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SOME IMPROPER INTEGRALS INVOLVING THE SQUARE OF THE TAIL OF THE SINE AND COSINE FUNCTIONS

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Abstract. A class of four improper integrals containing the square of the tail of the sine and cosine functions in their integrand are found using Fourier transform methods. Relations between the four improper integrals considered are given and an open problem concerning the general form of certain improper integrals of this type is raised.

1. Introduction and the main results

Recently the author proposed the following problem involving an improper integral containing the tail of the sine function [5]

$$\int_0^\infty \frac{1}{x^{2n+3}} \left(\sin x - \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right) dx = \frac{(-1)^{n+1}}{(2n+2)!} \frac{\pi}{2}. \tag{1}$$

A similar improper integral for the tail of the cosine function can also be proposed

$$\int_0^\infty \frac{1}{x^{2n+2}} \left(\cos x - \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} \right) dx = \frac{(-1)^{n+1}}{(2n+1)!} \frac{\pi}{2}. \tag{2}$$

In both cases n is a non-negative integer. Improper integrals related to (1) and (2) have appeared in the literature in the past [7, Problem 1914, p. 329], [6]. Inspired by these two improper integrals this paper is the product of an attempt to extend (1) and (2) to the case where the integrand is squared. The problem of evaluating a convergent improper integral after the integrand of some known convergent improper integral is squared is particularly challenging. Perhaps the most famous example of a similarly related improper integral to those we are going to consider here where the square of the integrand of a known improper integral is taken is the well-known Dirichlet integral and its square first found by Lindman in 1851 [4]

$$\int_0^\infty \frac{\sin x}{x} dx = \int_0^\infty \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}.$$

In this particular example, the integral and the integral with its integrand squared so happen to be equal.

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To help aid in writing more compactly the improper integrals we wish to consider, we introduce the following two sum functions

$$\sin_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \text{and} \quad \cos_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}. \tag{3}$$

Here n is a non-negative integer. We refer to these sum functions as the *sine sum function* and the *cosine sum function* respectively. Each is just the nth partial sum of the Maclaurin series expansion corresponding to the sine and cosine functions respectively.

In this paper we evaluate four classes of improper integrals involving the square of the tail of the sine and cosine functions. More precisely, using Fourier transform methods we find closed-form expressions for the following improper integrals:

$$I_n = \int_0^\infty \left(\frac{\cos x - \cos_n(x)}{x^{2n+1}} \right)^2 dx, \tag{4}$$

$$J_n = \int_0^\infty \left(\frac{\cos x - \cos_n(x)}{x^{2n+2}} \right)^2 dx, \tag{5}$$

$$\Lambda_n = \int_0^\infty \left(\frac{\sin x - \sin_n(x)}{x^{2n+2}} \right)^2 dx, \tag{6}$$

and

$$\Pi_n = \int_0^\infty \left(\frac{\sin x - \sin_n(x)}{x^{2n+3}} \right)^2 dx. \tag{7}$$

In all cases n is a non-negative integer.

The main results of this paper are contained in the following theorems.

THEOREM 1. *Let n be a non-negative integer and let I_n be the improper integral given in* (4). *Then*

$$I_n = \frac{\pi}{2(4n+1)[(2n)!]^2}.$$

THEOREM 2. *Let n be a non-negative integer and let Λ_n be the improper integral given in* (6). *Then*

$$\Lambda_n = \frac{\pi}{2(4n+3)[(2n+1)!]^2}.$$

Our last theorem gives the relationship between the square of the tail of the cosine improper integrals to their sine counterparts.

THEOREM 3. *Let n be a non-negative integer and let J_n and Π_n be the improper integrals given in* (5) *and* (7) *respectively. Then*

$$J_n = \Lambda_n \quad \textit{and} \quad \Pi_n = I_{n+1}.$$

2. Two lemmas and the proofs of the main results

We begin by recalling two special types of functions that will be needed. The first is the *signum function*. It is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1 & x < 0, \\ 0 & x = 0, \\ 1 & x > 0. \end{cases}$$

The second is the *indicator function* $\chi_I(x)$ on the interval I. It is defined to be equal to unity when $x \in I$ and zero otherwise. As we will be making use of the Fourier transform, for a function $f \in L^2(\mathbb{R})$ (a square-integrable function) the convention we adopt is

$$\hat{f}(\omega) = \mathscr{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx.$$

Here i is the imaginary unit. In our analysis we will also be making use of Plancherel's theorem. It states that if $g \in L^2(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} |g(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{g}(\omega)|^2\, d\omega. \tag{8}$$

We now present a lemma that establishes for what indices of x in the denominator improper integrals of the form (4) through to (7) converge.

