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UPPER BOUNDS FOR A GENERAL LINEAR FUNCTIONAL WITH APPLICATION TO ORTHOGONAL POLYNOMIAL EXPANSIONS

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Abstract. An upper bound on a linear functional satisfying several constraints is found, then used to provide a short and simple proof of convergence, for orthogonal polynomial expansions.

1. Introduction

Let f be integrable on $[-1,1]$ and denote by

$$\sum_{k=0}^{\infty} \frac{\langle f, \phi_k\rangle}{\langle \phi_k, \phi_k\rangle} \phi_k(x) \ \sim\ f(x)$$

its formal expansion via orthogonal polynomials, with respect to the weight function α implicit in $\langle\cdot,\cdot\rangle$, viz.:

$$\langle f, g\rangle \ =\ \int_{-1}^{1} f(t)g(t)\alpha(t)\,dt\,.$$

We assume that: (i) ϕ_k is of degree k, (ii) there is K such that $\phi_k(x) \leqslant K\ \forall x \in [-1,1]$, and

$$\text{(iii)} \qquad \sum_{k=0}^{n-1} \frac{1}{\langle \phi_k, \phi_k\rangle} \ =\ O(n^2) \quad \text{as} \quad n \to \infty.$$

These conditions hold (with $K = 1$) for the Legendre ($\alpha \equiv 1$) and Chebyshev ($\alpha(x) = 1/\sqrt{1-x^2}$) polynomials.

With these assumptions in place, we prove Theorem 1 below. Its first hypothesis is less general than many in the literature dealing, as it does, only with real analytic functions. However, virtues of Theorem 1 lie in the modesty of its second hypothesis, and in the brevity and simplicity of proof – which uses only real variables. Classical versions, wherein complex variables are used, can be found (for example) in [5] Theorem 9.1.1, [7] Section 15.41, and [6] Theorem 1.9. In the last case, the expansions can be treated as special cases in the Sturm-Liouville Theory. The earliest treatments via real variables are apparently [1, 2].

Our proof of Theorem 1 relies on Lemma 1, which is of independent interest. In these results, $||\cdot||$ denotes the usual sup-norm and $C[a,b]$ denotes the continuous functions on $[a,b]$.

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THEOREM 1. *Let f be real analytic on $[-1,1]$. If*

$$\frac{2^n O(n^2)}{(n-1)!}||f^{(n)}|| = o(1) \quad as \ n \to \infty,$$

then $\sum_{k=0}^{n} \frac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} \phi_k(x)$ *converges pointwise to $f(x)$ on $[-1,1]$.*

2. Proofs

We shall appeal to the following.

LEMMA 1. *Consider the bounded linear functional L on $C[a,b]$:*

$$L : f \mapsto \int_a^b f dw,$$

where w is of bounded variation. Suppose also that $L(x^j) = 0$ for $j = 0, 1, \cdots, n-1$, and that f is n times differentiable with bounded nth derivative. Then we have

$$|L(f)| \leqslant ||f^{(n)}|| \frac{2^n ||w||}{(n-1)!}.$$

Proof of Lemma 1: We may suppose that $[a,b] = [-1,1]$. By hypothesis, $\int_{-1}^{1} 1dw = 0$, so that $w(-1) = w(1)$. Replacing w with $w - w(1)$ does not change L, so we may assume that $w(-1) = w(1) = 0$. Likewise then, $\int_{-1}^{1} xdw = 0$, and so $\int_{-1}^{1} wdx = 0$. Write $W_1(x) = \int_{-1}^{x} w(t)dt$, so that $W_1(\pm 1) = 0$. Again likewise, $\int_{-1}^{1} x^2 dw = 0$, and so $\int_{-1}^{1} W_1 dx = 0$. Write $W_2(x) = \int_{-1}^{x} W_1(t)dt$, so that $W_2(\pm 1) = 0$. Continuing in this way we find that

$$\text{if } \ W_{k-1}(x) = \int_{-1}^{x} W_{k-2}(t)dt, \ \text{ then } \ W_{k-1}(\pm 1) = 0 \quad \text{for } k = 2, 3, \cdots, n.$$

