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NEW INTERESTING EULER SUMS

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Abstract. We present here some new and interesting Euler sums obtained by means of related integrals and elementary approach. We supplement Euler's general recurrence formula with two general formulas of the form $\sum_{n\geqslant 1} O_n^{(m)}\left(\frac{1}{(2n-1)^p}+\frac{1}{(2n)^p}\right)$ and $\sum_{n\geqslant 1}\frac{O_n}{(2n-1)^p(2n+1)^q}$, where $O_n^{(m)}=\sum_{j=1}^{n}\frac{1}{(2j-1)^m}$. Two formulas for $\zeta(5)$ are also derived.

1. Evolution of Euler sums

Euler, the most prolific and versatile mathematician, made notable contributions to all branches of mathematics but much of his significant work involves infinite series, especially his zeta function.

1.1. Euler sums

In response to a letter of 24th December, 1742 from Goldbach, Euler investigated sums involving zeta function and harmonic numbers and published his results many years later in [7] using a cumbersome notation:

$$\int \frac{1}{z^m}\left(\frac{1}{y^n}\right)=1+\frac{1}{2^m}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\dots.$$

Classical Euler sum $BW(p,q)$ is thus an infinite sum whose general term contains generalized harmonic number $H_n^{(p)}=\sum_{k=1}^{n}\frac{1}{k^p}$ in numerator and n^q with $q\geqslant 2$ in denominator. That is, $BW(p,q)=\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^q}$. The number $p+q$ is the *weight* of $BW(p,q)$. In the said paper, Euler employed three methods (which he called *Prima Methodus, Secunda Methodus* and *Tertia Methodus*) to discover formulas representing these sums in terms of zeta values. First, he multiplied the involved series to obtain the *reflection formula*:

$$\int \frac{1}{z^m}\left(\frac{1}{y^n}\right)+\int \frac{1}{z^n}\left(\frac{1}{y^m}\right)=\int \frac{1}{z^m}\int \frac{1}{z^n}+\int \frac{1}{z^{m+n}}$$

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which can be written as: $BW(n,m)+BW(m,n)=\zeta(m)\zeta(n)+\zeta(m+n)$ and straightway leads to the sum: $\sum_{n=1}^{\infty}\frac{H_n^{(m)}}{n^m}=\frac{1}{2}\zeta^2(m)+\frac{1}{2}\zeta(2m)$. Euler gave a general formula (without proof) expressing $EU(m)=BW(1,m)$ in terms of zeta values. We find in [7, §22] a list of formulas which, for $m\geqslant 2$, can be written as:

$$EU(m)=\sum_{n=1}^{\infty}\frac{H_n}{n^m}=\frac{m+2}{2}\zeta(m+1)-\frac{1}{2}\sum_{k=1}^{m-2}\zeta(m-k)\zeta(k+1),\ m=2,3,4,\ldots. \quad (1.1)$$

1.2. Post-Euler development

These recurrence relations hold [21]:

$$\zeta(2m)=\frac{2}{2m+1}\sum_{k=1}^{m-1}\zeta(2k)\zeta(2m-2k),\ m=2,3,4,\ldots,$$

$$(1-2^{-2m})\zeta(2m)=\frac{2}{2m-1}\sum_{k=1}^{m}\xi(2k-1)\xi(2m+1-2k),\ m=1,2,3,\ldots,$$

where $\xi(2m+1):=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)^{2m+1}}$.

Nielsen [11, pp.47–49] built on and supplemented Euler's work by supplying proof of the general formula using the method of partial fractions.

Georghiou and Philippou [9] established formula (1.1) and

$$\sum_{k=1}^{\infty}\frac{H_k}{k^{2n+1}}=\frac{1}{2}\sum_{j=2}^{2n}(-1)^j\zeta(j)\zeta(2n+2-j),\ n\geqslant 1. \quad (1.2)$$

For odd weight $p+q=2r+1$, Borweins [3] (see[8, Theorem 3.1]) established a correct version of Euler's formula relating to exponent $\geqslant 2$ of the harmonic series:

$$\begin{aligned}BW(p,q)=\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^q}&=\frac{1}{2}\left[1-(-1)^p\binom{2r}{p}-(-1)^p\binom{2r}{q}\right]\zeta(2r+1)\\&+\frac{1-(-1)^p}{2}\zeta(p)\zeta(q)+(-1)^p\sum_{k=1}^{\lfloor p/2\rfloor}\binom{2r-2k}{q-1}\zeta(2k)\zeta(2r+1-2k)\\&+(-1)^p\sum_{k=1}^{\lfloor q/2\rfloor}\binom{2r-2k}{p-1}\zeta(2k)\zeta(2r+1-2k+1).\end{aligned} \quad (1.3)$$

