Journal of Classical Analysis
Volume 15, Number 2 (2019), 61–69 doi:10.7153/jca-2019-15-06

BOUNDS ON COEFFICIENTS AND THIRD HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS RELATED WITH CERTAIN CONIC DOMAIN

JUGAL K. PRAJAPAT* AND RAJBALA

Abstract. In this paper, we obtain upper bounds on initial coefficients and third Hankel determinant

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

of the coefficients of analytic function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, belonging to the class $\mathscr{S}^*(q)$ in the open unit disk $\mathbb{D}$, which satisfies the subordination condition that

$$zf'(z)/f(z) \prec q(z) \quad (z \in \mathbb{D}),$$

where $q(z) = \sqrt{1+z^2}+z$. Several results are presented exhibiting improvement in earlier work.

1. Introduction

Let $\mathscr{H}(\mathbb{D})$ denote the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathscr{A}$ be the subclass of $\mathscr{H}(\mathbb{D})$ normalized by the condition $f(0) = f'(0) - 1 = 0$. This means that $f \in \mathscr{A}$ has the Taylor's series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{D}). \tag{1}$$

Further, let $\mathscr{P}$ be the class of *Carathéodory functions* $p \in \mathscr{H}(\mathbb{D})$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \qquad (z \in \mathbb{D}), \tag{2}$$

having the positive real part in $\mathbb{D}$. If f and g are two analytic functions in $\mathbb{D}$, then we say that function f is subordinate to g in $\mathbb{D}$ and write $f(z) \prec g(z)\,(z \in \mathbb{D})$, if there exists a Schwarz function $w(z)$ analytic in $\mathbb{D}$ with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z))\,(z \in \mathbb{D})$. In particular, if the function g is univalent in $\mathbb{D}$, then the above subordination is equivalent to:

$$f(z) \prec g(z) \Longleftrightarrow [f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D})].$$

Mathematics subject classification (2010): 30C45.
Keywords and phrases: Analytic functions, starlike functions, Fekete-Szegö functional, Hankel determinant, subordination.

* Corresponding author.

Let $\mathscr{S}^*(\phi)$ denote the class of functions $f \in \mathscr{A}$ satisfying $\dfrac{zf'}{f} \prec \phi(z)$, where ϕ is analytic with $\phi(0) = 1$. Note that, for $\phi(z) = (1+z)/(1-z)$, $\mathscr{S}^*(\phi)$ is the well known class of starlike functions. Also, Ma and Minda [14] proved some general results where ϕ is assumed to be univalent, and $\phi(\mathbb{D})$ is symmetric with respect to real axis and starlike with respect to $\phi(0) = 1$. Several other subclasses of $\mathscr{A}$ have been defined in the literature by choosing appropriately the arbitrary function ϕ in the class $\mathscr{S}^*(\phi)$ (see, [10, 18, 17]). Also we observe that, if ϕ in $\mathscr{S}^*(\phi)$ is not univalent, then obtaining geometric properties for such classes is much more difficult. Such class of functions was studied by Sokol and Stankiewicz [22] (see also, [2]) by taking $\phi(z) = \sqrt{1+z}$, also Raina and Sokol [21] have studied a class $\mathscr{S}^*(q)$ which is defined by

$$\mathscr{S}^*(q) = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{1+z^2} + z =: q(z) \right\}. \tag{3}$$

The branch of the square root in q is chosen to be $q(0) = 1$. Further, it was proved that $\mathscr{S}^*(q) \subset \mathscr{S}^*$ (see, [21, Lemma 2.1]).

