

(ii) If either $\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) \neq \sigma_{g_2,\Psi}^{(p,q,t)L}(f_1)$ *or* $\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_2,\Psi}^{(p,q,t)L}(f_1)$, f_1 *is of regular relative growth with respect to* g_1 *or* g_2 *and* $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$, *then*

$$\rho_{g_{1\pm g_2},\Psi}^{(p,q,t)L}(f_1) = \rho_{g_1,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_2,\Psi}^{(p,q,t)L}(f_1).$$

Proof. **First part:** As $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$. As $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1 \pm f_2) \leqslant \max\{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1), \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)\}$, we obtain that

$$\rho_{g_1,\Psi}^{(p,q,t)L}(f_1 \pm f_2) \leqslant \rho_{g_1,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_1,\Psi}^{(p,q,t)L}(f_2).$$

If possible, let $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1 \pm f_2) < \rho_{g_1,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$. Let $\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) \neq \sigma_{g_1,\Psi}^{(p,q,t)L}(f_2)$, then, by Theorem 1, we get that

$$\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) = \sigma_{g_1,\Psi}^{(p,q,t)L}(f_1 \pm f_2 \mp f_2) = \sigma_{g_1,\Psi}^{(p,q,t)L}(f_2),$$

which is a contradiction. Then

$$\rho_{g_1,\Psi}^{(p,q,t)L}(f_1 \pm f_2) = \rho_{g_1,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_1,\Psi}^{(p,q,t)L}(f_2).$$

Similarly, the result can be proved by using the condition $\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_2)$. This proves the first part of the theorem.

Second part: As $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$ and f_1 is of regular relative growth with respect to g_1 or g_2. As $\rho_{g_1 \pm g_2,\Psi}^{(p,q,t)L}(f_1) \geqslant \min\{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1), \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)\}$, we get that

$$\rho_{g_1 \pm g_2,\Psi}^{(p,q,t)L}(f_1) \geqslant \rho_{g_1,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_2,\Psi}^{(p,q,t)L}(f_1).$$

If possible, let $\rho_{g_1 \pm g_2,\Psi}^{(p,q,t)L}(f_1) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$. Let $\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) \neq \sigma_{g_2,\Psi}^{(p,q,t)L}(f_1)$. Then, by Theorem 1,

$$\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) = \sigma_{g_1 \pm g_2 \mp g_2,\Psi}^{(p,q,t)L}(f_1) = \sigma_{g_2,\Psi}^{(p,q,t)L}(f_1),$$

which is a contradiction. Then

$$\rho_{g_1 \pm g_2,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_1,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_2,\Psi}^{(p,q,t)L}(f_1).$$

Similarly, the result can be proved by using the condition $\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_2,\Psi}^{(p,q,t)L}(f_1)$. This proves the theorem. □

THEOREM 4. (Main) *Let f_1, f_2, g_1 and g_2 are entire functions and $\rho_{g_i,\Psi}^{(p,q,t)L}(f_j)$ are all finite and non zero for $i,j = 1,2$, $p,q \in N$, $t \in N \cup \{-1,0\}$.*

(A) If

(i) $\rho_{g_1,\Psi}^{(p,q,t)L}(f_i) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_j)$, $(i,j) = (1,2)$ *or* $(2,1)$,

(ii) g_1 *satisfies Property (A),*

then

$$\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1 \cdot f_2) = \sigma_{g_1,\Psi}^{(p,q,t)L}(f_i),$$
$$\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1 \cdot f_2) = \overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_i).$$

Similarly, if

(i) $\frac{f_1}{f_2}$ *is entire,*

(ii) $\rho_{g_1,\Psi}^{(p,q,t)L}(f_i) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_j),\ (i,j) = (1,2)\ or\ (2,1),$

(iii) g_1 *satisfies Property (A),*

then

$$\sigma_{g_1,\Psi}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) = \sigma_{g_1,\Psi}^{(p,q,t)L}(f_i),$$
$$\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) = \overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_i).$$

(B) If

(i) $\rho_{g_i,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_j,\Psi}^{(p,q,t)L}(f_1), (i,j) = (1,2)\ or\ (2,1),$