The sum $BW(p,q)$ also admits of representation in terms of zeta values when: $p=q$, and $(p,q)=(2,4)$ or $(4,2)$. For alternating Euler sums, Sitaramachandra Rao [15] gave the identity:

$$S(p,q):=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^{(p)}}{n^q},$$

when $p=1$ and for odd weight $1+q$ as,

$$2S(1,q)=(1+q)\eta(1+q)-\zeta(1+q)-2\sum_{j=1}^{\frac{q}{2}-1}\eta(2j)\ \zeta(1+q-2j). \tag{1.4}$$

Flajolet and Salvy [8] also gave the integral

$$S(1,1+2q)=\frac{1}{(2q)!}\int_0^1\frac{\ln^{2q}(x)\ln(1+x)}{x(1+x)}dx.$$

In the case where p and q are both positive integers and $p+q$ is an odd integer, they also [8] gave the identity:

$$\begin{aligned}2S(p,q)&=(1-(-1)^p)\zeta(p)\eta(q)+2(-1)^p\sum_{i+2k=q}\binom{p+i-1}{p-1}\zeta(p+i)\ \eta(2k)\\&+\eta(p+q)-2\sum_{j+2k=p}\binom{q+j-1}{q-1}(-1)^j\eta(q+j)\ \eta(2k),\end{aligned} \tag{1.5}$$

where $\eta(0)=\frac{1}{2}$, $\eta(1)=\ln 2$, $\zeta(1)=0$, and $\zeta(0)=-\frac{1}{2}$ in accordance with the analytic continuation of the Riemann zeta function. We define the alternating zeta function (or Dirichlet eta function) $\eta(z)$ as

$$\eta(z):=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^z}=\left(1-2^{1-z}\right)\zeta(z).$$

Sofo [17], further, developed the half integer Euler sums. For positive integers m,p and odd weight $m+p$,

$$\begin{aligned}W(m,p)&=\sum_{n\geqslant 1}\frac{H_{\frac{n}{2}}^{(m)}}{n^p}=(-1)^p\sum_{r=1}^{m}2^{m-1}\binom{m+p-1-r}{m-r}\begin{pmatrix}BW(r,m+p-r)\\-S(r,m+p-r)\end{pmatrix}\\&+(-1)^{p+1}\sum_{r=2}^{m}\frac{1}{2^{p-r}}\binom{m+p-1-r}{m-r}\zeta(r)\zeta(m+p-r)\\&+(-1)^{p+1}\sum_{k=2}^{p-1}\frac{(-1)^k}{2^{p-k}}\binom{m+p-1-k}{p-k}\zeta(k)\zeta(m+p-k).\end{aligned} \tag{1.6}$$

Also, since

$$\sum_{n\geqslant 1}\frac{H_n^{(m)}}{n^p}=2^{p-1}\sum_{n\geqslant 1}\frac{H_{\frac{n}{2}}^{(m)}}{n^p}\left(1-(-1)^{n+1}\right),$$

we obtain the alternating Euler identity at half integer value,

$$AW(m,p)=\sum_{n\geqslant 1}\frac{(-1)^{n+1}H_{\frac{n}{2}}^{(m)}}{n^p}=W(m,p)-2^{1-p}BW(m,p). \tag{1.7}$$

It is interesting to note that for (1.6) or (1.7) we can evaluate the difference (or sum) of two terms, in the following way. Let (m,p,r) be positive integers, with $p\geqslant 2,$ then

$$\begin{aligned} F(m,p,r) &= W(m,p,r) - W(m,p,r+2) = \sum_{n\geqslant 1} \frac{H_{\frac{n}{2}}^{(m)}}{(n+r)^p} - \sum_{n\geqslant 1} \frac{H_{\frac{n}{2}}^{(m)}}{(n+r+2)^p} \\ &= 2^m\zeta(m+p) - 2^m\eta(m), \text{ for } r=0. \end{aligned}$$

For $r>0$

$$\begin{aligned} 2^{-m}F(m,p,r) =& (-1)^{m+1}\sum_{j=1}^{p} \frac{\binom{m+p-1-j}{p-j} H_r^{(j)}}{r^{m+p-j}} - \frac{\eta(m)}{(1+r)^p} \\ &+(-1)^m \sum_{j=2}^{p} \frac{\binom{m+p-1-j}{p-j}\zeta(j)}{r^{m+p-j}} + (-1)^m \sum_{k=2}^{m} \frac{\binom{m+p-1-k}{m-k}\zeta(k)}{r^{m+p-k}}. \end{aligned}$$

$$\begin{aligned} F(5,4,0) =& 32\zeta(9) - 30\zeta(5), \\ F(2,4,2) =& \zeta(4) + \zeta(3) + \frac{79}{81}\zeta(2) - \frac{31}{8}. \end{aligned}$$

