Journal of Classical Analysis
Volume 15, Number 2 (2019), 87–97 doi:10.7153/jca-2019-15-09

ON APPROXIMATION PROPERTIES OF GENERALIZED q–BERNSTEIN–KANTOROVICH OPERATORS

LAKSHMI NARAYAN MISHRA* AND DHAWAL J. BHATT

Abstract. In this paper, we develop a generalization of q-Bernstein-Kantorovich type operators. We first study some fundamental properties of these operators and then investigate approximation properties of a sequence of these operators using Korovkin theorem. Finally, we estimate rate of approximation by modulus of continuity.

1. Introduction

For a real valued bounded function $f(x)$, which is defined on the closed interval $[0,1]$, the expression

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} \tag{1}$$

is called the Bernstein polynomial of order n of the function $f(x)$, which is discussed in [1]. If a function $f(x)$ is continuous on $[0,1]$ then $B_n(f;x)$ converges to $f(x)$ uniformly on $[0,1]$. To approximate integrable functions, Kantorovich introduced the following operators, called Bernstein-Kantorovich operators, defined as follows :
For $f \in C([0,1])$, $K_n : C([0,1]) \to C([0,1])$,

$$K_n(f;x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\, dt \tag{2}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}. \qquad (x \in [0,1])$

Some more generalizations of Bernstein polynomials (1) were discussed in [3, 7, 8, 10, 12].

In this paper, we use some q-analysis methods which are currently used in approximation theory. The important terms of q-analysis which are used in this paper are given below.

Mathematics subject classification (2010): 41A10, 41A25, 41A36.
Keywords and phrases: Korovkin theorem, Bernstein operator, q-integers, modulus of continuity.
* Corresponding author.

DEFINITION 1. *Given value of $q > 0$, we define the $q-$integer $[n]_q$ by*

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q} & ; \text{ if } q \neq 1 \\ n & ; \text{ if } q = 1 \end{cases},$$

for $n \in \mathbb{N}$.

In similar way, one can define $q-$real for any real number λ. In this case we denote it by $[\lambda]_q$.

DEFINITION 2. *For $q > 0$, we define the $q-$factorial $[n]_q!$ by*

$$[n]_q! = \begin{cases} [n]_q[n-1]_q \cdots [1]_q & ; \text{ if } n = 1,2,... \\ 1 & ; \text{ if } n = 0 \end{cases},$$

for $n \in \mathbb{N}$.

DEFINITION 3. *For $q > 0$, we define the $q-$binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \; 0 \leqslant k \leqslant n,$$

for $n \in \mathbb{N}$.

The $q-$binomial coefficient satisfies the following recurrence equations.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \tag{3}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q. \tag{4}$$

DEFINITION 4. *The $q-$analogue of $(1+x)_q^n$ is the polynomial*

$$(1+x)_q^n = \begin{cases} (1+x)(1+qx)\cdots(1+q^{n-1}x) & ; \text{ if } n = 1,2,... \\ 1 & ; \text{ if } n = 0 \end{cases}.$$

DEFINITION 5. *The $q-$derivative, $D_q f$ of a function f, is given by*

$$(D_q f)(x) = D_q\{f(x)\} = \begin{cases} \dfrac{f(x)-f(qx)}{(1-q)x} & ; \text{ if } x \neq 0 \\ f'(0) & ; \text{ if } x = 0 \end{cases}.$$

DEFINITION 6. *The definite q-integral of the function f is defined by*

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n \; ; \; a \in \mathbb{R}. \tag{5}$$

The series on the right-hand side in (5) is guaranteed to be convergent if the function f has the property $|f(x)| < Cx^{\alpha}$ in a right neighborhood of $x = 0$ for some $C > 0$, $\alpha > -1$.

The $q-$integral of the function f in a generic interval $[a,b]$ is defined in the following manner :

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx \, .$$

The following theorem is the fundamental theorem of quantum calculus.