Therefore,

$$L(f) = \int_{-1}^{1} f dw = -\int_{-1}^{1} w(t)f'(t)dt = \int_{-1}^{1} W_1(t)f''(t)dt = -\int_{-1}^{1} W_2(t)f^{(3)}(t)dt$$

$$= \cdots = (-1)^n \int_{-1}^{1} W_{n-1}(t)f^{(n)}(t)dt,$$

and so we have

$$|L(f)| \leqslant ||f^{(n)}|| \int_{-1}^{1} |W_{n-1}(t)|dt.$$

Now finally,

$$|W_{n-1}(x)| = \left| \int_{-1}^{x} \frac{(x-t)^{n-2}}{(n-2)!} w(t)\, dt \right| \leqslant ||w|| \int_{-1}^{x} \frac{(x-t)^{n-2}}{(n-2)!} dt \leqslant ||w|| \frac{2^{n-1}}{(n-1)!},$$

which completes our proof. □

Remarks: There is no requirement here that $L(x^n) \neq 0$, so that different upper bounds for $|L(f)|$ are possible, depending on the available values of n. Such may be the case, for example, if L comes from a quadrature formula. A similar result appears in [4]. We point out also that any bounded linear functional on $C[a,b]$ is given by such a w, by the Riesz Representation Theorem.

Proof of Theorem 1: We show that for each $x \in [-1,1]$,

$$L_n(f) := f(x) - \sum_{k=0}^{n-1} \frac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} \phi_k(x) = o(1) \quad \text{as } n \to \infty.$$

So fix $x \in [-1,1]$, and set $w_n(t) = p(t) - q_n(t)$, where

$$p(t) = \begin{cases} 0 & \text{if } \; -1 \leqslant t \leqslant x \\ 1 & \text{if } \quad x < t \leqslant 1 \end{cases}$$

and

$$q_n(t) = \int_{-1}^{t} \left(\sum_{k=0}^{n-1} \frac{\phi_k(x)}{\langle \phi_k, \phi_k \rangle} \phi_k(u) \right) du.$$

We may thus write

$$L_n(f) := \int_{-1}^{1} f dw_n,$$

with w_n being of bounded variation. Since the ϕ_k's are orthogonal and ϕ_k is of degree k (assumption (i)), we have

$$L_n(x^j) = 0 \quad \text{for} \quad j = 0, 1, \cdots, n-1,$$

and so applying Lemma 1, we get

$$|L_n(f)| \leqslant ||f^{(n)}|| \frac{2^n ||w_n||}{(n-1)!}.$$

Now using assumptions (ii) and (iii), we have

$$|w_n(t)| \leqslant 1 + |q_n(t)| = \int_{-1}^{t} O(n^2) du \leqslant \int_{-1}^{1} O(n^2) du = O(n^2). \qquad (*)$$

Therefore

$$|L_n(f)| \leqslant ||f^{(n)}|| \frac{2^n O(n^2)}{(n-1)!},$$

and the proof is complete. □

Remarks: As regards (iii), for the Chebyshev polynomials we have in fact

$$\sum_{k=0}^{n-1} \frac{1}{\langle \phi_k, \phi_k \rangle} = O(n) \quad \text{as } n \to \infty.$$

For the Legendre polynomials, the $O(n^2)$ in $(*)$ can be improved to $O(\sqrt{n})$ by applying estimate (4.1) from [3], but this would make the present investigation less elementary. In each of these cases, the second hypothesis in Theorem 1 could be relaxed accordingly. For Gegenbauer (or even Jacobi) polynomials, much more subtle versions of (ii) and (iii) would be required. Here again, the investigation would be non-elementary.

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