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A BOUNDS TAUBERIAN THEOREM

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Abstract. We weaken the hypothesis and the conclusion of a Hardy–Littlewood Tauberian theorem, and apply the new theorem to deduce asymptotic behavior of the coefficients of an exponentiated lacunary series.

1. Introduction

Most Tauberian theorems state an asymptotic condition on a function as the hypothesis and an asymptotic condition on a related function as the conclusion. It is sometimes useful to weaken the hypothesis and the conclusion. We prove the following Tauberian theorem and give an application.

THEOREM 1. *Suppose $c_n \geqslant 0$ for all n, that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < 1$ and that for some number $\alpha \geqslant 0$ and some positive constants k_1, k_2 we have that*

$$\frac{k_1}{(1-x)^{\alpha}} < f(x) < \frac{k_2}{(1-x)^{\alpha}} \quad (0 \leqslant x < 1).$$

Then there are positive constants k_3, k_4 such that

$$k_3 n^{\alpha} < \sum_{k=0}^{n} c_k < k_4 n^{\alpha} \quad (n \geqslant 1).$$

The asymptotic version of this theorem is a 1914 Tauberian theorem of G. H. Hardy and J. E. Littlewood [2, Theorem 8]. We cite the following form of their theorem, given in Korevaar [3, Theorem I.7.4]:

THEOREM 2. (Hardy–Littlewood) *Suppose $c_n \geqslant 0$ for all n, that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < 1$ and that for some number $\alpha \geqslant 0$ and some number C we have as $x \to 1^-$ that*

$$f(x) \sim \frac{C}{(1-x)^{\alpha}}.$$

Then we have as $n \to \infty$ that

$$\sum_{k=0}^{n} c_k \sim \frac{C}{\Gamma(\alpha+1)} n^{\alpha}.$$

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2. Proof of Theorem 1

This argument is modeled on Titchmarsh [6, §7.52], that is the case $\alpha = 1$.

Proof of Theorem 1.

First we need bounds for $1-e^{-u}$. For $0<u<1$ we have by the Mean value theorem that $1-e^{-u} = e^0 - e^{-u} = ue^{-v}$ for some v with $u<v<1$. In the range $u \geqslant 1$ we have trivially $1-e^{-u} < 1 \leqslant u$. Therefore we get inequalities for each range:

$$1-e^{-u} > \frac{u}{e} \quad \text{and so} \quad \frac{1}{1-e^{-u}} < \frac{e}{u} \quad (0<u<1); \tag{1}$$

$$1-e^{-u} < u \quad \text{and so} \quad \frac{1}{1-e^{-u}} > \frac{1}{u} \quad (1<u<\infty). \tag{2}$$

Write $s_n = \sum_{k=0}^{n} c_k$. To find an upper bound for s_n we observe

$$f(x) \geqslant \sum_{k=0}^{n} c_k x^k \geqslant x^n \sum_{k=0}^{n} c_k = x^n s_n. \tag{3}$$

Using this with $x = e^{-1/n}$, the hypothesis and (1) we have

$$e^{-1} s_n \leqslant f(e^{-1/n}) < \frac{k_2}{(1-e^{-1/n})^{\alpha}} < k_2 (en)^{\alpha}.$$

Setting $k_4 = k_2 e^{\alpha+1}$ we have $s_n < k_4 n^{\alpha}$.

