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# A MOMENT PROBLEM IN A WEIGHTED $L^2$ SPACE ON THE REAL LINE

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*Abstract.* For a class of sets with multiple terms

$$\{\lambda_n,\mu_n\}_{n=1}^{\infty} := \{\underbrace{\lambda_1,\lambda_1,\dots,\lambda_1}_{\mu_1-times},\underbrace{\lambda_2,\lambda_2,\dots,\lambda_2}_{\mu_2-times},\dots,\underbrace{\lambda_k,\lambda_k,\dots,\lambda_k}_{\mu_k-times},\dots\},$$

we consider a moment problem of the form

$$\int_{-\infty}^{\infty} e^{-2w(t)} t^k e^{\overline{\lambda_n} t} f(t)\,dt = d_{n,k}, \quad \forall\, n\in\mathbb{N} \quad \text{and} \quad k=0,1,2,\dots \mu_n-1,$$

in a weighted $L^2(-\infty,\infty)$ space. We obtain a solution $f$ which extends analytically as an entire function admitting a Taylor-Dirichlet series representation

$$f(z)=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\mu_n-1} c_{n,k} z^k\right) e^{\lambda_n z}, \quad c_{n,k}\in\mathbb{C}, \quad \forall\, z\in\mathbb{C}.$$

## 1. Introduction

This short note is a continuation of our work in [6] where we characterized the closed span of some exponential system

$$E_\Lambda := \{t^k e^{\lambda_n t} : n\in\mathbb{N},\ k=0,1,2,\dots,\mu_n-1\},$$

in certain weighted Banach spaces on the real line. This time we consider moment problems in weighted $L^2$ spaces on $\mathbb{R}$. We note that there is an extensive literature on various *Moment Problems* (see [3]) dealing with topics in Analysis, Statistics and Probability Theory (see [4]).

## 2. Notation and definitions

Following [6], let $w(t)$ be a non-negative real valued continuous function defined on the real line $\mathbb{R}$, not identically equal to zero, such that for some positive constants $a$ and $\beta$ we have

$$0 \leqslant w(t) \leqslant \begin{cases} at^2, & t\geqslant 0,\\ \beta|t|, & t<0. \end{cases} \tag{2.1}$$

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We denote by $L^2_w$ the weighted Banach space of complex-valued measurable functions $f$ defined on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |f(t)e^{-w(t)}|^2\,dt < \infty,$$

equipped with the norm

$$||f||_{L^2_w} := \left(\int_{-\infty}^{\infty} |f(t)e^{-w(t)}|^2\,dt\right)^{\frac{1}{2}}.$$

$L^2_w$ is also a Hilbert space when endowed with the inner product

$$\langle f,g\rangle := \int_{-\infty}^{\infty} f(t)\overline{g(t)}e^{-2w(t)}\,dt.$$

An exponential system $E_\Lambda$ is associated to a set with multiple terms

$$\{\lambda_n,\mu_n\}_{n=1}^{\infty} := \{\underbrace{\lambda_1,\lambda_1,\ldots,\lambda_1}_{\mu_1-times},\underbrace{\lambda_2,\lambda_2,\ldots,\lambda_2}_{\mu_2-times},\ldots,\underbrace{\lambda_k,\lambda_k,\ldots,\lambda_k}_{\mu_k-times},\ldots\},$$

where:

- $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of distinct complex or real numbers diverging to infinity, enumerated so that $0<|\lambda_n|\leqslant|\lambda_{n+1}|$ for all $n\in\mathbb{N}$ and $-\pi<\arg\lambda_n<\arg\lambda_{n+1}\leqslant\pi$ whenever $|\lambda_n|=|\lambda_{n+1}|$,
- $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of positive integers, not necessarily bounded.

We denote this set by $\Lambda$ and we say that $\mu_n$ is the multiplicity of $\lambda_n$ in the set $\Lambda$. We also denote by $\mathrm{span}(E_\Lambda)$ the set of all finite linear combinations of elements from $E_\Lambda$.

DEFINITION 2.1. We denote by $U$ the class of multiplicity sequences $\Lambda=\{\lambda_n,\mu_n\}_{n=1}^{\infty}$ whose terms satisfy the following three conditions:
$(A)$ $\sup_{n\in\mathbb{N}}|\arg\lambda_n|<\pi/4$,
$(B)$ there is a constant $\kappa>1$ so that

$$\frac{|\lambda_{n+1}|}{|\lambda_n|}>\kappa \qquad \forall\, n\in\mathbb{N}, \tag{2.2}$$

$(C)$ there are positive constants $\alpha$ and $c$, with $0<\alpha<1$ such that

$$\mu_n\leqslant c|\lambda_n|^{\alpha} \qquad \forall\, n\in\mathbb{N}. \tag{2.3}$$

EXAMPLE 2.1. $\Lambda=\{3^n,2^n\}_{n=1}^{\infty}$.

