Journal of
Classical
Analysis
Volume 15, Number 2 (2019), 71–77 doi:10.7153/jca-2019-15-07

SCHUR'S THEOREM FOR MODIFIED DISCRETE FOURIER TRANSFORM

N. O. KOTELINA AND A. B. PEVNYI*

Abstract. We find the eigenvalues of modified Fourier matrix S with entries $S_{kj} = \frac{1}{\sqrt{n}}\omega^{k(1-j)}$, $0 \leqslant k, j \leqslant n-1$, where $\omega = \exp\frac{2\pi i}{n}$. For this matrix $S^4 = \omega I$. The matrix has an interesting property: for $n = 4m$ eigenvalues have equal multiplicities. We prove a theorem giving the multiplicities of eigenvalues for all n. The theorem is similar to Schur's theorem (1921) for standard Fourier matrix. Our proofs are self-contained. In the proof we calculate modified Gauss sums by means of the classical analysis.

1. Introduction

In the paper [2] Schur considered the matrix $F = \left(F_{kj}\right)$ with $F_{kj} = \frac{1}{\sqrt{n}}\omega^{kj}$, $0 \leqslant k, j \leqslant n-1$, where $\omega = \exp\frac{2\pi i}{n}$.

For this matrix $F^4 = I$, therefore the eigenvalues of F are roots of fourth degree of unit. Denote by $\#(\lambda)$ the multiplicity of eigenvalue λ. Schur's theorem says that

$$\#(1) = \left[\frac{n+4}{4}\right], \#(-1) = \left[\frac{n+2}{4}\right], \#(i) = \left[\frac{n+1}{4}\right], \#(-i) = \left[\frac{n-1}{4}\right].$$

For $n = 4m$ we have $\#(1) = m+1, \#(-1) = m, \#(i) = m, \#(-i) = m-1$. We see that these multiplicities are not equal.

The problem of the standard Fourier transform from the spectral point of view is that for $n = 4m$ dimensions of eigenspaces are different, not equal. It leads to the lack of analogy with integral Fourier transform for which all eigenspaces are similiar. Because of that we modify matrix F and new modified matrix will have eigenvalues with equal multiplicities for $n = 4m$.

2. New version of DFT

The modified discrete Fourier transforms were suggested by S. M. Sitnik in series of works (see for example [3]). We study one new transform and prove that multiplicities of its eigenvalues are equal for $n = 4m$.

Mathematics subject classification (2010): 42A99.
Keywords and phrases: Schur's theorem, discrete Fourier transform, modified discrete Fourier transform, eigenvalues.
* Corresponding author.

Consider a matrix S with elements

$$S(k,j)=\tfrac{1}{\sqrt{n}}\omega^{k(1-j)}, \qquad 0\leqslant k,j\leqslant n-1, \tag{1}$$

where $\omega=\exp(2\pi i/n)$. For $n=4$ matrix S has the form:

$$S=\frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ \omega & 1 & \omega^{-1} & \omega^{-2}\\ \omega^2 & 1 & \omega^{-2} & \omega^{-4}\\ \omega^3 & 1 & \omega^{-3} & \omega^{-6}\end{bmatrix}.$$

In the zero row we see 1, in the first row – all roots of unity of the fourth degree. We take it from unit circle clockwise.

The matrices F and S are unitary. To study spectral properties of F we introduce matrix $P=S^2$. Let $\sqrt{\omega}$ denote $\exp(\pi i/n)$.

THEOREM 1. *Matrix $P=S^2$ has properties:*

1. $P^2=\omega I$ *for all* n.
2. *for even* n *trace* $\operatorname{tr}(P)$ *is equal to zero.*
3. *for odd* n, $n=2\nu+1$, *we have* $\operatorname{tr}(P)=-\sqrt{\omega}$ *and eigenvalues* $\pm\sqrt{\omega}$ *have multiplicities* $\#(\sqrt{\omega})=\nu$, $\#(-\sqrt{\omega})=\nu+1$.