THEOREM 1. If F is any anti $q-$derivative of the function f, namely, $D_qF = f$, continuous at $x = 0$, then

$$\int_0^a f(x)d_qx = F(a) - F(0) \, .$$

2. Construction of Operators

First, Lupas [13] defined a q-analogue of Bernstein operators and studied some approximation properties of them. Then, another generalizations of q-Bernstein operators are introduced and studied in [4, 5, 19, 22, 20, 21]. Dalmanoglu [4] gave the q-Bernstein-Kantorovich operators as follows :
For $f \in C([0,1])$, $K_{n,q} : C([0,1]) \to C([0,1])$,

$$K_{n,q}(f;x) = [n+1]_q \sum_{k=0}^{n} q^{-k}\, p_{n,k,q}(x) \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t)\, d_qt \tag{6}$$

where $p_{n,k,q}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1-q^s x). \qquad (x \in [0,1])$

For a real function of real variable $f \in C\left(\left[0, \dfrac{n+a}{n+b}\right]\right)$ Izgi [8] introduced the following operators

$$F_{n,a,b}(f;x) = \frac{(n+1)(n+b)}{(n+a)} \sum_{k=0}^{n} p_{n,k,a,b}(x) \int_{\frac{k(n+a)}{(n+1)(n+b)}}^{\frac{(k+1)(n+a)}{(n+1)(n+b)}} f(t)\, dt \tag{7}$$

where

$$p_{n,k,a,b}(x)=\left(\frac{n+b}{n+a}\right)^n\binom{n}{k}\,x^k\left(\frac{n+a}{n+b}-x\right)^{n-k}\left(0\leqslant x\leqslant\frac{n+a}{n+b},0\leqslant a\leqslant b\right).$$

To approximate the function $f(x)$ which satisfies the condition $|f(x)|<Kx^{\alpha}$ for some $K>0,\ \alpha>-1,$ in a right neighborhood of $x=0$, we introduce version of (7) in q-analysis and discuss some of its properties.

Let $a,b\in\mathbb{N}\cup\{0\},\ n\in\mathbb{N}$. For $f\in C\left(\left[0,\frac{[n+a]_q}{[n+b]_q}\right]\right)$, we define the following linear operator.

$$F^*_{n,a,b}:C\left(\left[0,\frac{[n+a]_q}{[n+b]_q}\right]\right)\to C\left(\left[0,\frac{[n+a]_q}{[n+b]_q}\right]\right)$$

$$F^*_{n,a,b}(f;x)=\frac{[n+1]_q[n+b]_q}{[n+a]_q}\sum_{k=0}^{n}q^{-k}\,p_{n,k,a,b}(x)\int_{\frac{[k]_q[n+a]_q}{[n+1]_q[n+b]_q}}^{\frac{[k+1]_q[n+a]_q}{[n+1]_q[n+b]_q}}f(t)\,d_qt \tag{8}$$

where

$$p_{n,k,a,b}(x)=\left(\frac{[n+b]_q}{[n+a]_q}\right)^n\begin{bmatrix}n\\k\end{bmatrix}_q x^k\prod_{s=0}^{n-k-1}\left(\frac{[n+a]_q}{[n+b]_q}-q^sx\right)\left(0\leqslant x\leqslant\frac{[n+a]_q}{[n+b]_q}\right).$$

3. Auxiliary Results

We first obtain the moments of the operators given in (8). The following lemma gives the central moment estimation of operators given in (8).

LEMMA 1. For $x\in\left[0,\frac{[n+a]_q}{[n+b]_q}\right]$ $(a,b\in\mathbb{N}\cup\{0\},a\leqslant b,n\in\mathbb{N})$ and the operators given in (8), the following equalities hold true :

1. $F^*_{n,a,b}(1;x)=1$.

2. $F^*_{n,a,b}(t;x)=\frac{[n]_q}{[n+1]_q}\cdot x+\frac{[n+a]_q}{[n+1]_q[n+b]_q}\cdot\frac{1}{1+q}$.

3. $F^*_{n,a,b}(t^2;x)=\frac{q[n]_q[n-1]_q}{[n+1]_q^2}\cdot x^2+\frac{2+3q+q^2}{1+q+q^2}\cdot\frac{[n]_q[n+a]_q^2}{[n+1]_q^3[n+b]_q^2}\cdot x$
$+\frac{1}{1+q+q^2}\left(\frac{[n+a]_q}{[n+1]_q[n+b]_q}\right)^2$.

Proof.