The Hankel determinant $H_{q,n}(f)$ of Taylor's coefficients for functions f of the form (1) is defined by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} \cdots & a_{n+q} \\ \vdots & \vdots \quad \cdots & \ddots \\ a_{n+q-1} & a_{n+q} \cdots & a_{n+2(q-1)} \end{vmatrix} \qquad (a_1 = 1; n, q \in \mathbb{N} = \{1, 2, \cdots\}). \tag{4}$$

The Hankel determinants are useful in the study of singularities and power series with integral coefficients [5]. Pommerenke [16] proved that the Hankel determinant of univalent functions satisfy $|H_{q,n}(f)| < kn^{-(1/2+\beta)q+3/2}$, where $\beta > 1/4000$ and k depends only on q. Later, Hayman [8] proved that $H_{2,n}(f) < An^{1/2}$ (A is an absolute constant) for a Areally mean univalent functions. The study of $H_{3,1}(f)$ for various subclasses of $\mathscr{A}$ are of interest for many researchers [3, 15, 19, 20, 23]. Note that $H_{2,1}(f) = a_3 - a_2^2$ be the classical *Fekete-Szegö* functional and $H_{2,2}(f) = a_2a_4 - a_3^2$ be the second Hankel determinant. The problems of calculating $max_{f\in F}|H_{2,1}(f)|$ and the $max_{f\in F}|H_{2,2}(f)|$ for various compact subfamilies of $\mathscr{A}$ was studied by many authors (see, [1, 4, 9, 11, 13]).

In the present paper, we obtain upper bounds on initial coefficients and third Hankel determinant $|H_{3,1}(f)|$ for functions $f \in \mathscr{S}^*(q)$.

We shall use the following lemmas.

LEMMA 1. (Duren [6]) *If $p \in \mathscr{P}$ is of the form (2), then*

$$|p_n| \leqslant 2 \qquad (n \in \mathbb{N}). \tag{5}$$

The inequality (5) is sharp and the equality holds for the function $p(z) = \dfrac{1+z}{1-z}$.

LEMMA 2. (Efraimidis [7]) *If $p \in \mathscr{P}$ is of the form (2) and $\mu \in \mathbb{C}$, then*

$$|p_n - \mu p_k p_{n-k}| \leqslant 2\max\{1, |2\mu - 1|\} \quad (1 \leqslant k \leqslant n-1).$$

If $|2\mu - 1| \geqslant 1$, then the inequality is sharp for the function $p(z) = \dfrac{1+z}{1-z}$ or its rotations. If $|2\mu - 1| < 1$, then the inequality is sharp for the function $p(z) = \dfrac{1+z^n}{1-z^n}$ or its rotations.

LEMMA 3. (Libra and Złotkiewicz [12]) *If $p \in \mathscr{P}$ is of the form (2) with $p_1 \geqslant 0$, then*

$$2p_2 = p_1^2 + x(4 - p_1^2) \tag{6}$$

and

$$4p_3 = p_1^3 + 2p_1 x(4 - p_1^2) - p_1 x^2 (4 - p_1^2) + 2(4 - p_1^2)(1 - |x|^2)z \tag{7}$$

for some x, z with $|x| \leqslant 1$ and $|z| \leqslant 1$.

LEMMA 4. (Raina and Sokoł [21]) *If the function defined by (1) belongs to the class $\mathscr{S}^*(q)$, then*

$$|a_3 - \lambda a_2^2| \leqslant \max\{1/2, |\lambda - 3/4|\} \qquad (\lambda \in \mathbb{C}).$$

Furthermore, this result is sharp.

2. Main Results

THEOREM 1. *Let the function $f \in \mathscr{S}^*(q)$ be given by (1). Then*

$$|a_2| \leqslant 1, \quad |a_3| \leqslant \frac{3}{4}, \quad |a_4| \leqslant \frac{1}{3} \quad \text{and} \quad |a_5| \leqslant \frac{1}{2}. \tag{8}$$

First two inequalities in (8) are sharp.