Proof. Calculate the entries of P:

$$P(k,l)=\frac{1}{n}\sum_{j=0}^{n-1}\omega^{k(1-j)}\omega^{j(1-l)}=\frac{\omega^k}{n}\sum_{j=0}^{n-1}\omega^{j(1-k-l)}.$$

Last sum is the sum of geometric progression with $q=\omega^{1-k-l}$. If $q\neq 1$ then sum=0 but if $q=1$ then sum=n. Introduce periodic delta-function $\delta_n(j)=1$ for $j\equiv 0$ (mod n) and $\delta_n(j)=0$ otherwise. Then we have

$$P(k,l)=\omega^k\delta_n(1-k-l)=\omega^k\delta_n(k+l-1), \quad 0\leqslant k,l\leqslant n-1.$$

For $n=6$ matrix P has the form

$$P=\begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0\\ \omega & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \omega^2\\ 0 & 0 & 0 & 0 & \omega^3 & 0\\ 0 & 0 & 0 & \omega^4 & 0 & 0\\ 0 & 0 & \omega^5 & 0 & 0 & 0\end{bmatrix}.$$

For even n the entries of main diagonal are equal to zero and $\operatorname{tr}(P)=0$. For odd n

$$\operatorname{tr}(P)=\omega^{\frac{n+1}{2}}=\exp\left(\frac{2\pi i}{n}\frac{n+1}{2}\right)=\exp(\pi i)\exp\frac{\pi i}{n}=-\sqrt{\omega}.$$

Now show that matrix $Q = P^2$ is equal to ωI. We have

$$Q(k,j) = \sum_{l=0}^{n-1} \omega^k \delta_n(k+l-1)\omega^l \delta_n(l+j-1).$$

For $k \neq j$ all summands are zero. For $k = j$ only one summand $\neq 0$, for example

$$Q(0,0) = \sum_{l=0}^{n-1} \omega^l \delta_n(j-1) = \omega,$$

$$Q(2,2) = \sum_{l=0}^{n-1} \omega^2 \delta_n(l+1)\omega^l = \omega^{n+1} = \omega.$$

As a result $Q = \omega I$, i. e. $P^2 = \omega I$.

Eigenvalues of P are $\sqrt{\omega}$ and $-\sqrt{\omega}$ with multiplicities k_1 and k_2. Sum of all eigenvalues is equal to trace of P. For even n $k_1\sqrt{\omega} - k_2\sqrt{\omega} = \mathrm{tr}(P) = 0$, hence $k_1 = k_2 = n/2$.

For $n = 2\nu+1$ $k_1\sqrt{\omega} - k_2\sqrt{\omega} = -\sqrt{\omega}$, hence $k_1 = \nu$, $k_2 = \nu+1$. Theorem is proved.

3. Eigenvalues of matrix S

By Theorem 1 equality $S^4 = \omega I$ holds. Therefore eigenvalues of S are ε, $i\varepsilon$, $-\varepsilon$, $-i\varepsilon$, where $\varepsilon = \sqrt[4]{\omega} = \exp\frac{\pi i}{2n}$. Our aim is to determine the multiplicities a, b, c, d of these eigenvalues, $a+b+c+d = n$. Here, the trace $\mathrm{tr}(S)$ plays an important role.

THEOREM 2. *Trace of S and multiplicities of eigenvalues are given in the Table 1.*

n	$\mathrm{tr}(S)$	$\#(\varepsilon)$	$\#(i\varepsilon)$	$\#(-\varepsilon)$	$\#(-i\varepsilon)$
$4m$	0	m	m	m	m
$4m+1$	$-i\varepsilon$	m	m	m	$m+1$
$4m+2$	$(1-i)\varepsilon$	$m+1$	m	m	$m+1$
$4m+3$	ε	$m+1$	$m+1$	m	$m+1$

Table 1: Trace and multiplicities of eigenvalues of matrix S

Proof. The multiplicities a, b, c, d of eigenvalues satisfy the equation

$$a\varepsilon + b\cdot i\varepsilon + c(-i\varepsilon) + d(-i\varepsilon) = \mathrm{tr}(S).$$