1. From (8), we have,

$$F^*_{n,a,b}(1;x) = \frac{[n+1]_q[n+b]_q}{[n+a]_q}\sum_{k=0}^{n} q^{-k}\, p_{n,k,a,b}(x) \int_{\frac{[k]_q[n+a]_q}{[n+1]_q[n+b]_q}}^{\frac{[k+1]_q[n+a]_q}{[n+1]_q[n+b]_q}} d_qt$$

$$F^*_{n,a,b}(1;x) = \frac{[n+1]_q[n+b]_q}{[n+a]_q}\sum_{k=0}^{n} q^{-k}\, p_{n,k,a,b}(x)\, q^k \left(\frac{[n+a]_q}{[n+1]_q[n+b]_q}\right)$$

$$= \sum_{k=0}^{n} p_{n,k,a,b}(x)$$

$$F^*_{n,a,b}(1;x) = 1\ .$$

2. Using (8), we have,

$$F^*_{n,a,b}(t;x) = \frac{[n+1]_q[n+b]_q}{[n+a]_q}\sum_{k=0}^{n} q^{-k}\, p_{n,k,a,b}(x) \int_{\frac{[k]_q[n+a]_q}{[n+1]_q[n+b]_q}}^{\frac{[k+1]_q[n+a]_q}{[n+1]_q[n+b]_q}} t\, d_qt$$

$$= \frac{[n+1]_q[n+b]_q}{[n+a]_q}\sum_{k=0}^{n} q^{-k}\, p_{n,k,a,b}(x)\times$$

$$\left[\frac{[n+a]_q^2}{[n+1]_q^2[n+b]_q^2}\cdot\frac{q^k}{1+q}\cdot\left([k]_q(1+q)+1\right)\right]$$

$$= \frac{[n+a]_q}{[n+1]_q[n+b]_q}\cdot\frac{1}{1+q}\sum_{k=0}^{n} p_{n,k,a,b}(x)\,\left([k]_q(1+q)+1\right)$$

$$= \frac{[n+a]_q}{[n+1]_q[n+b]_q}\sum_{k=0}^{n}[k]_q\, p_{n,k,a,b}(x) + \frac{[n+a]_q}{[n+1]_q[n+b]_q}\cdot\frac{1}{1+q}$$

$$= \frac{[n+a]_q}{[n+1]_q[n+b]_q}\left(\frac{[n+b]_q}{[n+a]_q}\right)^n [n]_qx\left(\frac{[n+a]_q}{[n+b]_q}\right)^{n-1}$$

$$+ \frac{[n+a]_q}{[n+1]_q[n+b]_q}\cdot\frac{1}{1+q}$$

$$F^*_{n,a,b}(t;x) = \frac{[n]_qx}{[n+1]_q} + \frac{[n+a]_q}{[n+1]_q[n+b]_q}\cdot\frac{1}{1+q}\ .$$

3. Similarly, we also have,

$$
\begin{aligned}
F^*_{n,a,b}(t^2;x) &= \frac{[n+1]_q[n+b]_q}{[n+a]_q}\sum_{k=0}^{n} q^{-k}\, p_{n,k,a,b}(x) \int\limits_{\frac{[k]_q[n+a]_q}{[n+1]_q[n+b]_q}}^{\frac{[k+1]_q[n+a]_q}{[n+1]_q[n+b]_q}} t^2\, d_qt \\
&= \frac{[n+1]_q[n+b]_q}{[n+a]_q}\sum_{k=0}^{n} q^{-k}\, p_{n,k,a,b}(x)\left[\frac{q^k}{1+q+q^2}\left(\frac{[n+a]_q}{[n+1]_q[n+b]_q}\right)^3\times\right. \\
&\qquad \left.\times((1+q+q^2)(q[k]_q[k-1]_q+[k]_q)+(1+2q)[k]_q+1)\right] \\
&= \left(\frac{[n+a]_q}{[n+1]_q[n+b]_q}\right)^2 \sum_{k=0}^{n} p_{n,k,a,b}(x)\times \\
&\qquad \times\left(q[k]_q[k-1]_q+[k]_q+\frac{1+2q}{1+q+q^2}\cdot[k]_q+\frac{1}{1+q+q^2}\right) \\
&= \left(\frac{[n+a]_q}{[n+1]_q[n+b]_q}\right)^2 \left[q\sum_{k=0}^{n} p_{n,k,a,b}(x)[k]_q[k-1]_q\right. \\
&\qquad \left.+\frac{2+3q+q^2}{1+q+q^2}\cdot\frac{[n]_qx}{[n+1]_q}+\frac{1}{1+q+q^2}\right] \\
&= \left(\frac{[n+a]_q}{[n+1]_q[n+b]_q}\right)^2 \left[q[n]_q[n-1]_qx^2\left(\frac{[n+b]_q}{[n+a]_q}\right)^2\right. \\
&\qquad \left.+\frac{2+3q+q^2}{1+q+q^2}\cdot\frac{[n]_qx}{[n+1]_q}+\frac{1}{1+q+q^2}\right] \\
F^*_{n,a,b}(t^2;x) &= \frac{q[n]_q[n-1]_q}{[n+1]_q^2}\cdot x^2+\frac{2+3q+q^2}{1+q+q^2}\cdot\frac{[n]_q[n+a]_q^2}{[n+1]_q^3[n+b]_q^2}\cdot x \\
&\qquad +\frac{1}{1+q+q^2}\cdot\left(\frac{[n+a]_q}{[n+1]_q[n+b]_q}\right)^2.
\end{aligned}
$$