Proof. Let $f \in \mathscr{S}^*(q)$. By definition there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$, such that

$$zf'(z) - w(z)f(z) = f(z)\sqrt{w^2(z) + 1}. \tag{9}$$

Assume that

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad (z \in \mathbb{D}). \tag{10}$$

Making use of (1) and (10) in (9), we get

$$\begin{aligned} f(z)\sqrt{w^2(z)+1} = {} & z + a_2 z^2 + \left(\frac{1}{2}c_1^2 + a_3\right) z^3 + \left(c_1 c_2 + \frac{1}{2}c_1^2 a_2 + a_4\right) z^4 \\ & + \left(\frac{1}{2}c_2^2 + c_1 c_3 - \frac{1}{8}c_1^4 + c_1 c_2 a_2 + \frac{1}{2}c_1^2 a_3 + a_5\right) z^5 + \cdots, \end{aligned} \tag{11}$$

and

$$
\begin{aligned}
zf'(z)-w(z)f(z) &= z+(2a_2-c_1)z^2+(3a_3-c_1a_2-c_2)z^3 \\
&\quad +(4a_4-c_1a_3-c_2a_2-c_3)z^4 \\
&\quad +(5a_5-c_1a_4-c_2a_3-c_3a_2-c_4)z^5+\cdots .
\end{aligned} \tag{12}
$$

Now equating the coefficients in (11) and (12), we have

$$
\begin{aligned}
&a_2=c_1, \quad a_3=\frac{1}{4}\left(2c_2+3c_1^2\right), \quad a_4=\frac{1}{12}\left(5c_1^3+10c_1c_2+4c_3\right), \\
&a_5=\frac{1}{24}\left(6c_2^2+14c_1c_3+4c_1^4+17c_1^2c_2+6c_4\right).
\end{aligned} \tag{13}
$$

If w is an alalytic function such that $w(0)=0$ and $|w(z)|<1$ $(z\in D)$, then we can easily write

$$
\frac{1+w(z)}{1-w(z)}=1+p_1z+p_2z^2+\cdots:=p(z), \tag{14}
$$

where $p(z)\in\mathscr{P}$. Using the series expansion of $w(z)$ given by (10) in (14) and equating the coefficients, we obtain

$$
\begin{aligned}
&c_1=\frac{1}{2}p_1, \quad c_2=\frac{1}{4}\left(2p_2-p_1^2\right), \quad c_3=\frac{1}{8}\left(4p_3-4p_1p_2+p_1^3\right), \\
&c_4=\frac{1}{16}\left(8p_4-8p_1p_3+6p_1^2p_2-4p_2^2-p_1^4\right), \\
&c_5=\frac{1}{32}\left(16p_5-16p_1p_4-16p_2p_3+12p_1p_2^2+12p_1^2p_3-8p_1^3p_2+p_1^5\right).
\end{aligned} \tag{15}
$$

Further using (15) in (13), we get

$$
\begin{aligned}
&a_2=\frac{1}{2}p_1, \quad a_3=\frac{1}{16}(4p_2+p_1^2), \quad a_4=\frac{1}{96}(-p_1^3+4p_1p_2+16p_3), \\
&a_5=\frac{1}{384}(p_1^4-10p_1^2p_2+8p_1p_3+48p_4).
\end{aligned} \tag{16}
$$

Now making use of Lemma 1, we obtain easily that $|a_2|\leqslant 1$. To obtain bound on a_3, we use Lemma 3, for some x and z such that $|x|\leqslant 1$ and $|z|\leqslant 1$, hence we obtain

$$
|a_3|=\frac{1}{16}\left|3p_1^2+2x(4-p_1^2)\right|.
$$

Taking into account of invariance of $|a_2|$ under rotation, we can assume that $p_1=2a_2$ is real. Assume that $p_1=p\in[0,2]$. Applying the triangle inequality in the above equation with $\mu=|x|$, we obtain

$$
|a_3|\leqslant\frac{1}{16}\left|3p^2+2\mu(4-p^2)\right|:=F(p,\mu).
$$

We easily observe that F is an increasing function of μ on $[0,1]$, hence it follows that

