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ON THE CHARACTERIZATION OF POLYNOMIALS AND RATIONAL FUNCTIONS USING DIVIDED DIFFERENCES

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Abstract. In this paper we present two conjectures about the characterization of functions by conditions on their divided differences. To analyze the conjectures and prove some results, we recall some facts about the Hermite interpolation problem including the computation of divided differences for positive and negative powers of x.

1. Introduction

This paper is concerned by *direct* and *inverse* results in the field of interpolation and approximation. The goal of this paper is to present conjectures about characterization of functions by a condition on its n-th order divided difference which is the coefficient of x^n of its Hermite interpolating polynomial. Many authors already worked on special cases of this conjecture. Let us remark that only classical analysis tools are required to obtain the presented results.

The Hermite interpolation problem is to look for a polynomial $p_n(x) \in \mathscr{P}_n$, the set of polynomials of degree at most n, which agrees with the function $f(x)$ in the following sense

$$p_n^{(l)}(x_i) = f^{(l)}(x_i) \quad (l = 0, \ldots, \alpha_i)$$

for $i = 0, \ldots, r$. The α_i's are $r+1$ non negative integers, and the x_i's are $r+1$ distinct points on the real line. Also

$$n = r + \sum_{i=0}^{r} \alpha_i,$$

and

$$\alpha = \max\{\alpha_i : i = 0, \ldots, r\}.$$

It is enough here that the function $f(x)$ be differentiable up to the order α. A survey of this problem is presented in [18].

For this problem we have the following existence and uniqueness result which has some different proofs.

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THEOREM 1. [7, 10, 11, 28] *There exists a unique polynomial* $p_n(x) \in \mathscr{P}_n$, *called Hermite polynomial, which agrees with the function* $f(x)$.

The Hermite polynomial can be expressed as $p_n(x) = \sum_{k=0}^{n} a_k x^k$. The coefficient a_n of x^n depends only on $f(x)$ and the sequence of points x_0, x_1, ..., x_r with their multiplicity $\alpha_0, \ldots, \alpha_r$. So we use the notation

$$a_n = f[\underbrace{x_0,\ldots,x_0}_{\alpha_0+1},\ldots,\underbrace{x_r,\ldots,x_r}_{\alpha_r+1}],$$

and call this coefficient the n-th order divided difference [14, 22].

As an application of the definition of divided differences we get the next two theorems which are, for our problem, *direct* theorems. These results concern the power functions x^l, for positive integer $l > 0$ and for $l = -1$. We consider two associated *inverse* theorems stated here as conjectures. Both results characterize power functions by the n-th order divided differences.

THEOREM 2. [6, 18] *For* $n \geqslant 1$ *we have*

$$x^l[\underbrace{x_0,\ldots,x_0}_{\alpha_0+1},\ldots,\underbrace{x_r,\ldots,x_r}_{\alpha_r+1}] = \begin{cases} 0 & \text{for } l = 0,\ldots,n-1, \\ 1 & \text{for } l = n, \\ \sum_{i=0}^{r}(\alpha_i+1)x_i & \text{for } l = n+1. \end{cases}$$

CONJECTURE 3. *Suppose that for* $f(x)$ *there exists a function* $G(x)$ *such that to any distinct* x_0, x_1, ..., x_r, *we have*

$$f[\underbrace{x_0,\ldots,x_0}_{\alpha_0+1},\ldots,\underbrace{x_r,\ldots,x_r}_{\alpha_r+1}] = G\left(\sum_{i=0}^{r}(\alpha_i+1)x_i\right).$$

Then $f(x)$ *is a polynomial of degree at most* $n+1$.

THEOREM 4. [18] *For* $f(x) = 1/x$ *we have*

$$\frac{1}{x}[\underbrace{x_0,\ldots,x_0}_{\alpha_0+1},\ldots,\underbrace{x_r,\ldots,x_r}_{\alpha_r+1}] = \frac{(-1)^n}{\prod_{i=0}^{r} x_i^{\alpha_i+1}}.$$

CONJECTURE 5. *Suppose that for* $f(x)$ *there exists a function* $G(x)$ *such that to any distinct non zero real numbers* x_0, x_1, ..., x_r, *we have*

$$f[\underbrace{x_0,\ldots,x_0}_{\alpha_0+1},\ldots,\underbrace{x_r,\ldots,x_r}_{\alpha_r+1}] = G\left(\prod_{i=0}^{r} x_i^{\alpha_i+1}\right).$$

Then there exists a constant a *such that* $h(x) = f(x) - \frac{a}{x}$ *is a polynomial of degree at most* n.