The following lemma gives the moment estimation of operators given in (8) about x.

LEMMA 2. For $x\in\left[0,\frac{[n+a]_q}{[n+b]_q}\right]$ $(a,b\in\mathbb{N}\cup\{0\}, a\leqslant b, n\in\mathbb{N})$ and the operators (8), the following equalities give p^{th} $(p=0,1,2)$ moments for the given operators about x.

1. $F^*_{n,a,b}(1;x)=1$.
2. $F^*_{n,a,b}(t-x;x)=\dfrac{[n+a]_q}{[n+1]_q[n+b]_q}\cdot\dfrac{1}{1+q}-\dfrac{q^nx}{[n+1]_q}$.
3. $F^*_{n,a,b}\left((t-x)^2;x\right)=\dfrac{q[n]_q[n-1]_q}{[n+1]_q^2}\cdot x^2+\dfrac{2+3q+q^2}{1+q+q^2}\cdot\dfrac{[n]_q[n+a]_q^2}{[n+1]_q^3[n+b]_q^2}\cdot x$

$$+\frac{1}{1+q+q^2}\left(\frac{[n+a]_q}{[n+1]_q[n+b]_q}\right)^2$$
$$-2x\left(\frac{[n]_q}{[n+1]_q}\cdot x+\frac{[n+a]_q}{[n+1]_q[n+b]_q}\cdot\frac{1}{1+q}\right)+x^2.$$

Proof.

1. From lemma 1, $F^*_{n,a,b}(1;x)=1$.
2. Using lemma 1 and linearity of $F^*_{n,a,b}$, we have,

$$F^*_{n,a,b}(t-x;x)=\frac{[n]_q}{[n+1]_q}\cdot x+\frac{[n+a]_q}{[n+1]_q[n+b]_q}\cdot\frac{1}{1+q}-x.$$
$$\therefore F^*_{n,a,b}(t-x;x)=\frac{[n]_q-[n+1]_q}{[n+1]_q}\cdot x+\frac{[n+a]_q}{[n+1]_q[n+b]_q}\cdot\frac{1}{1+q}.$$
$$\therefore F^*_{n,a,b}(t-x;x)=\frac{[n+a]_q}{[n+1]_q[n+b]_q}\cdot\frac{1}{1+q}-\frac{q^nx}{[n+1]_q}.$$

3. Proceeding in similar manner as above, we have,

$$F^*_{n,a,b}((t-x)^2;x)=\frac{q[n]_q[n-1]_q}{[n+1]_q^2}\cdot x^2+\frac{2+3q+q^2}{1+q+q^2}\cdot\frac{[n]_q[n+a]_q^2}{[n+1]_q^3[n+b]_q^2}\cdot x$$
$$+\frac{1}{1+q+q^2}\left(\frac{[n+a]_q}{[n+1]_q[n+b]_q}\right)^2$$
$$-2x\left(\frac{[n]_q}{[n+1]_q}\cdot x+\frac{[n+a]_q}{[n+1]_q[n+b]_q}\cdot\frac{1}{1+q}\right)+x^2.$$

We first note that the operator (8) are linear and positive operators. In case $a=b$, the operators (8) reduce to q-Bernstein-Kantorovich operators (6). Further, in case $q=1$, the operators (6) reduce to well-known Bernstein-Kantorovich operators (2).