LEMMA 1. *For $a \in \mathbb{R}$ and n a non-negative integer, convergence in the improper integrals is as follows:*

(a) $\displaystyle\int_0^{\infty} \frac{(\cos x - \cos_n(x))^2}{x^a}\, dx$ *where* $4n+1 < a < 4n+5$, *and*

(b) $\displaystyle\int_0^{\infty} \frac{(\sin x - \sin_n(x))^2}{x^a}\, dx$ *where* $4n+3 < a < 4n+7$.

Proof. For the improper integral in (a) write it as

$$\int_0^{\infty} \frac{(\cos x - \cos_n(x))^2}{x^a}\, dx = \int_0^{1} \frac{(\cos x - \cos_n(x))^2}{x^a}\, dx + \int_1^{\infty} \frac{(\cos x - \cos_n(x))^2}{x^a}\, dx.$$

The first of these improper integrals converges for $a < 4n+5$ since the integrand is $\mathscr{O}(x^{4n+4-a})$ as $x \to 0^+$. For the second improper integral it converges for $a > 4n+1$ since the integrand is $\mathscr{O}(x^{4n-a})$ as $x \to \infty$. On combining the two results, for the improper integral in (a) it converges for all $4n+1 < a < 4n+5$.

Similarly, for the integral in (b) write it as

$$\int_0^{\infty} \frac{(\sin x - \sin_n(x))^2}{x^a}\, dx = \int_0^{1} \frac{(\sin x - \sin_n(x))^2}{x^a}\, dx + \int_1^{\infty} \frac{(\sin x - \sin_n(x))^2}{x^a}\, dx.$$

The first of these improper integrals converges for $a < 4n+7$ since the integrand is $\mathscr{O}(x^{4n+6-a})$ as $x \to 0^+$. For the second improper integral it converges for $a > 4n+3$ since the integrand is $\mathscr{O}(x^{4n+2-a})$ as $x \to \infty$. On combining the two results, for the improper integral in (b) it converges for all $4n+3 < a < 4n+7$ and completes the proof. $\square$

From this Lemma it is immediate the improper integrals I_n, J_n, Λ_n, and Π_n converge for all non-negative n.

Before we prove the main results of this paper we need a lemma concerning two related binomial identities.

LEMMA 2. *If n is a non-negative integer and $x \in \mathbb{R}$, then the following binomial identities hold:*

(a) $\displaystyle\sum_{k=0}^{n} \binom{2n}{2k} x^{2k} = \frac{1}{2}\left[(x+1)^{2n} + (x-1)^{2n}\right]$, *and*

(b) $\displaystyle\sum_{k=0}^{n} \binom{2n}{2k} \frac{x^{2k}}{2k+1} = \frac{(x+1)^{2n+1} + (x-1)^{2n+1}}{2x(2n+1)}$.

Proof. Applying the binomial theorem we have

$$\begin{aligned}(x+1)^{2n} + (x-1)^{2n} &= \sum_{k=0}^{2n} \binom{2n}{k} x^k + \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k x^k \\ &= 2 \sum_{\substack{k=0 \\ k \in \text{even}}}^{2n} \binom{2n}{k} x^k = 2 \sum_{k=0}^{n} \binom{2n}{2k} x^{2k},\end{aligned}$$

after a shift in the index of $k \mapsto 2k$ is made, from which the result in (a) follows. For the result in (b), starting with the result in (a), replacing x with t before integrating with respect to t from 0 to x immediately gives the result. $\square$

We are now in a position to prove Theorem 1.

Proof. We commence by first finding the Fourier transform for the function

$$g_n(x) = \frac{\cos x - \cos_n(x)}{x^{2n+1}},$$

where n is a non-negative integer. As $g_n(x) \in L^2(\mathbb{R})$, from the linearity property of the Fourier transform we are able to write

$$\hat{g}_n(\omega) = \mathscr{F}\{g_n(x)\} = \mathscr{F}\left\{\frac{\cos x}{x^{2n+1}}\right\} - \sum_{k=0}^{n} \frac{(-1)^k}{(2n)!} \mathscr{F}\left\{\frac{1}{x^{2n-2k+1}}\right\}. \tag{9}$$

Consider the first of the Fourier transforms appearing to the right of the equality in (9). It can be found by applying the modulation property for the Fourier transform [3, p. A-14], namely

$$\mathscr{F}\{f(x)\cos x\} = \frac{\hat{f}(\omega+1) + \hat{f}(\omega-1)}{2}. \tag{10}$$

Here we will set $f_n(x) = 1/x^{2n+1}$. Since $\frac{1}{x^m}$ where m is a positive integer is not locally integrable its integral is to be interpreted as a Cauchy principal value (pv) integral which when differentiated with respect to x gives a Hadamard finite part (fp) integral. A homogeneous distribution can then be defined by the distributional derivative [1, pp. 241, 246]

$$\frac{1}{x^m} := \text{fp}\frac{1}{x^m} = \frac{(-1)^{m-1}}{(m-1)!}\frac{d^m}{dx^m}\log|x| = \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{dx^{m-1}}\text{pv}\frac{1}{x}. \tag{11}$$