(ii) g_i, g_1 *and* g_2 *satisfies Property (A),*

(iii) f_1 *is of regular relative growth with respect to* g_j,

then

$$\sigma_{g_1\cdot g_2,\Psi}^{(p,q,t)L}(f_1) = \sigma_{g_i,\Psi}^{(p,q,t)L}(f_1),$$
$$\overline{\sigma}_{g_1\cdot g_2,\Psi}^{(p,q,t)L}(f_1) = \overline{\sigma}_{g_i,\Psi}^{(p,q,t)L}(f_1).$$

Similarly, if

(i) $\rho_{g_i,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_j,\Psi}^{(p,q,t)L}(f_1), (i,j) = (1,2)\ or\ (2,1),$

(ii) g_i, g_1 *and* g_2 *satisfies Property (A),*

(iii) f_1 *is of regular relative growth with respect to* g_j,

(iv) $\frac{g_1}{g_2}$ *is entire function,*

then

$$\sigma_{\frac{g_1}{g_2},\Psi}^{(p,q,t)L}(f_1) = \sigma_{g_i,\Psi}^{(p,q,t)L}(f_1),$$
$$\overline{\sigma}_{\frac{g_1}{g_2},\Psi}^{(p,q,t)L}(f_1) = \overline{\sigma}_{g_i,\Psi}^{(p,q,t)L}(f_1).$$

(C) If

(i) $g_1 \cdot g_2$ *satisfies Property (A),*

(ii) $\rho_{g_i,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_j,\Psi}^{(p,q,t)L}(f_1), f_1$ *is of regular relative growth with respect to* $g_j,\ (i,j) = (1,2)\ or\ (2,1),$

(iii) $\rho_{g_i,\Psi}^{(p,q,t)L}(f_2) < \rho_{g_j,\Psi}^{(p,q,t)L}(f_2),\ f_2$ *is of regular relative growth with respect to* $g_j,\ (i,j) = (1,2)\ or\ (2,1),$

(iv) $\rho_{g_1,\Psi}^{(p,q,t)L}(f_i) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_j)$, $\rho_{g_2,\Psi}^{(p,q,t)L}(f_i) > \rho_{g_2,\Psi}^{(p,q,t)L}(f_j)$,

(v)

$$\rho_{g_m,\Psi}^{(p,q,t)L}(f_l) = \max\{\min\{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1), \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)\}, \min\{\rho_{g_1,\Psi}^{(p,q,t)L}(f_2), \rho_{g_2,\Psi}^{(p,q,t)L}(f_2)\}\},$$

then

$$\sigma_{g_1\cdot g_2,\Psi}^{(p,q,t)L}(f_1\cdot f_2) = \sigma_{g_m,\Psi}^{(p,q,t)L}(f_l),$$
$$\overline{\sigma}_{g_1\cdot g_2,\Psi}^{(p,q,t)L}(f_1\cdot f_2) = \overline{\sigma}_{g_m,\Psi}^{(p,q,t)L}(f_l).$$

Similarly, if

(i) $\frac{f_1}{f_2}, \frac{g_1}{g_2}$ *is entire function,*

(ii) $\frac{g_1}{g_2}$ *having Property (A),*

(iii) $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) \neq \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$, f_1 *is of regular relative growth with respect to* g_2,

(iv) $\rho_{g_1,\Psi}^{(p,q,t)L}(f_2) < \rho_{g_2,\Psi}^{(p,q,t)L}(f_2)$, f_2 *is of regular relative growth with respect to* g_2,

(v) $\rho_{g_1,\Psi}^{(p,q,t)L}(f_i) < \rho_{g_1,\Psi}^{(p,q,t)L}(f_j)$, $\rho_{g_2,\Psi}^{(p,q,t)L}(f_i) < \rho_{g_2,\Psi}^{(p,q,t)L}(f_j)$,