The restriction $r > 0$ comes from the fact that for $r = 0$, $p_n(x)$ is the Taylor polynomial of $f(x)$. So we have

$$\frac{f^{(n)}(x)}{n!} = f[\underbrace{x,\ldots,x}_{(n+1)-times}] = \begin{cases} G((n+1)x) & \text{for Conjecture 3,} \\ G\left(x^{1+n}\right) & \text{for Conjecture 5,} \end{cases}$$

which gives no restriction on $f(x)$.

For those two conjectures we have proofs for two special forms of the interpolating polynomial, namely for the Lagrange interpolating polynomial ($n = r$ and $\alpha_i = 0$ for all $i = 0,\ldots,n$), and for an almost Taylor expansion ($r = 1$, with $\alpha_0 = n-1$ and $\alpha_1 = 0$). So for many other situations proofs are waiting to be discovered.

Let us observe that these conjectures are associated to arithmetic mean for polynomial and geometric mean for rational function. We could ask if there exists similar conjectures for functions associated to other means.

2. Lagrange interpolation

For the Lagrange interpolation problem we have $r = n$ and $\alpha_i = 0$ for $i = 0,\ldots,n$. Several authors worked on Conjecture 3, see [1, 2, 3, 4, 5, 6, 8, 9, 12, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26], much few worked on Conjecture 5, see [27]. Both results are based on functional equations.

2.1. Lagrange interpolating polynomial

The Lagrange interpolating polynomial is

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x) \quad \text{where} \quad \ell_i(x) = \prod_{\substack{k=0 \\ k\neq i}}^{n} \frac{x - x_k}{x_i - x_k}.$$

So the coefficient of x^n is

$$f[x_0,\ldots,x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{\substack{k=0 \\ k\neq i}}^{n}(x_i - x_k)}.$$

It will be useful to assume that $f(x_k) = 0$ for $k = 0,\ldots,n$, except for two different indices i and j, because we could subtract to $f(x)$ the Lagrange interpolating polynomial of degree $n-2$ such that $p_{n-2}(x_k) = f(x_k)$ for $k = 0,\ldots,n$, except for i and j. This condition on $f(x)$ does not change the coefficient of x^n of $p_n(x)$ because

$$f[x_0,\ldots,x_n] = (f - p_{n-2})[x_0,\ldots,x_n].$$

Then we can write

$$f[x_0,\ldots,x_n] = \frac{f(x_i)}{\prod_{\substack{k=0 \\ k\neq i}}^{n}(x_i - x_k)} + \frac{f(x_j)}{\prod_{\substack{k=0 \\ k\neq j}}^{n}(x_j - x_k)}.$$

Let $f(x) = \ell_{ij}(x)\prod_{\substack{k=0\\k\neq i,j}}^{n}(x-x_k)$, so

$$f[x_0,\ldots,x_n] = \frac{\ell_{ij}(x_i)}{x_i - x_j} - \frac{\ell_{ij}(x_j)}{x_i - x_j}. \tag{1}$$

2.2. Characterization of a polynomial

The proof of Conjecture 3 for this case is based on the following functional equation lemma.

LEMMA 6. [3] *Suppose two functions $v(x)$ and $w(x)$ are such that*

$$v(x) - v(y) = (x-y)w(x+y). \tag{2}$$

Then $v(x) = ax^2 + bx + c$ and $w(x) = ax + b$.