2. Euler sums involving odd harmonic numbers

Let $O_n := \sum_{k=1}^{n} \frac{1}{2k-1}$. Then $O_n = H_{2n} - \frac{1}{2}H_n$.
One can compute the following sums with little effort:

$$\sum_{n=1}^{\infty} \frac{H_n}{(2n-1)(2n+1)} = \ln 2; \tag{2.1}$$

$$\sum_{n=1}^{\infty} \frac{O_n}{(2n-1)(2n+1)} = \frac{3}{8}\zeta(2); \tag{2.2}$$

$$\sum_{n=1}^{\infty} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3). \tag{2.3}$$

This sum noted in [15, p.14, eq(4.16)]: $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{O_n}{n^2} = \pi G - \frac{7}{4}\zeta(3)$ when added to (2.3) gives a nice double sum:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sum_{k=1}^{2n-1} \frac{1}{2k-1} = 1 + \frac{1}{3^2}\left(1+\frac{1}{3}+\frac{1}{5}\right) + \cdots = \frac{\pi G}{2},$$

where $G = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}$ is Catalan's constant. After some manipulation we can also isolate the identity,

$$\sum_{n=1}^{\infty} \frac{H_{4n-2}}{(2n-1)^2} = \frac{21}{32}\zeta(3) + \frac{\pi G}{2}$$

and the alternating double sum

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{2k-1}=1\,(1)-\frac{1}{2}\left(1-\frac{1}{3}\right)+\frac{1}{3}\left(1-\frac{1}{3}+\frac{1}{5}\right)-\cdots=G.$$

Sofo derives this formula [16, Corollary 1] for real $\alpha\neq-1,-2,-3,\ldots$:

$$\begin{aligned}2\sum_{n=1}^{\infty}\frac{H_n}{(n+\alpha+1)^m}=&2\zeta(m,\alpha+1)H_\alpha+m\zeta(m+1,\alpha+1)\\&-\sum_{k=1}^{m-2}\zeta(k+1,\alpha+1)\zeta(m-k,\alpha+1),\end{aligned}$$

which leads to

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^m}=&-\frac{(2^m-1)}{2^{m-1}}\zeta(m)\ln 2+\frac{m\,(2^{m+1}-1)}{2^{m+1}}\zeta(m+1)\\&-\sum_{k=1}^{m-2}\frac{(2^{k+1}-1)\,(2^{m-k}-1)}{2^{m+1}}\zeta(k+1)\,\zeta(m-k).\end{aligned}$$

If we, let $\kappa(m)=\frac{1}{2}(\zeta(m)+\eta(m))=\sum_{k\geqslant 1}\frac{1}{(2k-1)^m}$, and for even m only upon simplification one obtains

$$\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^m}=m\,\kappa(m+1)-\kappa(m)\ln 2-\frac{1}{2}\sum_{k=1}^{m-2}\kappa(k+1)\,\kappa(m-k).\tag{2.4}$$

By shifting index one gets sums with powers of $(2n-1)$ in the denominator, therefore we are able to obtain an identity for $\sum_{n=1}^{\infty}\frac{H_n}{(2n-1)^m}$.

2.1. Formulas with O_n in numerator and odd factors in denominator

The following formulas, valid only for even $m\in\mathbb{N}$, have been adapted from Jordan [10, 1]:

$$U(m):=\sum_{n=1}^{\infty}\frac{O_n}{(2n)^m}=\frac{2^{m+1}-1}{2^{m+2}}\zeta(m+1)-\frac{1}{2^{m+1}}\sum_{k=1}^{\frac{m}{2}-1}(2^{m-2k+1}-1)\,\zeta(2k)\,\zeta(m-2k+1).\tag{2.5}$$

Also, for even m only

$$\begin{aligned}V(m):=\sum_{n=1}^{\infty}\frac{O_n}{(2n-1)^m}=&\frac{1}{2}\kappa(m+1)+\kappa(m)\ln 2\\&-\frac{1}{2^{m+1}}\sum_{k=1}^{\frac{m}{2}-1}(2^{2k}-1)\,\zeta(2k)\,\zeta(m-2k+1).\end{aligned}\tag{2.6}$$

From (2.6), we know that

$$\sum_{n=1}^{\infty}\frac{O_n}{(2n-1)^{2m}}=\sum_{n=1}^{\infty}\frac{H_{2n}-\frac{1}{2}H_n}{(2n-1)^{2m}},$$

hence we have the new identity

$$\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n-1)^{2m}}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{(2n-1)^{2m}}+\frac{2^{2m+1}-1}{2^{2m+2}}\zeta(2m+1)+\frac{2^{2m}-1}{2^{2m}}\zeta(2m)\ln 2$$
$$-\frac{1}{2^{2m+1}}\sum_{k=1}^{m-1}(2^{2k}-1)\,\zeta(2k)\,\zeta(2m-2k+1).$$