Given a multiplicity sequence $\Lambda=\{\lambda_n,\mu_n\}_{n=1}^{\infty}$ in the $U$ class, we say that a doubly indexed family of functions $\{r_{n,k}: n\in\mathbb{N}, k=0,1,\ldots,\mu_n-1\}$ defined on $\mathbb{R}$ is a biorthogonal sequence to the exponential system $E_\Lambda$ in $L^2_w$, if

$$\int_{-\infty}^{\infty} r_{n,k}(t)t^l e^{\overline{\lambda_j}t}e^{-2w(t)}\,dt = \begin{cases} 1, & j=n,\ l=k,\\ 0, & j=n,\ l\in\{0,1,\ldots,\mu_n-1\}\setminus\{k\},\\ 0, & j\neq n,\ l\in\{0,1,\ldots,\mu_j-1\}.\end{cases}$$

## 3. The main results from [6] and our new one

Suppose now that $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$ is a multiplicity sequence belonging to the class $U$ and let $w(t)$ satisfy (2.1). We proved in [6, Theorem 1.1] that if $f$ is a function which belongs to the closed span of $E_\Lambda$ in $L^2_w$, then there exists an entire function $g(z)$ which admits a Taylor-Dirichlet series representation

$$g(z) = \sum_{n=1}^{\infty} \left( \sum_{k=0}^{\mu_n - 1} c_{n,k} z^k \right) e^{\lambda_n z}, \quad c_{n,k} \in \mathbb{C}, \quad \forall\, z \in \mathbb{C},$$

converging uniformly on compact subsets of $\mathbb{C}$, so that

$$f(x) = g(x) \qquad \text{for almost all} \quad x \in \mathbb{R}.$$

We also proved in [6, Theorem 5.1] that there is a family of functions

$$\{r_{n,k} :\ n \in \mathbb{N},\ k = 0, 1, \dots, \mu_n - 1\}$$

defined on $\mathbb{R}$, so that it is the unique biorthogonal sequence of functions to the system $E_\Lambda$ in $L^2_w$, belonging to its closed span in $L^2_w$. Moreover, there are positive constants $Q$ and $\xi$, independent of $n$ and $k$, so that

$$||r_{n,k}||_{L^2_w} \leqslant Q e^{-\xi (\Re \lambda_n)^2}, \qquad \forall\, n \in \mathbb{N}, \quad k = 0, 1, \dots, \mu_n - 1. \tag{3.1}$$

In this note we prove the following moment problem result.

THEOREM 3.1. *Let $w(t)$ be a function as in (2.1) and let $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty} \in U$. Consider an arbitrary doubly-indexed sequence of non-zero complex numbers*

$$\{d_{n,k} : n \in \mathbb{N}, k = 0, 1, \dots, \mu_n - 1\}$$

*with the following property: suppose that there is some*

$$0 \leqslant \gamma < 2 \tag{3.2}$$

*so that*

$$\limsup_{n \to \infty} \frac{\log A_n}{(\Re \lambda_n)^\gamma} < \infty, \qquad A_n = \max\{|d_{n,k}| : k = 0, 1, \dots, \mu_n - 1\}. \tag{3.3}$$

*Then there exists an entire function $f \in L^2_w$ so that*

$$\int_{-\infty}^{\infty} e^{-2w(t)} t^k e^{\overline{\lambda_n} t} f(t)\, dt = d_{n,k}, \qquad \forall\, n \in \mathbb{N} \quad \text{and} \quad k = 0, 1, 2, \dots \mu_n - 1. \tag{3.4}$$

*The function $f$ admits a Taylor-Dirichlet series representation*

$$f(z) = \sum_{n=1}^{\infty} \left( \sum_{k=0}^{\mu_n - 1} c_{n,k} z^k \right) e^{\lambda_n z}, \quad c_{n,k} \in \mathbb{C}, \quad \forall\, z \in \mathbb{C},$$

*with the series converging uniformly on compact subsets of the complex plane.*

*Proof.* Let $H$ be a separable Hilbert space endowed with an inner product $\langle\cdot\rangle$, and consider a sequence $\{f_n\}_{n=1}^{\infty} \subset H$. We say that (see [5, Chapter 4, Section 2]:

(i) $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence if there exists a constant $B > 0$ such that

$$\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 < B||f||^2 \quad \forall f \in H,$$

(ii) $\{f_n\}_{n=1}^{\infty}$ is a Riesz-Fischer sequence if the moment problem $\langle f, f_n\rangle = c_n$ has at least one solution in $H$ for every sequence $\{c_n\}$ in the space $l^2(\mathbb{N})$.

