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ON THE CALCULATION OF TWO ESSENTIAL HARMONIC SERIES WITH A WEIGHT 5 STRUCTURE, INVOLVING HARMONIC NUMBERS OF THE TYPE H_{2n}

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Abstract. The core of the present paper is represented by the calculation of two essential harmonic series with a weight 5 structure, involving harmonic numbers of the type H_{2n}. The two main series are evaluated by also exploiting results and strategies presented in the book, *(Almost) Impossible Integrals, Sums, and Series*, 2019.

1. Introduction

The central results of the paper are represented by two harmonic series, involving harmonic numbers of the type H_{2n}.

The nth generalized harmonic number of order m is defined by

$$H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m},$$

where m is a positive integer.

We call the two main series presented in the paper *essential* since they are usually a critical part in the derivation process of other such harmonic series.

In the classical sense, by the weight of a harmonic series we understand the value $W = a_1 + a_2 + \cdots + a_k + a$ we obtain from the summand of the harmonic series

$$\sum_{n=1}^{\infty} \frac{H_n^{(a_1)} H_n^{(a_2)} \cdots H_n^{(a_k)}}{n^a},$$

where $a_1, a_2, \ldots, a_k$, and a are positive integers.

By analogy with the series presented above, we may consider that the two main series have a weight 5 structure.

During the calculations we will also make use of results and strategies presented in the book, *(Almost) Impossible Integrals, Sums, and Series*.

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2. The lemmas and their proofs

LEMMA 1. *(Two useful series representations) The following equalities hold:*

i)

$$-\log(1+x)\log(1-x)=\sum_{n=1}^{\infty}x^{2n}\left(\frac{H_{2n}-H_n}{n}+\frac{1}{2n^2}\right),\ |x|<1;$$

ii)

$$\log(1-x)\operatorname{Li}_2(x)=3\sum_{n=1}^{\infty}\frac{x^n}{n^3}-2\sum_{n=1}^{\infty}x^n\frac{H_n}{n^2}-\sum_{n=1}^{\infty}x^n\frac{H_n^{(2)}}{n},\ |x|\leqslant 1\wedge x\neq 1,$$

where Li_n *denotes the Polylogarithm function.*

Proof. The proofs of both results are straightforward if we apply the Cauchy product of two series (e.g. check the proof presented in [2, Chapter 6, p. 344]).

LEMMA 2. *(Special logarithmic integrals) Let* n *be a positive integer. The following equalities hold:*

i)

$$\int_0^1 x^{n-1}\log(1-x)dx=-\frac{H_n}{n};$$

ii)

$$\int_0^1 x^{2n-1}\log(1+x)dx=\frac{H_{2n}-H_n}{2n}.$$

Proof. For a straightforward proof of $i)$, make use of the series representation, $\log(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}$. Then, for the point $ii)$ we may use that $\log(1+x)=\log(1-x^2)-\log(1-x)$, and we see immediately the resulting integrals may be calculated by using $i)$. For an alternative way to the integral $i)$, see [2, Chapter 3, p. 59].

LEMMA 3. *(An identity with harmonic numbers) The following equality holds:*

$$\sum_{k=1}^{\infty}\frac{H_k}{(k+1)(k+n+1)}=\frac{H_n^2+H_n^{(2)}}{2n}.$$

Proof. The present result may be viewed as a particular case of a more general result which may be found in [2, Chapter 4, p. 289] and proved in [2, Chapter 6, p. 372–374] by using *The Master Theorem of Series* defined in [3] and [2, Chapter 4, p. 288–289].

LEMMA 4. *(A powerful identity involving harmonic numbers) The following equality holds:*

$$\sum_{n=1}^{\infty} \frac{1}{(2k+2n+1)2n} = \frac{1}{(2k+1)^2} + \frac{H_{2k}}{2k+1} - \frac{H_k}{2(2k+1)} - \frac{\log(2)}{2k+1}.$$

Proof. The calculations are straightforward if we use the partial fraction expansion, add and subtract $1/(2n+1)$ inside the summand, and then split the series. The details of such an approach may be found in [2, Chapter 6, p. 531].