(vi)

$$\rho_{g_m,\Psi}^{(p,q,t)L}(f_l) = \max\{\min\{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1), \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)\}, \min\{\rho_{g_1,\Psi}^{(p,q,t)L}(f_2), \rho_{g_2,\Psi}^{(p,q,t)L}(f_2)\}\},$$

then

$$\sigma_{\frac{g_1}{g_2},\Psi}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) = \sigma_{g_m,\Psi}^{(p,q,t)L}(f_l),$$
$$\overline{\sigma}_{\frac{g_1}{g_2},\Psi}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) = \overline{\sigma}_{g_m,\Psi}^{(p,q,t)L}(f_l).$$

Proof. **Case 1:** Let $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$. By equation 3.1, for all sufficiently large values of r, we get

$$\begin{aligned}
&M_{f_1\cdot f_2}(r) \leqslant M_{f_1}(r)\cdot M_{f_2}(r)\\
&\leqslant M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]\\
&\quad\times M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_2)+\frac{\varepsilon}{2}\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_2)}\right]\right],
\end{aligned}$$

then as $r \to \infty$,

$$\frac{\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]}{\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_2)+\frac{\varepsilon}{2}\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_2)}\right]}\to\infty,$$

i.e.,

$$\begin{aligned}&M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]\\ &>M_{g_1}\left[\exp^{[p-1]}\left[\begin{array}{c}\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_2)+\frac{\varepsilon}{2}\right)\\ \cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_2)}\end{array}\right]\right].\end{aligned}\tag{3.5}$$

Then

$$\begin{aligned}&M_{f_1\cdot f_2}(r)\\ &\leqslant\left\{M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]\right\}^2.\end{aligned}$$

Now

$$\delta_1=\frac{\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\varepsilon}{\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}}>1.$$

Therefore we get that

$$\frac{\exp^{[p-2]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]}{\exp^{[p-2]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]}=\delta\,(say)>1.\tag{3.6}$$

Combining equations (4) and (3.6) and using Lemma 2, we get for all sufficiently large values of r that

$$\begin{aligned}&M_{f_1\cdot f_2}(r)\\ &\leqslant M_{g_1}\left[\left\{\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right\}^{\delta}\right]\\ &\leqslant M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right].\end{aligned}$$

As $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$ and $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1 \cdot f_2) \leqslant \max\{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1), \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)\}$, we get for all sufficiently large values of r that

$$M_{f_1 \cdot f_2}(r) \leqslant M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) + \varepsilon\right) \cdot \left(\log^{[q-1]}\Psi(r) \cdot \exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1 \cdot f_2)}\right]\right].$$

Therefore we get

$$\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1 \cdot f_2) \leqslant \sigma_{g_1,\Psi}^{(p,q,t)L}(f_1). \tag{3.7}$$

Next for the equality part let h, h_1, h_2 and k be four functions, $h = \frac{h_2}{h_1}$, $q > 1$ and k satisfies Property (A). Without any loss of generality let $\rho_{k,\Psi}^{(p,q,t)L}(h_1) < \rho_{k,\Psi}^{(p,q,t)L}(h_2)$. Now

$$T_h(r) = T_{\frac{h_2}{h_1}}(r) \leqslant T_{h_2}(r) + T_{h_1}(r).$$

Proceeding as above and using Lemma 3, we get

$$\log M_{\frac{h_2}{h_1}}(r) \leqslant 3\left[T_{h_1}(2r) + T_{h_2}(2r)\right].$$

Hence

$$\left[M_{\frac{h_2}{h_1}}\left(\frac{r}{2}\right)\right]^{\frac{1}{3}} \leqslant M_{h_1}(r) \cdot M_{h_2}(r).$$

Therefore we get that

$$M_{\frac{h_2}{h_1}}\left(\frac{r}{2}\right) < \left\{M_k\left[\exp^{[p-1]}\left[\left(\sigma_{k,\Psi}^{(p,q,t)L}(h_2) + \frac{\varepsilon}{2}\right) \cdot \left(\log^{[q-1]}\Psi(r). \exp^{[t+1]}L(r)\right)^{\rho_{k,\Psi}^{(p,q,t)L}(h_2)}\right]\right]\right\}^6.$$