It follows from [1, Proposition 2.3] that if two sequences $\{f_n\}$ and $\{g_n\}$ in $H$ are biorthogonal, that is,

$$\langle f_n, g_m\rangle = \begin{cases} 1, & m = n, \\ 0, & m \neq n, \end{cases}$$

and $\{f_n\}$ is a Bessel sequence, then $\{g_n\}$ is a Riesz-Fischer sequence.

Clearly $\overline{\text{span}}(E_\Lambda)$ in $L^2_w$ is a separable Hilbert space. Let us denote this space by $H_\Lambda$ and let $\{r_{n,k}\}$ be the biorthogonal sequence to $E_\Lambda$ from [6, Theorem 5.1] which satisfies (3.1). Define now for every $n \in \mathbb{N}$ and $k = 0, 1, \ldots, \mu_n - 1$ the following new pair of sequences,

$$U_{n,k}(t) := \lambda_n d_{n,k} r_{n,k}(t) \qquad \text{and} \qquad V_{n,k}(t) := \frac{t^k e^{\lambda_n t}}{\overline{\overline{\lambda_n d_{n,k}}}}.$$

Obviously $\{U_{n,k}\}$ and $\{V_{n,k}\}$ are biorthogonal sequences in $H_\Lambda$. We show below that $\{U_{n,k}\}$ and $\{V_{n,k}\}$ are Bessel and Riesz-Fischer sequences respectively in $H_\Lambda$.

First, by (3.1), (3.2) and (3.3), it follows that there is a positive constant $\delta$, with $\delta < \xi$, so that

$$||U_{n,k}||_{L^2_w} \leqslant e^{-\delta(\Re\lambda_n)^2}.$$

By the Cauchy-Schwartz inequality we get

$$|\langle U_{n,k}, U_{m,j}\rangle| \leqslant e^{-\delta(\Re\lambda_n)^2} \cdot e^{-\delta(\Re\lambda_m)^2}. \tag{3.5}$$

If we denote by $C_{n,k,m,j}$ the value of $\langle U_{n,k}, U_{m,j}\rangle$ and by $C$ the infinite dimensional hermitian matrix with entries the $C_{n,k,m,j}$'s, then $C$ is the Gram matrix associated with $\{U_{n,k}\}$. From (2.3) and (3.5), we get

$$\sum_{n=1}^{\infty} \sum_{k=0}^{\mu_n-1} \sum_{j=1}^{\infty} \sum_{l=0}^{\mu_j-1} |C_{n,k,j,l}| < \infty.$$

Hence the Gram matrix $C$ defines a bounded linear operator on the space of sequences $l^2(\mathbb{N})$ (see [2, Lemma 3.5.3] and [5, Section 4.2 Lemma 1]). It follows by [2, Lemma 3.5.1] that $\{U_{n,k}\}$ is a Bessel sequence in $H_\Lambda$.

Therefore, by [1, Proposition 2.3], its biorthogonal sequence $\{V_{n,k}\}$ is a Riesz-Fischer sequence in $H_\Lambda$. Hence the moment problem

$$\int_{-\infty}^{\infty} f(t)\overline{V_{n,k}(t)}e^{-2w(t)}\,dt = c_{n,k} \qquad \forall\, n\in\mathbb{N} \quad \text{and} \quad k=0,1,2,\dots\mu_n-1,$$

has a solution in $H_\Lambda$ whenever $\sum_{n=1}^{\infty}\sum_{k=0}^{\mu_n-1}|c_{n,k}|^2<\infty$. Since $\Lambda$ satisfies the relations (2.2) and (2.3), then we can take $c_{n,k}=1/\lambda_n$ for all $n\in\mathbb{N}$ and $k=0,1,\dots,\mu_n-1$. Recalling the definition of $V_{n,k}$, shows that there is some function $f\in H_\Lambda$ so that

$$\int_{-\infty}^{\infty} f(t)\left(\frac{t^k e^{\overline{\lambda_n}t}}{d_{n,k}\lambda_n}\right)e^{-2w(t)}\,dt = \frac{1}{\lambda_n} \qquad \forall\, n\in\mathbb{N} \quad \text{and} \quad k=0,1,2,\dots\mu_n-1.$$

Thus (3.4) holds. Finally, the analytic extension of $f$ and its series representation follows from [6, Theorem 1.1].

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