We found a remarkable recurrence for odd powers with $k\in\mathbb{N}$:

$$\sum_{n=1}^{\infty}\frac{O_n}{(2n-1)^{2k+1}}+\sum_{n=1}^{\infty}\frac{O_n}{(2n)^{2k+1}} \tag{2.7}$$
$$=\frac{1}{2}\kappa(2k+2)+\kappa(2k+1)\ln 2-\sum_{j=1}^{k-1}\frac{2^{2j+1}-1}{2^{2k+2}}\zeta(2j+1)\zeta(2k-2j+1).$$

Ramanujan [12, Ch.IX, p.104, 11.ii][2, p.257, (11.3)] recorded a wrong result:

$$\sum_{n=1}^{\infty}\frac{O_n}{(2n)^3}=\frac{\pi}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{(4k+1)^3}-\frac{\pi}{3\sqrt{3}}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}.$$

Sitaramachandrarao [15] notes:

$$\sum_{n=1}^{\infty}\frac{O_n}{(2n)^3}=\frac{3}{16}\zeta(4)-\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n^{(2)}}{(n+1)^2}.$$

The last term involves the constant

$$A_4=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(n+1)^2}\sum_{k=1}^{n}\frac{1}{k^2}\approx 0.1626546673974.$$

So putting the value of $\sum_{n=1}^{\infty}\frac{O_n}{(2n)^3}$ in the first formula in the chain, we get:

$$\sum_{n=1}^{\infty}\frac{O_n}{(2n-1)^3}=\frac{9}{32}\zeta(4)+\frac{7}{8}\zeta(3)\ln 2+\frac{A_4}{4}.$$

We find this value of the constant in [13]:

$$A_4=\frac{65}{16}\zeta(4)-\frac{7}{2}\zeta(3)\ln 2+\zeta(2)(\ln 2)^2-\frac{1}{6}(\ln 2)^4-4\mathrm{Li}_4\left(\frac{1}{2}\right),$$

where $\mathrm{Li}_4\left(\frac{1}{2}\right)=\sum_{n=1}^{\infty}\frac{1}{2^n n^4}\approx 0.517479061673899$. We then have [15, p.3, corrected]:

$$\sum_{n=1}^{\infty}\frac{O_n}{(2n)^3}=-\frac{53}{64}\zeta(4)+\frac{7}{8}\zeta(3)\ln 2-\frac{1}{4}\zeta(2)(\ln 2)^2+\frac{1}{24}(\ln 2)^4+\mathrm{Li4}\left(\frac{1}{2}\right)$$

and

$$LS(3):=\sum_{n=1}^{\infty}\frac{O_n}{(2n-1)^3}=\frac{83}{64}\zeta(4)+\frac{1}{4}\zeta(2)(\ln 2)^2-\frac{1}{24}(\ln 2)^4-\mathrm{Li}_4\left(\frac{1}{2}\right), \tag{2.8}$$

which through shift of index yields

$$\sum_{n=1}^{\infty}\frac{O_n}{(2n+1)^3}=\frac{23}{64}\zeta(4)+\frac{1}{4}\zeta(2)(\ln 2)^2-\frac{1}{24}(\ln 2)^4-\mathrm{Li}_4\left(\frac{1}{2}\right).$$

It is possible to generalize further the result (2.7) in the following way.

LEMMA 1. *Let (m,p) be positive integers with $p\geqslant 2$ and $(m+p)$ an odd integer. Put $O_n^{(m)}:=\sum_{k=1}^{n}\frac{1}{(2k-1)^m}$. Then*

$$O_n^{(m)}=H_{2n}^{(m)}-\frac{1}{2^m}H_n^{(m)}=\frac{1}{2^m}H_{n-\frac{1}{2}}^{(m)}+\eta(m). \tag{2.9}$$

Hence

$$\begin{aligned}X(m,p):=\sum_{n=1}^{\infty}O_n^{(m)}\left(\frac{1}{(2n-1)^p}+\frac{1}{(2n)^p}\right)&=\kappa(p)\eta(m)+\frac{1}{2^m}W(m,p)\\&+\left(\frac{2^{m+p}-4}{2^{m+p+1}}\right)BW(m,p)-\frac{1}{2}S(m,p),\end{aligned} \tag{2.10}$$

where $W(m,p)$ is given by (1.6), $BW(m,p)$ is given by (1.3) and $S(m,p)$ is given by (1.5).