Let n be fixed. To find a lower bound for s_n we start with the just-proved upper bound, and observe that s_n is non-decreasing, to get

$$\begin{aligned} f(x) &= (1-x)\sum_{m=0}^{\infty} s_m x^m \leqslant (1-x)s_n \sum_{m=0}^{n} x^m + k_4(1-x)\sum_{m=n+1}^{\infty} m^{\alpha} x^m \\ &\leqslant s_n + k_4(1-x)\sum_{m=n+1}^{\infty} m^{\alpha} x^m. \end{aligned} \tag{4}$$

We will bound the last sum above with an integral. Let λ be a large number, to be picked later, that is independent of n and such that $\lambda > \alpha$. We take $x = e^{-\lambda/n}$. The function $u^{\alpha} e^{-\lambda u/n}$ is decreasing for $u \geqslant n$, because the derivative of its logarithm is $\alpha/u - \lambda/n \leqslant \alpha/n - \lambda/n = (\alpha-\lambda)/n < 0$. Therefore $m^{\alpha} e^{-\lambda m/n} < \int_{m-1}^{m} u^{\alpha} e^{-\lambda u/n}\,du$ and so

$$\begin{aligned} \sum_{m=n+1}^{\infty} m^{\alpha} x^m &= \sum_{m=n+1}^{\infty} m^{\alpha} e^{-\lambda m/n} < \int_n^{\infty} u^{\alpha} e^{-\lambda u/n}\,du \quad \left(\text{substitute } u = \frac{n}{\lambda} v\right) \\ &= \int_{\lambda}^{\infty} \frac{n^{\alpha}}{\lambda^{\alpha}} v^{\alpha} e^{-v} \frac{n}{\lambda}\,dv = \left(\frac{n}{\lambda}\right)^{\alpha+1} \int_{\lambda}^{\infty} v^{\alpha} e^{-v}\,dv = \left(\frac{n}{\lambda}\right)^{\alpha+1} I(\lambda) \quad \text{(say)}. \end{aligned}$$

Using this with (4) and (2) we get

$$\begin{aligned} f(x) &< s_n + k_4(1-x)\left(\frac{n}{\lambda}\right)^{\alpha+1} I(\lambda) = s_n + k_4(1-e^{-\lambda/n})\left(\frac{n}{\lambda}\right)^{\alpha+1} I(\lambda) \\ &< s_n + k_4\left(\frac{\lambda}{n}\right)\left(\frac{n}{\lambda}\right)^{\alpha+1} I(\lambda) = s_n + k_4\left(\frac{n}{\lambda}\right)^{\alpha} I(\lambda). \end{aligned} \tag{5}$$

We also deduce from the hypothesis and (2) that

$$f(x) > \frac{k_1}{(1-x)^\alpha} = \frac{k_1}{(1-e^{-\lambda/n})^\alpha} > k_1\left(\frac{n}{\lambda}\right)^\alpha. \tag{6}$$

Combining (5) and (6) we get $s_n > (n/\lambda)^\alpha(k_1 - k_4 I(\lambda))$. Taking λ so large that $I(\lambda) < k_1/k_4$, we conclude that there is a $k_3 > 0$ such that $s_n > k_3 n^\alpha$, as claimed. □

3. An application

George Stoica [4, 5] proposed the following as Problem 11849 in the Problems column of the *American Mathematical Monthly*: Define numbers $a_0, a_1, \cdots$ by

$$\exp\left(\sum_{k=0}^\infty x^{2^k}\right) = \sum_{n=0}^\infty a_n x^n.$$

Prove that

$$\liminf_{n\to\infty} \frac{\ln a_n}{\ln n} \leqslant \frac{1}{\ln 2} - 1 \leqslant \limsup_{n\to\infty} \frac{\ln a_n}{\ln n}. \tag{7}$$

We will use Theorem 1 to show that there are positive constants k_3 and k_4 such that

$$k_3 n^{1/\ln 2} < \sum_{k=0}^n a_k < k_4 n^{1/\ln 2}. \tag{8}$$

Then we will show that this estimate implies (7).