LEMMA 5. *(Two classical Euler sums) The following equalities hold:*

i)

$$2\sum_{k=1}^{\infty} \frac{H_k}{k^m} = (m+2)\zeta(m+1) - \sum_{k=1}^{m-2} \zeta(m-k)\zeta(k+1), m \geqslant 2;$$

ii)

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{H_k}{k^{2m}} = \left(m+\frac{1}{2}\right)\eta(2m+1) - \frac{1}{2}\zeta(2m+1) - \sum_{i=1}^{m-1} \eta(2i)\zeta(2m-2i+1),$$
$$m \geqslant 1,$$

where ζ represents the Riemann zeta function and η denotes the Dirichlet eta function.

Proof. The result from the point $i)$ is known in the mathematical literature from old times. For example, an elementary solution may be found in [1, Chapter 2, pp. 103–105]. Another way to prove it is based on the identity in Lemma 3, by multiplying its both sides by n and then considering the differentiation with respect to n. A solution to the series result from the point $ii)$ may be found in [4].

LEMMA 6. *(A bunch of key harmonic series). The following equalities hold:*

i)

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3} = 3\zeta(2)\zeta(3) - \frac{9}{2}\zeta(5);$$

ii)

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} = \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3);$$

iii)

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^3} = \frac{11}{4}\zeta(4) - \frac{7}{4}\log(2)\zeta(3) + \frac{1}{2}\log^2(2)\zeta(2) - \frac{1}{12}\log^4(2)$$
$$-2\operatorname{Li}_4\left(\frac{1}{2}\right);$$

iv)

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^3}=\frac{5}{8}\zeta(2)\zeta(3)-\frac{11}{32}\zeta(5);$$

v)

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3}=\frac{2}{15}\log^5(2)-\frac{11}{8}\zeta(2)\zeta(3)-\frac{19}{32}\zeta(5)+\frac{7}{4}\log^2(2)\zeta(3)$$
$$-\frac{2}{3}\log^3(2)\zeta(2)+4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)+4\operatorname{Li}_5\left(\frac{1}{2}\right);$$

vi)

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_nH_n^{(2)}}{n^2}=\frac{23}{8}\zeta(5)-\frac{7}{4}\log^2(2)\zeta(3)+\frac{2}{3}\log^3(2)\zeta(2)+\frac{15}{16}\zeta(2)\zeta(3)$$
$$-\frac{2}{15}\log^5(2)-4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-4\operatorname{Li}_5\left(\frac{1}{2}\right),$$

where ζ represents the Riemann zeta function and Li_n denotes the Polylogarithm function.

Proof. All the series results are found in [2, Chapter 4, pp. 292–293, pp. 309–312].

3. The main theorems and their proofs

THEOREM 1. (Main results - the first part) *The following equality holds:*

$$\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{(2n)^3}=\frac{307}{128}\zeta(5)-\frac{1}{16}\zeta(2)\zeta(3)+\frac{1}{3}\log^3(2)\zeta(2)-\frac{7}{8}\log^2(2)\zeta(3)-\frac{1}{15}\log^5(2)$$
$$-2\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-2\operatorname{Li}_5\left(\frac{1}{2}\right),$$

where ζ represents the Riemann zeta function and Li_n denotes the Polylogarithm function.