4. Main Results

The following theorem shows the convergence of the sequence of operators (8) for a function $f\in C\left(\left[0,\frac{[n+a]_q}{[n+b]_q}\right]\right)$. Here $C\left(\left[0,\frac{[n+a]_q}{[n+b]_q}\right]\right)$ is endowed with the norm $||f||=\sup_{x\in\left[0,\frac{[n+a]_q}{[n+b]_q}\right]}|f(x)|$.

THEOREM 2. If a sequence of real numbers $\{q_n\}_{n=1}^{\infty}$ satisfies the conditions, $\lim_{n\to\infty}q_n=1$ and $\lim_{n\to\infty}\frac{1}{[n]_q}=0$ where $0<q_n<1$, then

$$||F^*_{n,a,b}(f;\cdot)-f(\cdot)||\to 0 \text{ as } n\to\infty$$

for every $f \in C\left(\left[0, \frac{[n+a]_q}{[n+b]_q}\right]\right)$; $a,b \in \mathbb{N} \cup \{0\}, a \leqslant b$.

Proof. From Lemma 1, we have,

$$F^*_{n,a,b}(1;x) = 1,$$
$$F^*_{n,a,b}(t;x) = \frac{[n]_q}{[n+1]_q} \cdot x + \frac{[n+a]_q}{[n+1]_q[n+b]_q} \cdot \frac{1}{1+q}$$

and

$$F^*_{n,a,b}(t^2;x) = \frac{q[n]_q[n-1]_q}{[n+1]_q^2} \cdot x^2 + \frac{2+3q+q^2}{1+q+q^2} \cdot \frac{[n]_q[n+a]_q^2}{[n+1]_q^3[n+b]_q^2} \cdot x + \frac{1}{1+q+q^2}\left(\frac{[n+a]_q}{[n+1]_q[n+b]_q}\right)^2.$$

Now, on replacing q by a sequence of real numbers $\{q_n\}$ such that $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} \frac{1}{[n]_q} = 0$ where $0 < q_n < 1$, it follows that $F^*_{n,a,b}(t^m;x) = x^m$ converges uniformly to x^m $(m = 0,1,2)$.

Hence, the result follows by Korovkin's theorem [11].

Above theorem states that we can approximate any function which is continuous on the interval $\left[0, \frac{[n+a]_q}{[n+b]_q}\right]$; $a,b \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}$.

For a function $f \in C([a,b])$, the modulus of continuity is defined as

$$\omega_f(\delta) \equiv \omega(f,\delta) = \sup_{\substack{x-\delta \leqslant t \leqslant x+\delta \\ a \leqslant x \leqslant b}} |f(t) - f(x)| \text{ ; where } \delta > 0.$$

Now, we estimate the rate of approximation of the sequence of operators (8). The following theorem gives the rate of approximation of the sequence of operators (8) in terms of modulus of continuity of a function $f \in C\left(\left[0, \frac{[n+a]_q}{[n+b]_q}\right]\right)$.

THEOREM 3. *If a sequence $\{q_n\}_{n=1}^{\infty}$ satisfies the conditions $\lim_{n\to\infty} q_n = 1$ and*

$$\lim_{n\to\infty} \frac{1}{[n]_q} = 0, \qquad (0 < q_n < 1),$$

then

$$||F^*_{n,a,b}(f;\cdot) - f(\cdot)|| \leqslant 2\,\omega(f, \sqrt{\delta_n})$$

for every $f \in C\left(\left[0, \frac{[n+a]_q}{[n+b]_q}\right]\right)$; $a,b \in \mathbb{N} \cup \{0\}$, $a \leqslant b$ and $\delta_n = F^*_{n,a,b}((t-x)^2;x)$ where $q = q_n$.