Proof. Consider, using (2.9)

$$\begin{aligned}X(m,p):=\sum_{n=1}^{\infty}O_n^{(m)}\left(\frac{1}{(2n-1)^p}+\frac{1}{(2n)^p}\right)&=\sum_{n=1}^{\infty}\left(\frac{\frac{1}{2^m}H_{n-\frac{1}{2}}^{(m)}+\eta(m)}{(2n-1)^p}+\frac{H_{2n}^{(m)}-\frac{1}{2^m}H_n^{(m)}}{(2n)^p}\right)\\&=\eta(m)\sum_{n=1}^{\infty}\frac{1}{(2n-1)^p}+\frac{1}{2^m}\sum_{n=1}^{\infty}\frac{H_{n-\frac{1}{2}}^{(m)}}{(2n-1)^p}+\sum_{n=1}^{\infty}\frac{H_{2n}^{(m)}}{(2n)^p}-\frac{1}{2^m}\sum_{n=1}^{\infty}\frac{H_n^{(m)}}{(2n)^p}\\&=\frac{1}{2^{m-1}}\kappa(p)\eta(m)+\frac{1}{2^{m+1}}\sum_{n=1}^{\infty}\left(\frac{H_{\frac{n}{2}}^{(m)}}{n^p}+\frac{(-1)^{n+1}H_{\frac{n}{2}}^{(m)}}{n^p}\right)\end{aligned}$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{H_n^{(m)}}{n^p}-\frac{(-1)^{n+1}H_n^{(m)}}{n^p}\right)-\frac{1}{2^{m+p}}\sum_{n=1}^{\infty}\frac{H_n^{(m)}}{n^p}.$$

Using (1.7) and simplifying,

$$X(m,p) = \kappa(p)\eta(m)+\frac{1}{2^{m+1}}\left(2W(m,p)-2^{1-p}BW(m,p)\right)+\frac{1}{2}\left(BW(m,p)-S(m,p)\right)$$
$$-\frac{1}{2^{m+p}}BW(m,p),$$

therefore collecting terms, (2.10) follows. □

EXAMPLE 1.

$$X(2,5)=\sum_{n=1}^{\infty}O_n^{(2)}\left(\frac{1}{(2n-1)^5}+\frac{1}{(2n)^5}\right)=\frac{11}{32}\zeta(4)\zeta(3)+\frac{29}{16}\zeta(5)\zeta(2)-\frac{635}{256}\zeta(7),$$

here we have used

$$\sum_{n\geqslant 1}\frac{H_{\frac{n}{2}}^{(2)}}{n^5}=\zeta(4)\zeta(3)-\frac{21}{16}\zeta(5)\zeta(2)+\frac{107}{64}\zeta(7)$$

and

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}H_{\frac{n}{2}}^{(2)}}{n^5}=\frac{7}{8}\zeta(4)\zeta(3)-\frac{13}{8}\zeta(5)\zeta(2)+\frac{147}{64}\zeta(7).$$

$$X(3,4)=\sum_{n=1}^{\infty}O_n^{(3)}\left(\frac{1}{(2n-1)^4}+\frac{1}{(2n)^4}\right)=\frac{127}{16}\zeta(7)+\frac{45}{64}\zeta(4)\zeta(3)-\frac{147}{32}\zeta(5)\zeta(2).$$

2.2. Sums with two odd factors in denominator

In this section we develop identities for Euler sums of the form

$$Y(p,q)=\sum_{n=1}^{\infty}\frac{O_n}{(2n-1)^p(2n+1)^q}, \tag{2.11}$$

which in turn we extract further Euler like sums of the type

$$\sum_{n=1}^{\infty}\frac{H_{2n}}{(4n^2-1)^p}. \tag{2.12}$$

LEMMA 2. *For positive integers p,q and $p\leqslant 4,\ q\leqslant 4$, we have*

$$Y(p,q)=\sum_{n=1}^{\infty}\frac{O_n}{(2n-1)^p(2n+1)^q}=\frac{3(-1)^{p+1}}{2^{p+q+1}}\binom{p+q-2}{p-1}\zeta(2)$$

$$+\sum_{k=2}^{q}\frac{(-1)^{p+1}}{2^{p+q-k}}\binom{p+q-k-1}{q-k}\kappa(k+1)$$
$$+\sum_{k=2}^{q}\frac{(-1)^{p}}{2^{p+q-k}}\binom{p+q-k-1}{q-k}V(k)$$
$$+\sum_{j=2}^{p}\frac{(-1)^{p-j}}{2^{p+q-j}}\binom{p+q-j-1}{p-j}V(j), \tag{2.13}$$

where $V(\cdot)$ is given by (2.6).

Proof. Consider the partial fraction decomposition

$$\frac{1}{(2n-1)^p(2n+1)^q}=\frac{(-1)^{p+1}}{2^{p+q-2}}\binom{p+q-2}{p-1}\frac{1}{(2n-1)(2n+1)}$$
$$+\sum_{j=2}^{p}\frac{(-1)^{p-j}}{2^{p+q-j}}\binom{p+q-j-1}{p-j}\frac{1}{(2n-1)^j}$$
$$+\sum_{k=2}^{q}\frac{(-1)^{p}}{2^{p+q-k}}\binom{p+q-k-1}{q-k}\frac{1}{(2n+1)^k},$$