Let's write $F(x) = \sum_{k=0}^\infty x^{2^k}$. We will apply Theorem 1 to the function $f(x) = \exp F(x)$, so that $c_n = a_n$. First we will show that $a_n \geqslant 0$ for all n. We observe that F satisfies the functional equation

$$F(x) = x + F(x^2)$$

and so f satisfies the functional equation

$$f(x) = e^x f(x^2).$$

We write this out and rearrange it as

$$f(x) = \sum_{n=0}^\infty a_n x^n = \left(\sum_{r=0}^\infty \frac{x^r}{r!}\right)\left(\sum_{s=0}^\infty a_s x^{2s}\right) = \sum_{n=0}^\infty x^n \sum_{s\leqslant n/2} \frac{a_s}{(n-2s)!}.$$

Equating the coefficients of x^n on both sides we get a recurrence for a_n:

$$a_n = \sum_{s\leqslant n/2} \frac{a_s}{(n-2s)!} \quad (n \geqslant 0).$$

Note that $a_0 = \exp(0) = 1$ and therefore by induction on the recurrence we have $a_n > 0$ for all n.

Next we prove the following simple asymptotic estimate for $F(x)$.

THEOREM 3.

$$F(x) = \frac{1}{\ln 2}\ln\frac{1}{1-x} + O(1) \quad (x \to 1^-).$$

Proof. We have

$$\frac{1}{1-x}F(x) = \sum_{n=1}^{\infty}(\lfloor \log_2 n \rfloor + 1)x^n$$

and

$$\frac{1}{1-x}\ln\frac{1}{1-x} = \sum_{n=1}^{\infty} H_n x^n,$$

where $H_n = \sum_{k=1}^{n}\frac{1}{k}$ is the harmonic number. Therefore

$$\frac{1}{1-x}\left(F(x) - \frac{1}{\ln 2}\ln\frac{1}{1-x}\right) = \sum_{n=1}^{\infty}\left(\lfloor \log_2 n \rfloor + 1 - \frac{1}{\ln 2}H_n\right)x^n.$$

The coefficients on the right-hand side are $O(1)$, because $H_n = \ln n + O(1)$, and $\lfloor \log_2 n \rfloor = \log_2 n + O(1) = (1/\ln 2)\ln n + O(1)$. Therefore the right-hand side is $O(1/(1-x))$. Multiplying both sides by $(1-x)$ we have the result. □

We exponentiate this result. The additive error term $O(1)$ becomes a multiplicative factor of $\exp(O(1))$, and this is bounded above and below by positive constants. Therefore we have that there are positive constants k_1 and k_2 such that

$$\frac{k_1}{(1-x)^{1/\ln 2}} < \exp F(x) < \frac{k_2}{(1-x)^{1/\ln 2}} \quad (0 \leqslant x < 1).$$

Therefore Theorem 1 applies with $\alpha = 1/\ln 2$, and (8) is proved.

Now we will show that (8) implies (7), by showing more generally that:

THEOREM 4. *Suppose that $c_n \geqslant 0$ for all n and that $\alpha \geqslant 1$. Suppose there are positive constants k_3, k_4 such that*

$$k_3 n^{\alpha} < \sum_{k=0}^{n} c_k < k_4 n^{\alpha} \quad (n \geqslant 1). \tag{9}$$

Then

$$\liminf_{n\to\infty}\frac{\ln c_n}{\ln n} \leqslant \alpha - 1 \leqslant \limsup_{n\to\infty}\frac{\ln c_n}{\ln n}. \tag{10}$$

Proof. Write $s_n = \sum_{k=0}^{n} c_k$. Write $L = \liminf_{n\to\infty}\ln c_n/\ln n$. If $L \leqslant 0$ there is nothing to prove, so assume $L > 0$. Then given ε with $0 < \varepsilon < L$ there is an N such that we have $\ln c_n/\ln n > L - \varepsilon$ (and so $c_n > n^{L-\varepsilon}$) for all $n \geqslant N$. Then for $M > N$ from the hypothesis we have

$$k_4 M^{\alpha} > s_M \geqslant \sum_{n=N}^{M} c_n > \sum_{n=N}^{M} n^{L-\varepsilon} > \int_{N-1}^{M} x^{L-\varepsilon}\,dx = \frac{M^{L+1-\varepsilon} - (N-1)^{L+1-\varepsilon}}{L+1-\varepsilon}.$$

Divide both sides by $M^{L+1-\varepsilon}$ and let $M\to\infty$ to get $\lim_{M\to\infty} k_3 M^{\alpha-L-1+\varepsilon} \geqslant 1/(L+1-\varepsilon)$ and so $\alpha-L-1+\varepsilon\geqslant 0$ for all $\varepsilon>0$ and so $\alpha-1\geqslant L$.