Proof. Let $f \in C\left(\left[0, \frac{[n+a]_q}{[n+b]_q}\right]\right)$.
From the linearity and monotonicity of $F^*_{n,a,b}(f;x)$, we can write,

$$\begin{aligned}|F^*_{n,a,b}(f;x) - f(x)| \\ &\leqslant \frac{[n+1]_q[n+b]_q}{[n+a]_q} \sum_{k=0}^{n} q^{-k} \left(\frac{[n+b]_q}{[n+a]_q}\right)^n \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \times \\ &\times \prod_{s=0}^{n-k-1} \left(\frac{[n+a]_q}{[n+b]_q} - q^s x\right) \cdot \int_{\frac{[k]_q[n+a]_q}{[n+1]_q[n+b]_q}}^{\frac{[k+1]_q[n+a]_q}{[n+1]_q[n+b]_q}} |f(t) - f(x)|\, d_q t. \qquad (9)\end{aligned}$$

From the definition of modulus of continuity, we have,

$$|f(t) - f(x)| \leqslant \omega(f, |t-x|).$$

Let $\delta > 0$ and choose $\lambda = \frac{|t-x|}{\delta}$. Then $\lambda \in \mathbb{R}^+$.
If $|t-x| < \delta$, it can be seen that

$$|f(t) - f(x)| \leqslant \left(1 + \frac{(t-x)^2}{\delta^2}\right) \omega(f, \delta). \qquad (10)$$

If $|t-x| \geqslant \delta$ then from the property of modulus of continuity, we get

$$\omega(f, \lambda\delta) \leqslant (1+\lambda)\, \omega(f,\delta) \leqslant (1+\lambda^2)\, \omega(f,\delta). \qquad (11)$$

Therefore, by (10) and (11), we have

$$|f(t) - f(x)| \leqslant \left(1 + \frac{(t-x)^2}{\delta^2}\right) \omega(f, \delta). \qquad (12)$$

Consequently by (9) and (12), we get

$$\begin{aligned}|F^*_{n,a,b}(f;x) - f(x)| &\leqslant \frac{[n+1]_q[n+b]_q}{[n+a]_q} \sum_{k=0}^{n} q^{-k} \left(\frac{[n+b]_q}{[n+a]_q}\right)^n \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \times \\ &\times \prod_{s=0}^{n-k-1} \left(\frac{[n+a]_q}{[n+b]_q} - q^s x\right) \int_{\frac{[k]_q[n+a]_q}{[n+1]_q[n+b]_q}}^{\frac{[k+1]_q[n+a]_q}{[n+1]_q[n+b]_q}} \left(1 + \frac{(t-x)^2}{\delta^2}\right) \omega(f,\delta)\, d_q t \\ &= \left(F^*_{n,a,b}(1;x) + \frac{1}{\delta^2} F^*_{n,a,b}((t-x)^2;x)\right) \omega(f,\delta).\end{aligned}$$

Now, on replacing q by a sequence of real numbers $\{q_n\}$ such that $\lim_{n\to\infty} q_n = 1$ and

$\lim_{n\to\infty} \frac{1}{[n]_q} = 0$ where $0 < q_n < 1$, from the lemma (2), it follows that

$$\lim_{n\to\infty} F^*_{n,a,b}((t-x)^2;x) = 0.$$

Letting $\delta_n = F^*_{n,a,b}((t-x)^2;x)$ (with $q = q_n$) and taking $\delta = \sqrt{\delta_n}$, we get

$$|F^*_{n,a,b}(f;x) - f(x)| \leqslant 2\ \omega(f, \sqrt{\delta_n}). \qquad \left(x \in \left[0, \frac{[n+a]_q}{[n+b]_q}\right]\right)$$

which completes the proof.

REFERENCES

[1] S. N. BERNSTEIN, *Demonstration du Theorem de Weierstrass Fondee sur le Calculu des Probabilites*, Comm. Soc. Mat. Charkow Ser., **13**(1912), 1–2.

[2] H. BOHMAN, *On Approximation of Continuous and Analytic Functions*, Ark. Math., **2**(1952), 43–57.

[3] J. D. CAO, *A generalization of the Bernstein Polynomials*, J. Math. Anal. and Appl., **209**(1997), 140–146.