Proof. Replace y by $-y$ in (2) to obtain

$$v(x) - v(-y) = (x+y)w(x-y). \tag{3}$$

Subtract (2) from (3) to get

$$v(y) - v(-y) = (x+y)w(x-y) - (x-y)w(x+y). \tag{4}$$

Replacing x by $-y$ in (2), we have

$$v(y) - v(-y) = 2yw(0). \tag{5}$$

Since $2y = (x+y) - (x-y)$, (4) and (5) lead to

$$\begin{aligned}((x+y)-(x-y))w(0) &= 2yw(0)\\ &= v(y) - v(-y)\\ &= (x+y)w(x-y) - (x-y)w(x+y),\end{aligned}$$

or

$$(x+y)[w(x-y) + w(0)] = (x-y)[w(x+y) + w(0)].$$

Set $\xi = x+y$ and $\zeta = x-y$, then

$$\frac{w(\xi) + w(0)}{\xi} = \frac{w(\zeta) + w(0)}{\zeta},$$

which means that the ratio $\frac{w(x)+w(0)}{x}$ is a constant, say a. With $w(0) = -b$, we get

$$w(x) = ax + b.$$

Finally, set $y = 0$ in (2), we get

$$v(x) - v(0) = xw(x) = ax^2 + bx$$

and the result follows with $c = v(0)$. $\square$

THEOREM 7. *Suppose that for $f(x)$ there exists a function $G(x)$ such that to any distinct x_0, $x_1, \dots, x_n$, we have*

$$f[x_0,\dots,x_n] = G\left(\sum_{i=0}^{n} x_i\right).$$

Then $f(x)$ is a polynomial of degree at most $n+1$.

Proof. We can write

$$G\left([x_i + x_j] + \sum_{\substack{k=0 \\ k\neq i,j}}^{n} x_k\right) = g\left(x_i + x_j\right).$$

Let $x_i = x$ and $x_j = y$ in (1), then we have

$$\ell_{ij}(x) - \ell_{ij}(y) = (x-y)g\left(x+y\right).$$

So we can conclude from Lemma 6 and (1) that $f(x)$ is a polynomial of degree $n+1$. □

2.3. Characterization of a rational function

As for polynomial, the proof of this case of Conjecture 5 is based on the next functional equation lemma.

LEMMA 8. [3] *Suppose the two functions $v(x)$ and $w(x)$ are defined for $x \neq 0$. Suppose we have*

$$v(x) - v(y) = (x-y)w\left(xy\right). \tag{6}$$

Then $w(x) = -\frac{a}{x} + b$ and $v(x) = \frac{a}{x} + bx + c$.

Proof. Changing y by $1/y$ in (6), we get

$$v(x) - v(1/y) = (x - 1/y)w(x/y). \tag{7}$$

Subtract (6) from (7) to get

$$v(y) - v(1/y) = (x - 1/y)w((x/y) - (x-y)w(xy). \tag{8}$$

Using $1/y$ instead of x in (6), we get

$$v(y) - v(1/y) = (y - 1/y)w(1). \tag{9}$$

Since $y - 1/y = (y-x) + (x - 1/y)$, (8) and (9) lead to

$$\begin{aligned} ((y-x) + (x-1/y))w(1) &= (y-1/y)w(1) \\ &= v(y) - v(1/y) \\ &= (x-1/y)w\left(x/y\right) - (x-y)w\left(xy\right)), \end{aligned}$$

so

$$(x-1/y)\left[w(x/y)-w(1)\right]=(x-y)\left[w(xy)-w(1)\right].$$

Set $\xi=xy$ and $\zeta=y/x$, then

$$\xi\frac{[w(\xi)-w(1)]}{\xi-1}=\zeta\frac{[w(\zeta)-w(1)]}{\zeta-1},$$

which means that $x\frac{[w(x)-w(1)]}{x-1}$ is constant, say a. So, we get

$$w(x)=\frac{a}{x}(x-1)+w(1)=-\frac{a}{x}+b.$$

where $b=a+w(1)$. Finally, set $y=1$ in (6) to get

$$v(x)-v(1)=(x-1)w(x)=(x-1)\left[-\frac{a}{x}+b\right]=\frac{a}{x}+bx-(a+b),$$

and

$$v(x)=\frac{a}{x}+bx+c,$$

where $c=v(1)-(a+b)$, and the result follows. □

THEOREM 9. *Suppose that for $f(x)$ there exists a function $G(x)$ such that to any distinct nonzero x_0, x_1, ..., x_n, we have*

$$f[x_0,\ldots,x_n]=G\left(\prod_{i=0}^{n}x_i\right).$$

Then there exists a constant a such that $f(x)-\frac{a}{x}$ is a polynomial of degree at most n.