$$
\max_{0\leqslant\mu\leqslant 1}F(p,\mu)=F(p,1)=\frac{1}{16}(p^2+8).
$$

Again, we observe that $F(p,1)$ is also an increasing function of $p\,(0\leqslant p\leqslant 2)$. Hence the maximum value of $F(p,\mu)$ is

$$\max_{\Omega} F(p,\mu)=F(2,1)=\frac{3}{4}.$$

To show the sharpness of first two inequalities, consider

$$\frac{zf_1'(z)}{f_1(z)}=q(z)=\sqrt{1+z^2}+z. \tag{17}$$

Then $f_1\in\mathscr{S}^*(q)$. Consequently

$$\begin{aligned} f_1(z)&=z\exp\int_0^z\frac{\sqrt{1+t^2}+t-1}{t}dt\\ &=z+z^2+\frac{3}{4}z^3+\frac{5}{12}z^4+\frac{2}{9}z^5+\cdots\quad(z\in\mathbb{D}).\end{aligned}$$

This gives that $|a_2|=1$ and $|a_3|=\dfrac{3}{4}$.

To obtain bound on a_4, applying Lemma 3, for some x and z such that $|x|\leqslant 1$ and $|z|\leqslant 1$, we get

$$|a_4|=\frac{1}{96}\left|5p_1^3+\left(10p_1x-4p_1x^2+8(1-|x|^2)z\right)(4-p_1^2)\right|. \tag{18}$$

Applying the triangle inequality in (18) with $\mu=|x|$, we obtain

$$|a_4|\leqslant\frac{1}{96}\left[5p^3+\left(10p\mu+4p\mu^2+8(1-\mu^2)\right)(4-p^2)\right]:=G(p,\mu).$$

Let $\Omega=\{(p,\mu):0\leqslant p\leqslant 2$ and $0\leqslant\mu\leqslant 1\}$. Differentiating G with respect to p and μ, respectively, we have

$$\frac{\partial G}{\partial\mu}:=\frac{1}{48}(4-p^2)(5p+4\mu(p-2)),\text{ and} \tag{19}$$

$$\frac{\partial G}{\partial p}:=\frac{1}{192}\{6p^2(5-10\mu-4\mu^2)+32\mu^2(p+1)+16(5\mu+2p)\}. \tag{20}$$

Solving (19) and (20), we obtain that $(0,0)$ is a point of extremum. Since Ω is closed and bounded and G is continuous on Ω, the maximum shall be attained on the boundary of Ω. It is easy to see that on the boundary line $L_1=\{(0,\mu):\ 0\leqslant\mu\leqslant 1\}$, we have $G(0,\mu)=\dfrac{1}{3}(1-\mu^2)$, that gives critical point $(0,0)$. On the boundary line $L_2=\{(2,\mu):\ 0\leqslant\mu\leqslant 1\}$, we have $G(2,\mu)=\dfrac{5}{12}$, which is a constant. On the boundary line $L_3=\{(p,0):\ 0\leqslant p\leqslant 2\}$, we have $G(p,0)=\dfrac{1}{96}(5p^3-8p^2+32)$, which gives the critical points $(0,0)$ and $(16/15,0)$. On the line $L_4=\{(p,1):\ 0\leqslant p\leqslant 2\}$, we have $G(p,1) = \frac{1}{24}(8p - p^3)$, which gives $(0,0)$ and $(\sqrt{56/9}, 1)$ as critical points. Comparing all these results, we observe that the maximum of $G(p,\mu)$ exists at $(0,0)$, that is

$$\max_{\Omega} G(p,\mu) = G(0,0) = \frac{1}{3}.$$

In the same way for $|a_5|$, from (16) we have

$$|a_5| \leqslant |A| + |B|,$$

where $A = \frac{1}{384}p_1^4 - \frac{5}{192}p_1^2 p_2$ and $B = \frac{1}{8}\left(p_4 + \frac{1}{6}p_1 p_3\right)$. Using Lemma 2, we have $|B| \leqslant \frac{1}{3}$. Now using Lemma 3, for some x and z such that $|x| \leqslant 1$ and $|z| \leqslant 1$, we have