now by summing over the integers n

$$Y(p,q)=\sum_{n=1}^{\infty}\frac{O_n}{(2n-1)^p(2n+1)^q} \tag{2.14}$$
$$=\frac{(-1)^{p+1}}{2^{p+q-2}}\binom{p+q-2}{p-1}\sum_{n=1}^{\infty}\frac{O_n}{(2n-1)(2n+1)}$$
$$+\sum_{j=2}^{p}\frac{(-1)^{p-j}}{2^{p+q-j}}\binom{p+q-j-1}{p-j}\sum_{n=1}^{\infty}\frac{O_n}{(2n-1)^j}$$
$$+\sum_{k=2}^{q}\frac{(-1)^{p}}{2^{p+q-k}}\binom{p+q-k-1}{q-k}\sum_{n=1}^{\infty}\frac{O_n}{(2n+1)^k}.$$

Consider the third term in (2.14) and make a change in the summation index, so that

$$\sum_{k=2}^{q}\frac{(-1)^{p}}{2^{p+q-k}}\binom{p+q-k-1}{q-k}\sum_{n=1}^{\infty}\frac{O_n}{(2n+1)^k}$$
$$=\sum_{k=2}^{q}\frac{(-1)^{p}}{2^{p+q-k}}\binom{p+q-k-1}{q-k}\sum_{n=1}^{\infty}\frac{O_n}{(2n-1)^k}$$
$$+\sum_{k=2}^{q}\frac{(-1)^{p+1}}{2^{p+q-k}}\binom{p+q-k-1}{q-k}\kappa(k+1). \tag{2.15}$$

Substituting (2.15) into (2.14), we have

$$Y(p,q)=\frac{(-1)^{p+1}}{2^{p+q-2}}\binom{p+q-2}{p-1}\left(\frac{3}{8}\zeta(2)\right)+\sum_{k=2}^{q}\frac{(-1)^{p+1}}{2^{p+q-k}}\binom{p+q-k-1}{q-k}\kappa(k+1)$$

$$+\sum_{k=2}^{q}\frac{(-1)^p}{2^{p+q-k}}\binom{p+q-k-1}{q-k}V(k)+\sum_{j=2}^{p}\frac{(-1)^{p-j}}{2^{p+q-j}}\binom{p+q-j-1}{p-j}V(j),$$

where $V(\cdot)$ is given by (2.6), therefore the identity (2.13) follows. □

REMARK 1. While the identity (2.13) is numerically correct for all integer values $p,q\geqslant 1$, we restrict its application to $(p,q)\leqslant 4$ because we do not have closed form identities of $V(2m+1)$, for $m\geqslant 2$.

EXAMPLE 2.

$$Y(2,4)=\sum_{n=1}^{\infty}\frac{O_n}{(2n-1)^2(2n+1)^4}=\frac{3}{16}\zeta(2)\ln 2-\frac{3}{32}\zeta(2)+\frac{15}{64}\zeta(4)\ln 2-\frac{15}{64}\zeta(4)$$
$$+\frac{1}{4}LS(3)-\frac{7}{128}\zeta(3)-\frac{31}{256}\zeta(5)-\frac{1}{64}\zeta(2)\zeta(3),$$
$$Y(4,2)=\sum_{n=1}^{\infty}\frac{O_n}{(2n-1)^4(2n+1)^2}=\frac{3}{16}\zeta(2)\ln 2-\frac{3}{32}\zeta(2)+\frac{15}{64}\zeta(4)\ln 2+\frac{7}{128}\zeta(3)$$
$$-\frac{1}{4}LS(3)+\frac{31}{256}\zeta(5)-\frac{3}{128}\zeta(2)\zeta(3).$$

An important corollary of Lemma 2.13 is the case of $q=p$.

COROLLARY 1. *From (2.13), for the case $q=p$ and $p\in\mathbb{N}\setminus\{1\}$ we have*

$$\begin{aligned}Y(p,p)=\sum_{n=1}^{\infty}\frac{O_n}{(4n^2-1)^p}&=\frac{3(-1)^{p+1}}{2^{2p+1}}\binom{2p-2}{p-1}\zeta(2)\\&+\sum_{j=2}^{p}\frac{(-1)^{p+1}}{2^{2p-j}}\binom{2p-j-1}{p-j}\kappa(j+1)\\&+\sum_{j=2}^{p}\frac{(-1)^{p}}{2^{2p-j}}\binom{2p-j-1}{p-j}\left(1+(-1)^j\right)V(j).\end{aligned}\tag{2.16}$$

REMARK 2. From (2.16) we have,

$$Y(p,p)=\sum_{n=1}^{\infty}\frac{O_n}{(4n^2-1)^p}=\sum_{n=1}^{\infty}\frac{H_{2n}-\frac{1}{2}H_n}{(4n^2-1)^p}$$

from which we obtain the new identity

$$\sum_{n=1}^{\infty}\frac{H_{2n}}{(4n^2-1)^p}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{(4n^2-1)^p}+\frac{3(-1)^{p+1}}{2^{2p+1}}\binom{2p-2}{p-1}\zeta(2)$$
$$+\sum_{j=2}^{p}\frac{(-1)^{p+1}}{2^{2p-j}}\binom{2p-j-1}{p-j}\kappa(j+1)$$

$$+\sum_{j=2}^{p}\frac{(-1)^p}{2^{2p-j}}\binom{2p-j-1}{p-j}\left(1+(-1)^j\right)V(j). \tag{2.17}$$

We note that identities (2.16) and (2.17) hold for all integer p values bigger than one, because we do not require identities of $V(2m+1)$ for $m\geqslant 2$.