Proof. We can write

$$G\left([x_ix_j]\prod_{\substack{k=0\\k\neq i,j}}^{n}x_k\right)=g(x_ix_j).$$

Let $x_i=x$ and $x_j=y$ in (1), then we have

$$\ell_{ij}(x)-\ell_{ij}(y)=(x-y)g(xy).$$

So we can conclude from the Lemma 8 and (1) that there exists a constant a such that $h(x)=f(x)-\frac{a}{x}$ is a polynomial of degree at most n. □

3. Almost Taylor's expansion

For $r=1$, $\alpha_0=n-1$ and $\alpha_1=0$, Hermite interpolating polynomial looks like a Taylor's expansion of $f(x)$. Conjecture 3 was solved in [26], but we present a different proof here. The result we present about Conjecture 5 is a new one.

3.1. Taylor's expansion and interpolating polynomial

In this case the Hermite interpolating polynomial is

$$p_n(x) = \sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j + f[\underbrace{x_0,\dots,x_0}_{n-\text{times}},x_1](x-x_0)^n,$$

so we obtain the following form of the Taylor's expansion

$$f(x_1) = p_n(x_1) = \sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{j!}(x_1-x_0)^j + f[\underbrace{x_0,\dots,x_0}_{n-\text{times}},x_1](x_1-x_0)^n. \tag{10}$$

3.2. Characterization of a polynomial

We establish the Conjecture 3 for which we present a modified version of the proof given in [26].

THEOREM 10. *Suppose that for $f(x)$ there exists a function $G(x)$ such that to any two distinct x_0 and x_1 we have*

$$f[\underbrace{x_0,\dots,x_0}_{n-times},x_1] = G(nx_0+x_1). \tag{11}$$

Then $f(x)$ is a polynomial of degree at most $n+1$.

Proof. From (10) and the condition (11), we have

$$f(x_1) = \sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{j!}(x_1-x_0)^j + G(nx_0+x_1)(x_1-x_0)^n \tag{12}$$

Set

$$nx_0+x_1 = (n+1)x,$$

or

$$(x_1-x_0) = (n+1)(x-x_0).$$

After a substitution, we have

$$f(x_0+(n+1)(x-x_0)) = \sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{j!}(n+1)^l(x-x_0)^j + G((n+1)x)(n+1)^n(x-x_0)^n.$$

From this expression, we can conclude that $G(x)$ is differentiable, and hence continuous, up to order $n-1$. From the same expression we also obtain that $f^{(n-1)}(x)$ is differentiable, so $f^{(n)}(x)$ exists everywhere.

Now reconsider (12) and set $x_1 = x_0 u$ to get

$$f(x_0 u) = \sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{j!} x_0^j (u-1)^j + G(x_0(n+u)) x_0^n (u-1)^n$$

Now differentiate this expression with respect to u to obtain

$$\begin{aligned} f'(x_0 u) = &\sum_{j=0}^{n-2} \frac{f^{(j+1)}(x_0)}{j!} x_0^j (u-1)^j \\ &+ G'(x_0(n+u)) x_0^n (u-1)^n + G(x_0(n+u)) n x_0^{n-1} (u-1)^{n-1}, \end{aligned}$$

and then with respect to x_0

$$\begin{aligned} f'(x_0 u) = &\sum_{j=0}^{n-2} \frac{f^{(j+1)}(x_0)}{j!} x_0^j (u-1)^j + \frac{f^{(n)}(x_0)}{(n-1)!} x_0^{n-1} \frac{(u-1)^{n-1}}{u} \\ &+ G'(x_0(n+u))(n+u) x_0^n \frac{(u-1)^n}{u} + G(x_0(n+u)) n x_0^{n-1} \frac{(u-1)^n}{u}. \end{aligned}$$

Now subtracting, we have

$$\frac{f^{(n)}(x_0)}{n!} = G(x_0(n+u)) - G'(x_0(n+u))(u-1)x_0.$$

Now set $x = x_0(n+u)$ to get

$$\begin{aligned} \frac{f^{(n)}(x_0)}{n!} &= G(x) - (x - (n+1)x_0) G'(x) \\ &= \left[G(x) - x G'(x) \right] + x_0 (n+1) G'((n+1)x). \end{aligned}$$