[4] O. DALMANOGLU, *Approximation by Kantorovich type q-Bernstein operators*, in Proceedings of the 12th WSEAS International Conference on Applied Mathematics, Cairo, Egypt, (2007), 113–117.

[5] A. R. GAIROLA, DEEPMALA AND L. N. MISHRA, *Rate of approximation by finite iterates of q-Durrmeyer operators*, Proc. Natl. Acad. Sci. India Sect. A, **86**, 2(2016), 229–234.

[6] H. GAUCHMAN, *Integral Inequalities in Calculus*, Comput. Math. Appl., **47**, 2-3(2004), 281–300.

[7] A. ILINSKII AND S. OSTROVSKA, *Convergence of generalized Bernstein polynomials*, J. Approx. Theory, **116**(2002), 100–112.

[8] A. IZGI, *Approximation by a Class of New Type Bernstein Polynomials of one and two variables*, Global Journal of Pure and Applied Mathematics, **9**, 1(2013), 55–71.

[9] F. H. JACKSON, *On the q-definite integrals*, Quart. J. Pure Appl. Math., **41**(1910), 193–203.

[10] L. V. KANTOROVICH, *Sur certains developments suivant les polynomes de la forms de S. Bernstein I, II*, C. R. Acad. Sci. USSR, **20**(1930), 563–568.

[11] P. P. KOROVKIN, *On Convergence of Linear Positive Operators in the Space of Continuous Functions*, Dokl. Akad. Nauk SSSR, **90**(1953), 961–964.

[12] G. G. LORENTZ, *Bernstein polynomials*, Chelsea, New York, 1986.

[13] A. LUPAS, *A q-analogue of the Bernstein operator*, Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca, **9**(1987), 85–92.

[14] V. N. MISHRA, K. KHATRI, L. N. MISHRA, *Statistical approximation by Kantorovich type Discrete $q-$ Beta operators*, Adv Differ Equ 2013, **345** (2013), doi:10.1186/1687-1847-2013-345.

[15] H. ORUC AND N. TUNCER, *On the Convergence and Iterates of q-Bernstein Polynomials*, J. Approx. Theory, **117**, 2(2002), 301–313.

[16] S. OSTROVSKA, *On the Lupas q-analogue of the Bernstein Operator*, Rocky Mountain. J. Math., **36**, 5(2006), 1615–1629.

[17] S. OSTROVSKA, *q-Bernstein Polynomials and Their Iterates*, J. Approx. Theory, **123**, 2(2003), 232–255.

[18] G. M. PHILIPS, *Bernstein Polynomials Based on the q-integers*, Ann. Numer. Math., **4**(1997), 511–518.

[19] G. M. PHILIPS, *A Generalization of the Bernstein Polynomials Based on the q-integers*, Anziam J.,**42**(2000), 79–86.

[20] M. A. SIDDIQUI, R. R. AGRAWAL AND N. GUPTA, *On a class of modified new Bernstein operators*, Adv. Stud. Contemp. Math. (Kyungshang), **24**, 1(2014), 97–107.

[21] K. K. SINGH, A. R. GAIROLA AND DEEPMALA, *Approximation theorems for q-analouge of a linear positive operator by A. Lupas*, Int. J. Anal. Appl., **12**, 1(2016), 30–37.

[22] V. N. MISHRA, K. KHATRI, L. N. MISHRA, DEEPMALA, *Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators*, Journal of Inequalities and Applications 2013, 2013:586. doi:10.1186/1029-242X-2013-586.

(Received March 10, 2019)

Lakshmi Narayan Mishra
Department of Mathematics, School of Advanced Sciences
Vellore Institute of Technology (VIT) University
Vellore 632 014, Tamil Nadu, India
L. 1627 Awadh Puri Colony, Beniganj, Phase IIIrd
Opposite Industrial Training Institute (I.T.I.)
Ayodhya Main Road, Ayodhya 224 001, Uttar Pradesh, India
e-mail: lakshminarayanmishra04@gmail.com,
l_n_mishra@yahoo.co.in

Dhawal J. Bhatt
Department of Mathematics
St. Xavier's College
Navrangpura, Ahmedabad-380 009 (Gujarat), India
e-mail: dhawalbhatt_1031@yahoo.com, *dhawal.bhatt@sxca.edu.in*