$$|A| = \frac{1}{384}\left|-4p_1^4 - 5p_1^2 x(4 - p_1^2)\right|.$$

We may assume without restriction that $p_1 = p \in [0,2]$ and $\mu = |x| \in [0,1]$. Consequently

$$|A| \leqslant \frac{1}{384}\{4p^4 + 5p^2\mu(4 - p^2)\} := H(p,\mu). \tag{21}$$

We observe that H is an increasing function of μ on $[0,1]$, hence it follows that

$$\max_{0 \leqslant \mu \leqslant 1} H(p,\mu) = H(p,1) = \frac{1}{384}(-p^4 + 20p^2).$$

Clearly $H(p,1)$ is also an increasing function of p $(0 \leqslant p \leqslant 2)$. Hence the maximum value of $H(p,\mu)$ is

$$\max_{\Omega} H(p,\mu) = H(2,1) = \frac{1}{6}.$$

This implies that $|A| \leqslant \frac{1}{6}$, and hence we get $|a_5| \leqslant \frac{1}{2}$. This completes the proof.

THEOREM 2. *Let the function $f \in \mathscr{S}^*(q)$ be given by* (1). *Then*

$$|a_2 a_3 - a_4| \leqslant \frac{2}{9}\sqrt{\frac{8}{3}} \quad \text{and} \quad |a_2 a_4 - a_3^2| \leqslant \frac{1}{4}. \tag{22}$$

Proof. On making use of (16), we have

$$|a_2 a_3 - a_4| = \frac{1}{24}|p_1^3 + 2p_1 p_2 - 4p_3|. \tag{23}$$

Using Lemma 3, for some x and z such that $|x| \leqslant 1$ and $|z| \leqslant 1$, we have

$$|a_2 a_3 - a_4| = \frac{1}{24}\left|p_1^3 + (4 - p_1^2)\{-p_1 x + p_1 x^2 - 2(1 - |x|^2)z\}\right|.$$

Since we have $|p_1| \leqslant 2$, so we may assume without restriction that $p_1 = p \in [0,2]$. Applying the triangle inequality with $\mu = |x|$, the above equation yields

$$\begin{aligned}|a_2a_3 - a_4| &\leqslant \frac{1}{24}[p^3 + (4-p^2)\{p\mu + p\mu^2 + 2(1-\mu^2)\}] \\ &= \frac{1}{24}[p^3 + (4-p^2)\{(p-2)\mu^2 + p\mu + 2\}] \\ &:= J(p,\mu).\end{aligned}$$

Now differentiating J with respect to p and μ, respectively, we have

$$\frac{\partial J}{\partial \mu} := \frac{1}{24}(4-p^2)(p + 2\mu(p-2)), \tag{24}$$

and

$$\frac{\partial J}{\partial p} := \frac{1}{24}[3p^2(1-\mu) - \mu^2(3p^2 - 4p - 4) - 4(p-\mu)]. \tag{25}$$

Solving (24) and (25) we obtain that $(0,0)$ is the point of extremum. Now we shall examine extremum of J on the boundary. Taking the boundary line $L_1 = \{(0,\mu) : 0 \leqslant \mu \leqslant 1\}$, we have $J(0,\mu) = \frac{1}{3}(1-\mu^2)$, that gives critical point $(0,0)$. On the boundary line $L_2 = \{(2,\mu) : 0 \leqslant \mu \leqslant 1\}$, we have $J(2,\mu) = \frac{1}{3}$, which is a constant. On the boundary line $L_3 = \{(p,0) : 0 \leqslant p \leqslant 2\}$, we have $J(p,0) = \frac{1}{24}(p^3 + 2(4-p^2))$, which gives the critical points $(0,0)$ and $(4/3,0)$. On the line $L_4 = \{(p,1) : 0 \leqslant p \leqslant 2\}$, we have $J(p,1) = \frac{1}{24}(8p - p^3)$, which gives $(\sqrt{8/3},1)$ as critical point. Comparing these results, we observe that