In this last expression $f^{(n)}(x_0)$ is linear with respect to x_0. The two coefficients of this expression must be constant, so we can write

$$\begin{cases} G(x) - x G'(x) = c, \\ (n+1) G'(x) = d. \end{cases}$$

Using the second equation in the first one leads to

$$G(x) = c + \frac{d}{n+1} x.$$

Then $G(x)$ is a polynomial of degree at most 1, and consequently $f(x)$ is a polynomial of degree at most $n+1$. $\square$

3.3. Characterization of a rational function

We can adapt the preceding proof to obtain a proof of the Conjecture 5.

THEOREM 11. *Suppose that for $f(x)$ there exists a function $G(x)$ such that to any two non zero distinct x_0 and x_1 we have*

$$f[\underbrace{x_0,\ldots,x_0}_{n-times},x_1] = G\left(x_0^n x_1\right). \tag{13}$$

Then there exists a constant a such that $h(x) = f(x) - \frac{a}{x}$ is a polynomial of degree at most n.

Proof. From (10) and the condition (13), we have

$$f(x_1) = \sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{j!}(x_1 - x_0)^j + G\left(x_0^n x_1\right)(x_1 - x_0)^n. \tag{14}$$

Set

$$x_0^n x_1 = x^{n+1},$$

to obtain

$$x_1 = x_0 \left(\frac{x}{x_0}\right)^{n+1}.$$

Replacing x_1 by this expression in (14) allow us to conclude that $G(x)$ is differentiable, and hence continuous, up to order $n-1$. Moreover $f^{(n-1}(x)$ is differentiable, so $f^{(n}(x)$ exists.

Reconsidering (14) and set $x_1 = x_0 u$. After a substitution, we have

$$f\left(x_0 u\right) = \sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{j!} x_0^j (u-1)^j + G\left(x_0^{n+1} u\right) x_0^n (u-1)^n.$$

Let us compute the first derivative of $f(x_0 u)$ with respect to u to obtain

$$f'\left(x_0 u\right) = \sum_{j=0}^{n-2} \frac{f^{(j+1)}(x_0)}{j!} x_0^j (u-1)^j + G'\left(x_0^{n+1} u\right) x_0^{2n} (u-1)^n + G\left(x_0^{n+1} u\right) n x_0^{n-1} (u-1)^{n-1},$$

and with respect to x_0 to obtain

$$f'(x_0 u) = \sum_{j=0}^{n-2} \frac{f^{(j+1)}(x_0)}{j!} x_0^j (u-1)^j + \frac{f^n(x_0)}{(n-1)!} x_0^{n-1} \frac{(u-1)^{n-1}}{u} + G'\left(x_0^{n+1} u\right)(n+1) x_0^{2n} \frac{(u-1)^n}{u^\theta} + G\left(x_0^{n+1} u\right) n x_0^{n-1} \frac{(u-1)^n}{u}.$$

From those two expressions we get

$$\frac{f^n(x_0)}{n!} = G\left(x_0^{n+1}u\right) - G'\left(x_0^{n+1}u\right)x_0^{n+1}u(u-1).$$

Let us introduce $\xi = x_0^{n+1}u$ to replace u, which is $u = \frac{\xi}{x_0^{n+1}}$. We obtain

$$\frac{f^n(x_0)}{n!} = \left[G(\xi) + \xi G'(\xi)\right] - G'(\xi)\left(\frac{\xi^2}{x_0^{n+1}}\right).$$

So $f^n(x_0)$ as a function of x_0 leads to the system

$$\begin{cases} G(\xi) + \xi G'(\xi) = c, \\ G'(\xi)\xi^2 \quad = d, \end{cases}$$

for two constant c and d. Using the second equation in the first, we get

$$G(\xi) = c - \frac{d}{\xi}.$$

Then $\frac{f^n(x_0)}{n!} = c - \frac{d}{x_0^{n+1}}$. So there exists a constant a such that $h(x) = f(x) - \frac{a}{x}$ is a polynomial of degree at most n. □