In addition matrix $P = S^2$ has eigenvalues $\sqrt{\omega}$ and $-\sqrt{\omega}$ with multiplicities k_1 and k_2 determined in Theorem 1. We have $\varepsilon^2 = (-\varepsilon)^2 = \sqrt{\omega}$, $(i\varepsilon)^2 = (-i\varepsilon)^2 = -\sqrt{\omega}$, hence $k_1 \geqslant a+c$, $k_2 \geqslant b+d$. Adding these inequalities we obtain

$$n = k_1 + k_2 \geqslant a+b+c+d = n.$$

Whence $a+c=k_1$, $b+d=k_2$. By formula (1)

$$\mathrm{tr}(S)\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\omega^{k(1-k)}=\frac{1}{\sqrt{n}}\overline{R(n)},$$

where

$$R(n)=\sum_{k=0}^{n-1}\omega^{k(k-1)}. \tag{2}$$

The sum (2) is similar to the classical Gauss sum in which k^2 stands instead $k(k-1)$. In next lemma we calculate the sum $R(n)$ with the help of Dirichlet method.

LEMMA 1. *For all $n\geqslant 1$ the equalities hold*

$$R(n)=\begin{cases}0, & n=4m,\\ \bar{\varepsilon}i\sqrt{n}, & n=4m+1,\\ \bar{\varepsilon}(1+i)\sqrt{n}, & n=4m+2,\\ \bar{\varepsilon}\sqrt{n}, & n=4m+3,\end{cases}$$

where $\varepsilon=\exp\frac{\pi i}{2n}$, $\bar{\varepsilon}=\exp\left(-\frac{\pi i}{2n}\right)$.

Proof. We have $k(k-1)=(k-\frac{1}{2})^2-\frac{1}{4}$. Hence, $R(n)=\bar{\varepsilon}Q(n)$, where

$$Q(n)=\sum_{k=0}^{n-1}\exp\Big(\frac{2\pi i}{n}\big(k-\tfrac{1}{2}\big)^2\Big).$$

Following Dirichlet, let's introduce function

$$g(x)=\exp\Big(\frac{2\pi i}{n}\big(x-\tfrac{1}{2}\big)^2\Big),\qquad x\in\Big[\frac{1}{2},n+\frac{1}{2}\Big].$$

Notice that $g(\frac{1}{2})=g(\frac{1}{2}+n)$ and continue $g(x)$ to $\mathbb{R}$ with period n and decompose into Fourier series

$$g(x)=\sum_{k\in\mathbb{Z}}c_k\exp\Big(\frac{2\pi ikx}{n}\Big),\qquad c_k=\frac{1}{n}\int_{\frac{1}{2}}^{\frac{1}{2}+n}g(t)\exp\Big(\frac{-2\pi ikt}{n}\Big)\,dt.$$

In the definition of c_k under the integral sign there is an n-periodic function, so we can take integral over any interval with the length equal to n. Function $g(x)$ is continuous and piecewise-differentiable, therefore Fourier series converges to $g(x)$ at any point. Hence

$$Q(n)=\sum_{m=0}^{n-1}g(m)=\sum_{k\in\mathbb{Z}}c_k\sum_{m=0}^{n-1}\exp\Big(\frac{2\pi ikm}{n}\Big).$$

Last sum is non-zero only at k divisible by n, therefore

$$Q(n) = n\sum_{k\in\mathbb{Z}} c_{kn} = \sum_{k\in\mathbb{Z}} \int\limits_{\frac{1}{2}}^{\frac{1}{2}+n} g(t)\exp(-2\pi ikt)\,dt =$$

$$= \sum_{k\in\mathbb{Z}} \int\limits_{\frac{1}{2}}^{\frac{1}{2}+n} \exp\left(-2\pi ikt + \tfrac{2\pi i}{n}(t-\tfrac{1}{2})^2\right)dt.$$

Further, substitute $\tau = t - \frac{1}{2}$, $\tau \in [0,n]$:

$$Q(n) = \sum_{k\in\mathbb{Z}} \int\limits_0^n \exp\left(\frac{2\pi i}{n}\tau^2 - 2\pi ik\tau - \pi ik\right)d\tau.$$