From equation (3.6) we have

$$M_{\frac{h_2}{h_1}}\left(\frac{r}{2}\right) < M_k\left[\exp^{[p-1]}\left[\left(\sigma_{k,\Psi}^{(p,q,t)L}(h_2) + \varepsilon\right) \cdot \left(\log^{[q-1]}\Psi(r) \cdot \exp^{[t+1]}L(r)\right)^{\rho_{k,\Psi}^{(p,q,t)L}(h_2)}\right]\right],$$

i.e.,

$$\sigma_{k,\Psi}^{(p,q,t)L}(h) = \sigma_{k,\Psi}^{(p,q,t)L}\left(\frac{h_2}{h_1}\right) \leqslant \sigma_{k,\Psi}^{(p,q,t)L}(h_2). \tag{3.8}$$

Let $f = f_1 \cdot f_2$, $\rho_{g_1,\Psi}^{(p,q,t)L}(f_2) < \rho_{g_1,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_1,\Psi}^{(p,q,t)L}(f)$, then by equation (3.7), we obtain that

$$\sigma_{g_1,\Psi}^{(p,q,t)L}(f) = \sigma_{g_1,\Psi}^{(p,q,t)L}(f_1 \cdot f_2) \leqslant \sigma_{g_1,\Psi}^{(p,q,t)L}(f_1).$$

As $f_1 = \frac{f}{f_2}$, then by equation (3.7), we get that

$$\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) \leqslant \sigma_{g_1,\Psi}^{(p,q,t)L}(f) = \sigma_{g_1,\Psi}^{(p,q,t)L}(f_1 \cdot f_2).$$

Hence,

$$\sigma_{g_1,\Psi}^{(p,q,t)L}(f) = \sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) \Rightarrow \sigma_{g_1,\Psi}^{(p,q,t)L}(f_1 \cdot f_2) = \sigma_{g_1,\Psi}^{(p,q,t)L}(f_1),$$

provided $q > 1$. Similarly, the result can be proved if we consider $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$.

Subcase 1: Let $\rho_{g_1,\Psi}^{(p,q,t)L}(f_2) < \rho_{g_1,\Psi}^{(p,q,t)L}(f_1)$. Now we have

$$\rho_{g_1,\Psi}^{(p,q,t)L}(f_2) < \rho_{g_1,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_1,\Psi}^{(p,q,t)L}(f).$$

As $f_1 = f \cdot f_2$, then

$$\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) = \sigma_{g_1,\Psi}^{(p,q,t)L}(f) = \sigma_{g_1,\Psi}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right),$$

where $q > 1$. Thus subcase 1 is established.

Subcase 2: Let $\rho_{g_1,\Psi}^{(p,q,t)L}(f_2) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_1)$. Again we have

$$\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_1,\Psi}^{(p,q,t)L}(f_2) = \rho_{g_1,\Psi}^{(p,q,t)L}(f).$$

By (3.8) we get that

$$\sigma_{g,\Psi}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) \leqslant \sigma_{g_1,\Psi}^{(p,q,t)L}(f_2).$$

Again $f_2 = \frac{f_1}{f}$, so we have

$$\sigma_{g_1,\Psi}^{(p,q,t)L}(f_2) \leqslant \sigma_{g_1,\Psi}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right).$$

So,

$$\sigma_{g_1,\Psi}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) = \sigma_{g_1,\Psi}^{(p,q,t)L}(f_2),$$

where $q > 1$.

Case 2: Let $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$, g_1 having Property (A). By equations (3.1) and (3.4), we get for a sequence of values of $r \to \infty$

$$\begin{aligned} &M_{f_1 \cdot f_2}(r) \leqslant M_{f_1}(r) \cdot M_{f_2}(r) \\ &\leqslant M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) + \frac{\varepsilon}{2}\right) \cdot \left(\log^{[q-1]}\Psi(r) \cdot \exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right] \end{aligned}$$

$$\times M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_2)+\frac{\varepsilon}{2}\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_2)}\right]\right],$$

then as $r\to\infty$

$$\frac{\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]}{\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_2)+\frac{\varepsilon}{2}\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_2)}\right]}\to\infty,$$

i.e.,

$$M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]$$
$$>M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_2)+\frac{\varepsilon}{2}\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_2)}\right]\right].$$

Then

$$M_{f_1\cdot f_2}(r)$$
$$\leqslant\left\{M_{g_1}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]\right\}^2.$$

Proceeding in the way used in case 1 we can prove the result

$$\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1\cdot f_2)=\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1),$$
$$\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right)=\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_i).$$

Similarly, if we take $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)<\rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$, then we can prove the result

$$\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1\cdot f_2)=\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_2),$$

where $q>1$.