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REFERENCES

[1] J. ACZÉL, *A mean value property of the derivative of quadratic polynomials – without mean values and derivatives*, Mathematics Magazine, 58:1 (1985), 42–45.
[2] J. ACZÉL, *Problem 296: Solution of $f(x) - g(x) = (x-y)h(x+y)$*, Two Year College Mathematical Journal, 18:4 (1987), 344–345.
[3] J. ACZÉL AND M. KUCZMA, *On two mean value properties and functional equations associated with them*, Aequationes Mathematicae, 38:2–3 (1989), 216–235.
[4] K. M. ANDERSEN, *A characterization of polynomials*, Mathematics Magazine, 69:2 (1996), 137–142.
[5] D. F. BAILEY, *A mean value property of cubic polynomials – without mean values*, Mathematics Magazine, 65:2 (1992), 123–124.
[6] D. F. BAILEY AND G. J. FIX, *A generalization of the mean value theorem*, Appl. Math. Lett., 1:4 (1988), 327–330.
[7] S. D. CONTE AND C. DE BOOR, *Elementary Numerical Analysis*, McGraw-Hill, New York, 1965.
[8] J. K. CHUNG AND P. K. SAHOO, *Characterizations of polynomials with mean value properties*, Appl. Math. Lett., 6:3 (1993), 97–100.
[9] G. E. CROSS AND P. L. KANNAPPAN, *A functional identity characterizing polynomials*, Aequationes Mathematicae, 34:2–3 (1987), 147–152.

[10] M. CROUZEIX AND A. L. MIGNOT, *Analyse numérique des équations différentielles*, Masson, Paris, 1984.
[11] P. J. DAVIES, *Interpolation & Approximation*, Dover, New York, 1963.
[12] R. O. DAVIES AND G. ROUSSEAU, *A divided difference characterization of polynomials over a general field*, Aequationes Mathematicae, 55:1–2 (1998), 73–78.
[13] C. DE BOOR, *A Practical Guide to Splines*, Springer-Verlag, New York, 1978.
[14] C. DE BOOR, *Divided Differences*, Surveys in Approximation Theory, 1 (2005), 46–69.
[15] E. DEEBA AND P. SIMEONOV, *Characterization of polynomials using divided differences*, Mathematics Magazine, 76:1 (2003), 61–66.
[16] D. E. DOBBS, *Average Values and Linear Functions*, Two Year College Mathematical Journal, 16:2 (1985), 132–135.
[17] H. H. DOWNING, *Note on the law of the mean*, The American Mathematical Monthly, 27:12 (1920), 467–469.
[18] F. DUBEAU, *On Hermite interpolation and divided differences*, Surveys in Mathematics and its Applications, 15 (2020), 257–277.
[19] S. HARUKI, *A property of quadratic polynomials*, The American Mathematical Monthly, 86:7 (1979), 577–579.
[20] P. L. KANNAPPAN AND P. K. SAHOO, *Characterization of polynomials and divided difference*, Proc. Indian Acad. Sci. (Math. Sci.), 105 (1995), 287–290.
[21] M. S. KLAMKIN AND D. J. NEWMAN, *On Some Inverse Problems in Potential Theory*, Quarterly Applied Math., 26 (1968), 277–280.
[22] E. T. Y. LEE, *A remark on divided differences*, The American Mathematical Monthly, 96:7 (1989), 618–622.
[23] M. Z. NASHED, *A Functional Equation Which Characterizes Polynomials with Applications to Uniqueness*, Applicable Analysis, 5 (1975), 55–63.
[24] T. L. SAATY, *Modern Nonlinear Equations*, McGraw-Hill, New York, 1967.
[25] J. SCHWAIGER, *On a characterization of polynomials by divided differences*, Aequationes Mathematicae, 48:2–3 (1994), 317–323.
[26] B. SHAWYER AND B. WATSON, *Problem 312: Mean value theorem for polynomials*, Two Year College Mathematical Journal, 18:4 (1987), 344–345.
[27] P. SIMEONOV, *Characterization of a class of functions using divided differences*, Abstract and Applied Analysis, 5:2 (2000), 85–90.
[28] A. SPITZBART, *A generalization of Hermite's interpolation formula*, The American Mathematical Monthly, 67:1 (1960), 42–46.

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