Proof. Based on Lemma 1, the point $i)$, we obtain by integration that

$$-\int_0^x\frac{\log(1+y)\log(1-y)}{y}\mathrm{d}y=\sum_{n=1}^{\infty}x^{2n}\left(\frac{H_{2n}-H_n}{2n^2}+\frac{1}{4n^3}\right),$$

and if we multiply both sides by $\log(1+x)/x$ and integrate from $x=0$ to $x=1$, using Lemma 2, the point $ii)$, we have

$$-\int_0^1\frac{\log(1+x)}{x}\left(\int_0^x\frac{\log(1+y)\log(1-y)}{y}\mathrm{d}y\right)\mathrm{d}x=\sum_{n=1}^{\infty}\frac{H_{2n}-H_n}{2n}\left(\frac{H_{2n}-H_n}{2n^2}+\frac{1}{4n^3}\right).\tag{1}$$

If in (1) we integrate by parts, then use $\sum_{n=1}^{\infty} a_{2n} = \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty}(-1)^{n-1}a_n\right)$ for the series in the right-hand side, and afterwards rearrange, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{(2n)^3}\\
=&\frac{7}{32}\sum_{n=1}^{\infty}\frac{H_n}{n^4}-\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4}+\frac{5}{16}\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}-\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3}-\frac{5}{64}\zeta(2)\zeta(3)\\
&+\frac{1}{4}\int_0^1\frac{\log(1+x)\log(1-x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x\\
=&\frac{23}{16}\zeta(5)-\frac{9}{64}\zeta(2)\zeta(3)+\frac{1}{6}\log^3(2)\zeta(2)-\frac{7}{16}\log^2(2)\zeta(3)-\frac{1}{30}\log^5(2)\\
&-\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-\operatorname{Li}_5\left(\frac{1}{2}\right)+\frac{1}{4}\int_0^1\frac{\log(1+x)\log(1-x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x, \qquad (2)
\end{aligned}$$

where in the calculations we also used Lemma 5, the point $i)$ with $m=4$, the point $ii)$ with $m=2$, and Lemma 6, the points $ii)$ and $v)$.

On the other hand, using Lemma 1, the point $ii)$, Lemma 2, the point $i)$, and then the series results from Lemma 5, the point $ii)$ with $m=2$, and Lemma 6, the points $v)$ and $vi)$, the integral in (2) may be written as

$$\begin{aligned}
&\int_0^1\frac{\log(1+x)\log(1-x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x\\
=&\int_0^1\left(2\sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1}\frac{H_n}{n^2}+\sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1}\frac{H_n^{(2)}}{n}-3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n-1}}{n^3}\right)\log(1-x)\mathrm{d}x\\
=&3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4}-2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3}-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_nH_n^{(2)}}{n^2}\\
=&\frac{123}{32}\zeta(5)+\frac{5}{16}\zeta(2)\zeta(3)+\frac{2}{3}\log^3(2)\zeta(2)-\frac{7}{4}\log^2(2)\zeta(3)-\frac{2}{15}\log^5(2)\\
&-4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-4\operatorname{Li}_5\left(\frac{1}{2}\right). \qquad (3)
\end{aligned}$$

By combining (2) and (3) the desired result follows.

THEOREM 2. (Main results - the second part) *The following equality holds:*

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{(2n+1)^3}=&\frac{1}{12}\log^5(2)+\frac{31}{128}\zeta(5)-\frac{1}{2}\log^3(2)\zeta(2)+\frac{7}{4}\log^2(2)\zeta(3)\\
&-\frac{17}{8}\log(2)\zeta(4)+2\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right),
\end{aligned}$$

where ζ represents the Riemann zeta function and Li_n denotes the Polylogarithm function.

Proof. Using the identity with harmonic numbers in Lemma 3 where we replace n by $2n$, then multiply both sides by $1/n^2$, and consider the summation from $n=1$ to ∞, we have