Then the first part (part A) of the theorem is proved.

Case 3: Let $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)<\rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$ and $g_1\cdot g_2$, g_1 satisfy Property (A) and f_1 is of regular relative growth w.r.t. g_2. As $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)<\rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$, then for all large values of r,

$$\exp^{[p-1]}\left[\left(\sigma_{g_2,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_2,\Psi}^{(p,q,t)L}(f_1)}\right]$$

$$> \exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]$$

$$\Rightarrow M_{g_2}\left[\exp^{[p-1]}\left[\left(\sigma_{g_2,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_2,\Psi}^{(p,q,t)L}(f_1)}\right]\right]$$

$$> M_{g_2}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right].$$

Then by equations (3.2) and (3.3) and above we get for a sequence of values of $r\to\infty$

$$M_{g_1\cdot g_2}\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]$$
$$\leqslant\left[M_{f_1}(r)\right]^2.$$

Now using Lemma 2, we get for a sequence of values of $r\to\infty$,

$$M_{g_1\cdot g_2}\left[\left[\exp^{[p-1]}\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]\frac{1}{\delta}\right]$$
$$<M_{f_1}(r).$$

Now making $\delta\to 1+$

$$\left[\left(\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]<\log^{[p-1]}M_{g_1\cdot g_2}^{-1}M_{f_1}(r).$$

As $\varepsilon>0$ is arbitrary, we have that

$$\sigma_{g_1\cdot g_2,\Psi}^{(p,q,t)L}(f_1)\geqslant\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1). \tag{3.9}$$

Next for the equality part let $h,\ h_1,\ h_2$ and k be four entire functions, $h=\frac{h_2}{h_1},\ p>1$ and h satisfies Property (A) and k is regular relative growth with respect to h_2. Without any loss of generality let $\rho_{h_1,\Psi}^{(p,q,t)L}(k)<\rho_{h_2,\Psi}^{(p,q,t)L}(k)$. Now

$$T_h(r)=T_{\frac{h_2}{h_1}}(r)\leqslant T_{h_2}(r)+T_{h_1}(r).$$

Proceeding as above proof and using Lemma 3, we have that

$$\log M_{\frac{h_2}{h_1}}(r)\leqslant 3\left[T_{h_1}(2r)+T_{h_2}(2r)\right],\quad \text{i.e.,}$$

$$\left[M_{\frac{h_2}{h_1}}\left(\frac{r}{2}\right)\right]^{\frac{1}{3}}\leqslant M_{h_1}(r)\cdot M_{h_2}(r).$$

Using same method as in the above proof we have

$$\sigma_{h,\Psi}^{(p,q,t)L}(k)=\sigma_{\frac{h_2}{h_1},\Psi}^{(p,q,t)L}(k)\geqslant\sigma_{h_1,\Psi}^{(p,q,t)L}(k). \tag{3.10}$$

Let $g = g_1 \cdot g_2$, $\rho_{g,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_1,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$, then by equation (3.9), we have

$$\sigma_{g,\Psi}^{(p,q,t)L}(f_1) = \sigma_{g_1 \cdot g_2,\Psi}^{(p,q,t)L}(f_1) \geqslant \sigma_{g_1,\Psi}^{(p,q,t)L}(f_1).$$

As $g_1 = \frac{g}{g_2}$ so by equation (3.10) we get that

$$\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) \geqslant \sigma_{g,\Psi}^{(p,q,t)L}(f_1) = \sigma_{g_1 \cdot g_2,\Psi}^{(p,q,t)L}(f_1).$$

Hence we get that

$$\sigma_{g,\Psi}^{(p,q,t)L}(f_1) = \sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) \Rightarrow \sigma_{g_1 \cdot g_2,\Psi}^{(p,q,t)L}(f_1) = \sigma_{g_1,\Psi}^{(p,q,t)L}(f_1),$$

provided $q > 1$. Similarly, the result can be proved if we consider $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_1,\Psi}^{(p,q,t)L}(f_2)$.