Write $U=\limsup_{n\to\infty}\ln c_n/\ln n$. If $U=+\infty$ there is nothing to prove, so assume $U<\infty$. Then given $\varepsilon>0$ there is an N such that we have $\ln c_n/\ln n<U+\varepsilon$ (and so $c_n<n^{U+\varepsilon}$) for all $n\geqslant N$. Then for $M>N$ from the hypothesis we have

$$k_3 M^{\alpha} < s_M = \sum_{n=0}^{N-1} c_n + \sum_{n=N}^{M} c_n < \sum_{n=0}^{N-1} c_n + \sum_{n=N}^{M} n^{U+\varepsilon} < \sum_{n=0}^{N-1} c_n + M^{U+1+\varepsilon}.$$

Divide both sides by $M^{U+1+\varepsilon}$ and let $M\to\infty$ to get $\lim_{M\to\infty} k_3 M^{\alpha-U-1-\varepsilon}\leqslant 1$ and so $\alpha-U-1-\varepsilon\leqslant 0$ for all $\varepsilon>0$ and so $\alpha-1\leqslant U$. □

REMARK 1. The converse of Theorem 4 is not true in general, that is, (10) does not imply (9). In this sense our result (8) is stronger than the original result (7). To see that the converse need not hold, consider the example

$$c_n = \begin{cases} 1, & \text{if } n \text{ is a power of 2,} \\ n, & \text{otherwise.} \end{cases}$$

Then $\liminf_{n\to\infty}\frac{\ln c_n}{\ln n}=0$ and $\limsup_{n\to\infty}\frac{\ln c_n}{\ln n}=1$, so (10) is true for any α in the range $1\leqslant\alpha\leqslant 2$. But (writing $k=2^r$ when k is a power of 2)

$$\sum_{k=0}^{n} c_k = \sum_{k=0}^{n} k + \sum_{r\leqslant \ln n/\ln 2} (1-2^r) = \tfrac{1}{2}n^2 + O(n).$$

Therefore (9) is true for $\alpha=2$, but not for any other α.

REMARK 2. The series $\sum_{k=0}^{\infty} x^{2^k}$ is a lacunary series and has been studied by several authors. In particular in 1907 G. H. Hardy [1] made an extensive study of this series (in §12) and the related series $\sum_{k=0}^{\infty}(-1)^k x^{2^k}$. Hardy proved more precise asymptotics for $F(x)$, specifically that

$$F(x) = -\frac{\ln\ln(1/x)}{\ln 2} + \lambda(x),$$

where $\lambda(x)$ is a bounded, oscillating function. We can rewrite this as

$$F(x) = -\frac{\ln(1-x)}{\ln 2} + \frac{1}{\ln 2}\ln\frac{1-x}{\ln(1/x)} + \lambda(x) = -\frac{\ln(1-x)}{\ln 2} + \mu(x) \quad \text{(say)}.$$

As $x\to 1^-$ we have that $(1-x)/\ln(1/x)\to 1$, so $\mu(x)$ is a bounded function that does not go to a limit. Writing $f(x)=\exp F(x)=\sum_{n=0}^{\infty} a_n x^n$ we have

$$f(x) = \frac{1}{(1-x)^{1/\ln 2}} e^{\mu(x)}.$$

The factor $e^{\mu(x)}$ is bounded above and below by positive constants, but does not go to a limit, so the hypothesis of Theorem 2 is not met for our function.

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