EXAMPLE 3. Some miscellaneous examples are highlighted.

$$\sum_{n=1}^{\infty}\frac{H_n}{(4n^2-1)^3}=-\frac{7}{16}\zeta(3)-\frac{\pi^2}{64}(5-6\ln 2)+\ln 2, \tag{2.18}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{(4n^2-1)^4}=&\frac{31}{64}\zeta(5)-\frac{\pi^4}{768}(2\ln 2-1)-\ln 2-\frac{7}{256}\zeta(3)(\pi^2-8)\\&+\frac{\pi^2}{128}(11-10\ln 2),\end{aligned} \tag{2.19}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{O_n}{(4n^2-1)^4}=&\frac{\pi^4}{768}(\ln 2-1)-\frac{\pi^2}{512}\zeta(3)+\frac{5\pi^2}{256}(2\ln 2-1),\\\sum_{n=1}^{\infty}\frac{O_n}{(4n^2-1)^5}=&\frac{\pi^6}{30720}+\frac{5\pi^4}{12288}(3-4\ln 2)+\frac{5\pi^2}{2048}\zeta(3)-\frac{35\pi^2}{2048}(2\ln 2-1),\end{aligned} \tag{2.20}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{O_n}{(4n^2-1)^6}=&\left(\frac{189}{1024}\zeta(2)+\frac{315}{2048}\zeta(4)+\frac{63}{2048}\zeta(6)\right)\ln 2-\frac{189}{4096}\zeta(6)\\&-\left(\frac{189}{2048}+\frac{63}{4096}\zeta(3)+\frac{3}{4096}\zeta(5)\right)\zeta(2)\\&-\left(\frac{105}{1024}+\frac{15}{4096}\zeta(3)\right)\zeta(4),\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{O_n}{(2n-1)^3(2n)^3}=&\frac{17}{8}\zeta(4)-\frac{7}{8}\zeta(3)(3+\ln 2)+\frac{\pi^2}{24}(2\ln^2 2-9\ln 2+12)\\&-\frac{\ln^4 2}{12}-2Li_4\left(\frac{1}{2}\right),\end{aligned} \tag{2.21}$$

$$\sum_{n=1}^{\infty}\frac{H_{2n}}{(4n^2-1)^3}=\frac{15}{128}\zeta(4)-\frac{7}{32}\zeta(3)-\frac{3}{32}\zeta(2)+\frac{1}{2}\ln 2,$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_{2n}}{(4n^2-1)^6}=&\frac{381}{4096}\zeta(7)-\frac{63}{2048}\zeta(6)+\frac{527}{2048}\zeta(5)+\frac{135}{4096}\zeta(4)-\frac{1}{2}\ln 2\\&-\frac{7}{1024}\zeta(3)+\frac{195}{1024}\zeta(2)-\frac{3}{128}\zeta(2)\zeta(5)-\frac{15}{572}\zeta(4)\zeta(3)\\&-\frac{63}{512}\zeta(2)\zeta(3).\end{aligned}$$

2.3. Euler sums involving powers of harmonic numbers

Euler did not treat sums that involve powers of harmonic numbers. H. F. Sandham, an Irish mathematician[1], introduced the first quadratic double sum in 1948 as a problem in the *American Mathematical Monthly* [14].

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2 = \frac{17\pi^4}{360}. \tag{2.22}$$

Apparently, it went unnoticed. It was recorded by Castellanos in his 1988 survey article [5, I, p.86], rightly attributed to H. F. Sandham but with a wrong entry in the bibliography.