Subcase 1: Let $\rho_{g_2,\Psi}^{(p,q,t)L}(f_1) > \rho_{g_1,\Psi}^{(p,q,t)L}(f_1)$. We have

$$\rho_{g,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_1,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_2,\Psi}^{(p,q,t)L}(f_1).$$

As $g_1 = g \cdot g_2$,

$$\sigma_{g_1,\Psi}^{(p,q,t)L}(f_1) = \sigma_{g,\Psi}^{(p,q,t)L}(f_1) = \sigma_{\frac{g_1}{g_2},\Psi}^{(p,q,t)L}(f_1),$$

where $p > 1$. Thus subcase 1 is established.

Now let $g = \frac{g_1}{g_2}$ and we will prove the following subcases.

Subcase 2: Let $\rho_{g_2,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_1,\Psi}^{(p,q,t)L}(f_1)$, then we have

$$\rho_{g,\Psi}^{(p,q,t)L}(f_1) = \rho_{g_2,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_1,\Psi}^{(p,q,t)L}(f_1).$$

By equation (3.10), we obtain that

$$\sigma_{\frac{g_1}{g_2},\Psi}^{(p,q,t)L}(f_1) \leqslant \sigma_{g_2,\Psi}^{(p,q,t)L}(f_1).$$

Again $g_2 = \frac{g_1}{g}$ so we have

$$\sigma_{g_2,\Psi}^{(p,q,t)L}(f_1) \leqslant \sigma_{\frac{g_1}{g_2},\Psi}^{(p,q,t)L}(f_1).$$

So,

$$\sigma_{\frac{g_1}{g_2},\Psi}^{(p,q,t)L}(f_1) = \sigma_{g_2,\Psi}^{(p,q,t)L}(f_1),$$

where $p > 1$.

Case 4: Let $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$ and $g_1 \cdot g_2$ and g_1 having Property (A), f_1 is of regular relative growth w.r.t. g_2.

As $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) < \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$, then for all large values of r we have that

$$\exp^{[p-1]}\left[\left(\overline{\sigma}_{g_2,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_2,\Psi}^{(p,q,t)L}(f_1)}\right]$$

$$> \exp^{[p-1]}\left[\left(\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]$$

$$\Rightarrow M_{g_2}\left[\exp^{[p-1]}\left[\left(\overline{\sigma}_{g_2,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_2,\Psi}^{(p,q,t)L}(f_1)}\right]\right]$$

$$> M_{g_2}\left[\exp^{[p-1]}\left[\left(\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right].$$

Then by equation (3.2) and above we get for all sufficiently large values of $r \to \infty$ that

$$M_{g_1\cdot g_2}\left[\exp^{[p-1]}\left[\left(\overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1)-\varepsilon\right)\cdot\left(\log^{[q-1]}\Psi(r)\cdot\exp^{[t+1]}L(r)\right)^{\rho_{g_1,\Psi}^{(p,q,t)L}(f_1)}\right]\right]$$
$$\leqslant \left[M_{f_1}(r)\right]^2.$$

Now by applying the method using in case 3 we can easily get the required result

$$\overline{\sigma}_{g_1\cdot g_2,\Psi}^{(p,q,t)L}(f_1) = \overline{\sigma}_{g_1,\Psi}^{(p,q,t)L}(f_1),$$
$$\overline{\sigma}_{\frac{g_1}{g_2},\Psi}^{(p,q,t)L}(f_1) = \overline{\sigma}_{g_i,\Psi}^{(p,q,t)L}(f_1),$$

where $i = 1,2$. Similarly, if we consider $\rho_{g_1,\Psi}^{(p,q,t)L}(f_1) > \rho_{g_2,\Psi}^{(p,q,t)L}(f_1)$, then we can prove

$$\overline{\sigma}_{g_1\cdot g_2,\Psi}^{(p,q,t)L}(f_1) = \overline{\sigma}_{g_2,\Psi}^{(p,q,t)L}(f_1).$$

So the second part (part B) of the theorem is proved.