$$J(0,1) < J(4/3,0) < J(0,0) < J(\sqrt{8/3},1).$$

Thus we get,

$$\max_{\Omega} J(p,\mu) = J(\sqrt{8/3},1) = \frac{2}{9}\sqrt{\frac{8}{3}}.$$

Further, in view of (16), we get

$$|a_2a_4 - a_3^2| = \frac{1}{768}\left|-7p_1^4 - 8p_1^2p_2 + 64p_1p_3 - 48p_2^2\right|. \tag{26}$$

In (26), using Lemma 3, for some x and z such that $|x| \leqslant 1$ and $|z| \leqslant 1$, we have

$$|a_2a_4 - a_3^2| = \left|-\frac{7}{768}p_1^4 - \frac{(4-p_1^2)}{192}\{-p_1^2x + p_1^2x^2 + 12x^2 - 8p_1(1-|x|^2)z\}\right|. \tag{27}$$

We assume without restriction that $p_1 = p \in [0,2]$. Applying the triangle inequality with $\mu = |x|$ in (27) yields

$$|a_2a_4 - a_3^2| \leqslant \frac{7}{768}p^4 + \frac{(4-p^2)}{192}\{p^2\mu + p^2\mu^2 + 12\mu^2 + 8p(1-\mu^2)\}.$$

Further, the remaining part of the proof is similar to that of first inequality in (22), hence we omit the detail and this completes the proof.

REMARK 1. For $f \in \mathscr{S}^*(q)$ of the form (1), Raina *et. al.* [21] obtained upper bounds for the initial coefficient a_4 and the functional $a_2a_4 - a_3^2$. Here we observe that Theorems 1 and 2 provide an improvement on these bounds.

THEOREM 3. *Let the function $f \in \mathscr{S}^*(q)$ be given by* (1). *Then*

$$|H_{3,1}(f)| \leqslant \frac{7}{16} + \frac{2}{27}\sqrt{\frac{8}{3}}.$$

Proof. By the definition of $H_{3,1}(f)$, we have

$$|H_{3,1}(f)| \leqslant |a_3|\,|a_2a_4 - a_3^2| + |a_4|\,|a_4 - a_2a_3| + |a_5|\,|a_3 - a_2^2|.$$

Now, using Lemma 1, Theorems 1 and 2, we have

$$|H_{3,1}(f)| \leqslant \frac{3}{4}.\frac{1}{4} + \frac{1}{3}.\frac{2}{9}\sqrt{\frac{8}{3}} + \frac{1}{2}.\frac{1}{2} = \frac{7}{16} + \frac{2}{27}\sqrt{\frac{8}{3}}.$$

Hence, it completes the proof.

REFERENCES

[1] Y. ABU MUHAN, L. LI AND S. PONNUSAMY, *Extremal problems on the class of convex functions of order -1/2*, Arch. Math. (Basel) **103**(6)(2014), 461–471.

[2] R. M. ALI, N. E. CHO, V. RAVICHANDRAN AND S. S. KUMAR, *First order differential subordinations for functions associated with the lemniscate of Bernoulli*, Taiwanese J. Math. **16**(2012), 1017–1026.

[3] D. BANSAL, S. MAHARANA AND J. K. PRAJAPAT, *Third order Hankel determinant for certain univalent functions*, J. Korean Math. Soc. **52**(6)(2015), 1139–1148.

[4] B. BHOWMIK, S. PONNUSAMY AND K. J. WIRTHS, *On the Fekete–Szegö problem for concave univalent functions*, J. Math. Anal. Appl. **373**(2)(2011), 432–438.

[5] D. G. CANTOR, *Power series with integral coefficients*, Bull. Amer. Math. Soc. **69**(1963), 362–366.

[6] P. L. DUREN, *Univalent Functions*, Springer Verlag, New York Inc. 1983.