$$
\begin{aligned}
&2\sum_{n=1}^{\infty}\frac{H_{2n}^2}{(2n)^3}+2\sum_{n=1}^{\infty}\frac{H_{2n}^{(2)}}{(2n)^3}=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k}{(k+1)(k+2n+1)n^2}\right)\\
=&\frac{1}{4}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_{2k-1}}{k(k+n)n^2}\right)+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_{2k}}{(2k+1)(2k+2n+1)n^2}\right)\\
=&\frac{1}{4}\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_{2k-1}}{k(k+n)n^2}\right)+\sum_{k=1}^{\infty}\frac{H_{2k}}{2k+1}\left(\sum_{n=1}^{\infty}\frac{1}{(2k+2n+1)n^2}\right)\\
=&\frac{1}{4}\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_{2k-1}}{k^2n^2}\right)-\frac{1}{4}\sum_{k=1}^{\infty}\frac{H_{2k-1}}{k^2}\left(\sum_{n=1}^{\infty}\frac{1}{n(k+n)}\right)+\sum_{k=1}^{\infty}\frac{H_{2k}}{(2k+1)^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)\\
&-4\sum_{k=1}^{\infty}\frac{H_{2k}}{(2k+1)^2}\left(\sum_{n=1}^{\infty}\frac{1}{(2k+2n+1)2n}\right)\\
=&\frac{1}{4}\zeta(2)\sum_{k=1}^{\infty}\frac{H_{2k}-1/(2k)}{k^2}-\frac{1}{4}\sum_{k=1}^{\infty}\frac{H_k(H_{2k}-1/(2k))}{k^3}+\zeta(2)\sum_{k=1}^{\infty}\frac{H_{2k+1}-1/(2k+1)}{(2k+1)^2}\\
&-4\sum_{k=1}^{\infty}\frac{H_{2k}}{(2k+1)^2}\left(\frac{1}{(2k+1)^2}+\frac{H_{2k}}{2k+1}-\frac{H_k}{2(2k+1)}-\frac{\log(2)}{2k+1}\right)\\
=&\frac{1}{8}\sum_{n=1}^{\infty}\frac{H_n}{n^4}+\zeta(2)\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^2}+\zeta(2)\sum_{n=1}^{\infty}\frac{H_{2n+1}}{(2n+1)^2}+4\log(2)\sum_{n=1}^{\infty}\frac{H_{2n+1}}{(2n+1)^3}\\
&+4\sum_{n=1}^{\infty}\frac{H_{2n+1}}{(2n+1)^4}-4\sum_{n=1}^{\infty}\frac{H_{2n+1}^2}{(2n+1)^3}+2\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{(2n+1)^3}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{n^3}\\
&+\zeta(2)-\zeta(2)\zeta(3)-\frac{15}{4}\log(2)\zeta(4)+4\log(2),
\end{aligned}
$$

and using that $\sum_{n=1}^{\infty}a_n=a_1+\sum_{n=1}^{\infty}a_{2n}+\sum_{n=1}^{\infty}a_{2n+1}$ and $\sum_{n=1}^{\infty}a_{2n}=\frac{1}{2}\left(\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}(-1)^{n-1}a_n\right)$, we arrive at

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{(2n+1)^3}\\
=&\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}-\frac{1}{2}\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2}-\log(2)\sum_{n=1}^{\infty}\frac{H_n}{n^3}-\frac{17}{16}\sum_{n=1}^{\infty}\frac{H_n}{n^4}+\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{(2n)^3}\\
&+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3}-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^3}-\log(2)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4}\\
&+\frac{15}{8}\log(2)\zeta(4)+\frac{1}{2}\zeta(2)\zeta(3)\\
=&\frac{1}{12}\log^5(2)+\frac{31}{128}\zeta(5)-\frac{1}{2}\log^3(2)\zeta(2)+\frac{7}{4}\log^2(2)\zeta(3)-\frac{17}{8}\log(2)\zeta(4)
\end{aligned}
$$

$$+2\log(2)\,\mathrm{Li}_4\left(\frac{1}{2}\right).$$

In the calculations we have used results from Lemma 4, Lemma 5, the point $i)$ with $m=2,3,4$, the point $ii)$ with $m=2$, Lemma 6, the points $i)$, $ii)$, $iii)$, $iv)$, $v)$, and Theorem 1.

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