Part 3: This part is a combined form of Part A and Part B. So the proof is omitted. This completes the proof of the theorem. □

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REFERENCES

[1] R. P. AGARWAL, S. K. DATTA, T. BISWAS AND P. SAHOO, *On the growth analysis of iterated entire functions*, Advanced Studies in Contemporary Mathematics., **26**, 1 (2016), 93–137.

[2] L. BERNAL, *Orden relativ de crecimiento de funciones enteras*, Collect Mth., **46**, 1 (1917), 157–159.

[3] T. BISWAS, *Some results relating to sum and product theorems of relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th type of entire functions*, Korean J. Math., **26**, 2 (2018), 215–269.

[4] I. CHYZHYKOV, J. HEITTOKANGAS AND J. RÄTTYÄ, *Finiteness of φ-order of solutions of linear differential equations in the unit disc*, J. Anal. Math., (2009), 109: 209–229, 10.1007/s11854-009-0030-3.

[5] S. K. DATTA, T. BISWAS AND C. GHOSH, *On relative (p,q)-th order based growth measure of entire functions*, Filomat, **30**, 7 (2016), 1723–1735.

[6] S. K. DATTA, T. BISWAS AND D. DUTTA, *Generalized relative type and generalized weak type of entire functions*, Journal of Complex Analysis, (2016), Article ID 3468354, 11 pages.

[7] W. K. HAYMAN, *Meromorphic functions*, The Calendron Press, (1946).

[8] O. P. JUNEJA, G. P. KAPOOR AND S. K. BAJPAI, *On the (p,q)-order and lower (p,q)-order of an entire function*, J. Reine Angew. Math., **282** (1976), 53–67.

[9] O. P. JUNEJA, G. P. KAPOOR AND S. K. BAJPAI, *On the (p,q)-type and lower (p,q)-type of an entire function*, J. Reine Angew. Math., **290** (1977), 180–190.

[10] L. M. SANCHEZ RUIZ, S. K. DATTA, T. BISWAS AND G. K. MANDAL, *On the (p,q)-th relative order and oriented growth properties of entire functions*, Abstr. Appl. Anal., (2014), Article ID 826137, 8 pages, http://dx.doi.org/10.1155/2014/826137.

[11] H. M. SRIVASTAVA, S. K. DATTA, T. BISWAS AND D. DUTTA, *Sum and product theorems depending on the (p,q)-th order and (p,q)-th type of entire functions*, Cogent Mathematics, (2015), 2 : 1107951, 1–22.

[12] X. SHEN, J. TU AND H. Y. XU, *Complex oscillation of a second order linear differential equation with entire coefficients of $[p,q]$ - φ ' order*, Advances in Difference Equations, (2014), 2014 : 200.

[13] D. SOMASUNDARAM AND R. THAMIZHARASI, *A note on the entire functions of L-bounded index and L-type*, Indian J. Pure. Appl. Math., **19**, 3 (1988), 284–293.

[14] G. VALIRON, *Lectures on the general theory of Integral functions*, Chelsea Publishing Company, (1949).

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Sanjib Kumar Datta
Department of Mathematics
University of Kalyani
P.O.: Kalyani, Dist.: Nadia, PIN: 741235, West Bengal, India
e-mail: sanjibdatta05@gmail.com

Banani Dutta
Department of Mathematics
Darjeeling Government College
Lebong cart road, Dist.: Darjeeling, PIN: 734101, West Bengal, India
e-mail: duttabananimath@gmail.com

Nityagopal Biswas
Department of Mathematics
Chakdaha College
Chakdaha, Nadia, West Bengal, India
e-mail: nityamaths@gmail.com