De Doelder [6] evaluated this associated sum in 1991 without any reference to Sandham's sum:

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n+1}\right)^2 = \frac{11\pi^4}{360}. \tag{2.23}$$

For this, he used the *psi function.*

The comparison of the terms in the two series in the following expression makes it clear that

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n(n+1)} = \sum_{n=1}^{\infty}\frac{H_n^2}{n} - \sum_{n=1}^{\infty}\frac{H_n^2}{n+1} = \sum_{n=1}^{\infty}\frac{H_n^2 - H_{n-1}^2}{n}.$$

Now $H_n^2 - H_{n-1}^2 = (H_n - H_{n-1})(H_n + H_{n-1}) = \frac{1}{n}(2H_n - \frac{1}{n}) = 2\frac{H_n}{n} - \frac{1}{n^2}$. Hence, we get

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n(n+1)} = 2\sum_{n=1}^{\infty}\frac{H_n}{n^2} - \sum_{n=1}^{\infty}\frac{1}{n^3} = 3\zeta(3). \tag{2.24}$$

We also have:

$$\sum_{n=1}^{\infty}\frac{H_{n+1}^2}{n(n+1)} = \zeta(2) + 3.$$

We can deduce De Doelder's sum from that of Sandham as follows. On comparing the terms of the two series, we notice that

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2 - \sum_{n=1}^{\infty}\left(\frac{H_n}{(n+1)}\right)^2 = \sum_{n=1}^{\infty}\frac{H_n^2 - H_{n-1}^2}{n^2}.$$

Therefore,

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2 - \sum_{n=1}^{\infty}\left(\frac{H_n}{(n+1)}\right)^2 = 2\sum_{n=1}^{\infty}\frac{H_n}{n^3} - \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{3}{2}\zeta(4) = \frac{\pi^4}{60}.$$

[1]Henry Francis Sandham (1917–1963) studied mathematics at Trinity College, Dublin (Ireland) and Queen's University Belfast, received a Ph.D. in 1958 on his thesis *Products of the Hypergeometric Functions*, and taught at Trinity College. In September 1952, he joined the School of Theoretical Physics as a lecturer under the Nobel laureate Erwin Schrödinger at Dublin Institute for Advanced Studies which he left in 1956 to work with English Electric Labs (Staffordshire) until his death in April 1963. In addition to half a dozen papers and problems published in various journals, he presented a note on the *Perimeter of an Ellipse* at the International Congress of Mathematicians, Amsterdam, September 2–9, 1954.

Sandham's sum was conjectured in April 1993 by Enrico Au-Yeung, a student of J. Borwein at the University of Waterloo, on the basis of his computations and was established (by means of generating functions and Parseval's identity for Fourier series) by Borweins [4], who referred to De Doelder's paper but not to Sandham. Other related identities for Euler sums are available in [18], [19] and [20].

2.4. Few miscellaneous sums

De Doelder's formulas [6, (15) and (22)] are worthy of note:

$$\sum_{k=1}^{\infty} \frac{\psi(\frac{1}{2} \pm k) - \psi(\frac{1}{2})}{k^2} = \frac{7}{2}\zeta(3), \quad \sum_{k=1}^{\infty} \frac{(\psi(\frac{1}{2} \pm k) - \psi(\frac{1}{2}))^2}{k^2} = \frac{\pi^4}{16}.$$

Now $O_n = \frac{1}{2}\left(\psi\left(n+\frac{1}{2}\right) - \psi\left(\frac{1}{2}\right)\right)$. Thus we get [4]:

$$\sum_{n=1}^{\infty} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3), \quad \sum_{n=1}^{\infty} \left(\frac{O_n}{n}\right)^2 = \frac{\pi^4}{32}.$$

The first occurs in Ramanujan's Manuscript2 [12, Ch. IX, p.104, Entry12, Ex.iii]. We have this associated sum:

$$\sum_{n=1}^{\infty} \frac{O_n}{(2n-1)^2} = \frac{7}{16}\zeta(3) + \frac{3}{4}\ln 2\zeta(2).$$

We could obtain:

$$\sum_{n=1}^{\infty} \frac{O_n}{n^4} = \frac{31}{4}\zeta(5) - \frac{7}{2}\zeta(3)\zeta(2),$$

as well as

$$\sum_{n=1}^{\infty} \frac{H_{2n-1}}{(2n-1)^4} = \frac{155}{64}\zeta(5) - \frac{3}{4}\zeta(3)\zeta(2), \quad \sum_{n=1}^{\infty} \frac{H_{2n}}{(2n)^4} = \frac{37}{64}\zeta(5) - \frac{1}{4}\zeta(3)\zeta(2)$$

and

$$\sum_{n=1}^{\infty} \frac{H_{2n-1}}{(2n)^4} = \frac{35}{64}\zeta(5) - \frac{1}{4}\zeta(3)\zeta(2), \quad \sum_{n=1}^{\infty} \frac{H_{2n}}{(2n+1)^4} = \frac{93}{64}\zeta(5) - \frac{3}{4}\zeta(3)\zeta(2).$$

The previous four results yield two nice formulas:

$$\sum_{n=1}^{\infty} \frac{(1+(-1)^n 2)H_n}{n^4} = -\frac{11}{16}\zeta(5); \tag{2.25}$$

$$\sum_{n=1}^{\infty} \frac{(1-(-1)^n 2)H_n}{(n+1)^4} = \frac{3}{16}\zeta(5). \tag{2.26}$$

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