We take out the multiplier $\exp(-\pi ik) = (-1)^k$ and make a substitution $\tau = nt$:

$$Q(n) = n\sum_{k\in\mathbb{Z}} (-1)^k \int\limits_0^1 \exp\left(2\pi int^2 - 2\pi inkt + 2\pi in\frac{k^2}{4} - \frac{\pi ink^2}{2}\right)dt =$$

$$= n\sum_{k\in\mathbb{Z}} (-1)^k \exp\left(-\frac{\pi ink^2}{2}\right) \int\limits_0^1 \exp\left(2\pi in\left(t - \frac{k}{2}\right)^2\right)dt.$$

And now substitute $\tau = t - \frac{k}{2}$:

$$Q(n) = n\sum_{k\in\mathbb{Z}} (-1)^k \exp\left(-\frac{\pi ink^2}{2}\right) \int\limits_{-\frac{k}{2}}^{-\frac{k}{2}+1} \exp(2\pi in\tau^2)\,d\tau.$$

Divide the sum into 2 parts, one part consisting of summands with even numbers $2k$ and another with odd:

$$Q(n) = n\sum_{k\in\mathbb{Z}} (-1)^{2k} \exp(-2\pi ink^2) \int\limits_{-k}^{-k+1} \exp(2\pi in\tau^2)\,d\tau +$$

$$+ n\sum_{k\in\mathbb{Z}} (-1)^{2k+1} \exp\left(-\frac{\pi in}{2}(4k^2+4k+1)\right) \int\limits_{-k-\frac{1}{2}}^{-k+\frac{1}{2}} \exp(2\pi in\tau^2)\,d\tau.$$

The multiplier before the first integral is equal to 1, and before the second $-i^{-n}$. Hence,

$$Q(n) = n\big(1 - i^{-n}\big) \int_{-\infty}^{\infty} \exp\big(2\pi i n \tau^2\big)\, d\tau.$$

Now apply the formula for Fresnel integral:

$$\int_{-\infty}^{\infty} \exp(iat^2)\, dt = \sqrt{\frac{\pi}{2a}}(1+i), \quad a > 0.$$

With $a = 2\pi n$ we finally get

$$Q(n) = \sqrt{n}\frac{(1-i^{-n})(1+i)}{2},$$

whence the statement of the Lemma follows.

We now proceed to prove the theorem. We have

$$\operatorname{tr}(S) = \overline{R(n)} = \varepsilon\frac{1-i}{2}\big[1 - i^n\big].$$

See values of $\operatorname{tr}(S)$ in the Table 1.

For $n = 4m$ we have $a + c = k_1 = 2m$, $b + d = k_2 = 2m$,

$$\varepsilon[(a-c) + i(b-d)] = \operatorname{tr}(S) = 0.$$

As a result $a = b = c = d = m$.

For $n = 4m+1$ we obtain $a + c = 2m$, $b + d = 2m+1$,

$$\varepsilon[(a-c) + i(b-d)] = -i\varepsilon$$

and $a = c = b = m$, $d = m+1$.

For $n = 4m+2$ we have $a + c = 2m+1$, $b + d = 2m+1$,

$$\varepsilon[(a-c) + i(b-d)] = (1-i)\varepsilon.$$

Hence $a = m+1$, $c = m$, $b = m$, $d = m+1$.

For $n = 4m+3$ we obtain $a + c = 2m+1$, $b + d = 2m+2$,

$$\varepsilon[(a-c) + i(b-d)] = \varepsilon$$

and $a = m+1$, $c = m$, $b = d = m+1$. Theorem is proved.

REFERENCES

[1] B. C. BERNDT, R. J. EVANS AND K. S. WILLIAMS, *Gauss and Jacobi sums*, 598 pp. Wiley, New York, 1998.

[2] I. SCHUR, *Über die Gaussschen summen*, Nach. Gessel. Göttingen, Math-Phys Klasse, pp. 147–153 (1921).

[3] S. M. SITNIK, *Modified discrete Fourier transform*, [in Russian]. Vestnik of Voronezh institute of MVD of Russia. **36**, 7 (2006), 196–201.

(Received February 25, 2019)

N. O. Kotelina
Syktyvkar State University
Syktyvkar 167001, Russia
e-mail: nkotelina@gmail.com

A. B. Pevnyi
Syktyvkar State University
Syktyvkar 167001, Russia
e-mail: pevnyi@syktsu.ru