[7] I. EFRAIMIDIS, *A generalization of Livingston's coefficient inequalities for functions with positive real part*, J. Math. Anal. Appl. **435**(2016), 369–379.

[8] W. K. HAYMAN, *On the second Hankel determinant of mean univalent functions*, Proc. Lond. Math. Soc. **18**(3)(1968), 77–94.

[9] F. R. KEOGH AND E. P. MERKES, *A coefficients inequalities for certain classes of analytic functions*, Proc. Amer. Math. Soc. **20**(1969), 8–12.

[10] Y. C. KIM, S. PONNUSAMY AND T. SUGAWA, *Mapping properties of nonlinear integral operators and pre-Schwarzian derivatives*, J. Math. Anal. Appl. **29**(2) (2004), 433–447.

[11] S. K. LEE, V. RAVICHANDRAN AND S. SUBRAMANIAN, *Bounds for the second Hankel determinant of certain univalent functions*, J. Inequal. Appl. **2013**(2013), Article 281, 17pp.

[12] R. J. LIBRA AND E. J. ZŁOTKIEWICZ, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc. **85**(2)(1982), 225–230.

[13] L. LI AND S. PONNUSAMY, *On the generalized Zalcman functional $\lambda a_n^2 - 2_{2n-1}$ in the close-to-convex family*, Proc. Amer. Math. Soc. **145**(2017), 833–846.

[14] W. MA AND D. MINDA, *A unified treatment of some special classes of univalent functions*, Proceedings of the Conference on complex Analysis (Tianjin, 1992), 157–169, Conf. Proc. Lecture Notes Anal I, International Press. Cambridge, MA 1994.

[15] A. K. MISHRA, J. K. PRAJAPAT AND S. MAHARANA, *Bounds on Hankel determinant for starlike and convex functions with respect to symmetric points*, Cogent Math. Stat. **3**(2016), 1160557.

[16] C. POMMERENKE, *On the coefficients and Hankel determinant of univalent functions*, J. Lond. Math. Soc. **44**(1966), 111–122.

[17] S. PONNUSAMY, *Differential subordination and starlike functions*, Complex Variables: Theory and Appl. **19**(1992), 185–194.

[18] S. PONNUSAMY, *Differential subordination concerning starlike functions*, Proc. Ind. Acad. Sci. (Math. Sci.) **104**(1994), 397–411.

[19] J. K. PRAJAPAT, D. BANSAL, A. SINGH AND A. K. MISHRA, *Bounds on third Hankel determinant for close-to-convex functions*, Acta Univ. Sapientiae Math. **7**(2)(2015), 210–219.

[20] J. K. PRAJAPAT, S. MAHARANA AND DEEPAK BANSAL, *Bounds on third Hankel Determinant for certain classes of analytic functions*, Stud. Univ. Babes-Bolyai Math. **62**(2)(2017), 183–195.

[21] R. K. RAINA AND J. SOKOŁ, *On coefficient estimates for a certain class of starlike functions*, Hacet. J. Math. Stat. **44**(6)(2015), 1427–1433.

[22] J. SOKOŁ AND J. STANKIEWICZ, *Radius of convexity of some subclasses of strongly starlike functions*, Zeszyty Nauk. Politech. Rzeszowskiej Mat. **19**(1996), 101–105.

[23] P. ZAPRAWA, *Third Hankel determinants for subclasses of Univalent functions*, Mediterr. J. Math. **14**(19)(2017), 1–10.

(Received June 23, 2018)

Jugal K. Prajapat
Department of Mathematics
Central University of Rajasthan
Bandarsindri, Kishangarh-305817, Dist.- Ajmer, Rajasthan, India
e-mail: jkprajapat@gmail.com

Rajbala
Department of Mathematics
Central University of Rajasthan
Bandarsindri, Kishangarh-305817, Dist.- Ajmer, Rajasthan, India
e-mail: rajbalachoudary9@gmail.com