If one now restricts the domain of the Fourier transform to the space

$$\mathscr{X} = \{\varphi \in \mathscr{S}(\mathbb{R}) : \varphi(0) = 0\},$$

where φ is any test function within the Schwartz space $\mathscr{S}(\mathbb{R})$ (the space of functions of $\mathscr{C}^\infty$ class that, along with all their derivatives, rapidly decay) then the Fourier transform of (11) is well defined as the Fourier transform of a tempered distribution [1, pp. 300–304]. Indeed one has [3, p. A-6]

$$\mathscr{F}\left\{\frac{1}{x^m}\right\} = \mathscr{F}\left\{\text{pv}\frac{1}{x^m}\right\} = -i\sqrt{\frac{\pi}{2}}\frac{(-i\omega)^{m-1}}{(m-1)!}\,\text{sgn}(\omega). \tag{12}$$

On setting $m = 2n+1$ we have

$$\mathscr{F}\left\{\frac{1}{x^{2n+1}}\right\} = -i\sqrt{\frac{\pi}{2}}\frac{(-i\omega)^{2n}}{(2n)!}\,\text{sgn}(\omega).$$

Combining this result with the modulation property given in (10) gives

$$\mathscr{F}\left\{\frac{\cos x}{x^{2n+1}}\right\} = -\frac{i}{2}\sqrt{\frac{\pi}{2}}\frac{(-1)^n}{(2n)!}\left[(\omega+1)^{2n}\text{sgn}(\omega+1) + (\omega-1)^{2n}\text{sgn}(\omega-1)\right]. \tag{13}$$

Next consider the second of the Fourier transforms appearing to the right of the equality in (9). From (12), on setting $m = 2n-2k+1 \geqslant 1$, as $n \geqslant k \geqslant 0$, we see that

$$\mathscr{F}\left\{\frac{1}{x^{2n-2k+1}}\right\} = -i\sqrt{\frac{\pi}{2}}\frac{(-1)^{n-k}\omega^{2n-2k}}{(2n-2k)!}\,\text{sgn}(\omega). \tag{14}$$

Thus

$$\sum_{k=0}^{n}\frac{(-1)^k}{(2k)!}\mathscr{F}\left\{\frac{1}{x^{2n-2k+1}}\right\} = -i\sqrt{\frac{\pi}{2}}\frac{(-1)^n\omega^{2n}}{(2n)!}\,\text{sgn}(\omega)\sum_{k=0}^{n}\binom{2n}{2k}\frac{1}{\omega^{2k}}.$$

Setting $x = 1/\omega$ in the first of the binomial identities given in Lemma 2 leads to

$$\sum_{k=0}^{n}\frac{(-1)^k}{(2k)!}\mathscr{F}\left\{\frac{1}{x^{2n-2k+1}}\right\} = -\frac{i}{2}\sqrt{\frac{\pi}{2}}\frac{(-1)^n}{(2n)!}\left[(\omega+1)^{2n} + (\omega-1)^{2n}\right]\text{sgn}(\omega). \tag{15}$$

Combining the results found in (13) and (15) into (9) one has

$$\hat{g}_n(\omega) = -\frac{i}{2}\sqrt{\frac{\pi}{2}}\frac{(-1)^n}{(2n)!}\Big[(\omega+1)^{2n}(\operatorname{sgn}(\omega+1)-\operatorname{sgn}(\omega)) \\ +(\omega-1)^{2n}(\operatorname{sgn}(\omega-1)-\operatorname{sgn}(\omega))\Big],$$

giving

$$|\hat{g}_n(\omega)|^2 = \frac{\pi}{2[(2n)!]^2}(1-|\omega|)^{4n}\chi_{[-1,1]}(\omega).$$

We are now in a position to apply Plancherel's theorem to the function $g_n(x)$. Doing so yields

$$\int_{-\infty}^{\infty}\left(\frac{\cos x-\cos_n(x)}{x^{2n+1}}\right)^2 dx = \frac{\pi}{2[(2n)!]^2}\int_{-1}^{1}(1-|\omega|)^{4n}\,d\omega = \frac{\pi}{(4n+1)[(2n)!]^2}.$$

As the integrand of the integral appearing to the left of the equality is even between symmetric limits, the desired result for I_n then follows. □

The proof of Theorem 2 proceeds in an almost identical manner as to Theorem 1.

Proof. We commence by first finding the Fourier transform for the function

$$g_n(x) = \frac{\sin x - \sin_n(x)}{x^{2n+2}},$$

where n is a non-negative integer. As $g_n(x) \in L^2(\mathbb{R})$, from the linearity property of the Fourier transform we are able to write

$$\hat{g}_n(\omega) = \mathscr{F}\{g_n(x)\} = \mathscr{F}\left\{\frac{\sin x}{x^{2n+2}}\right\} - \sum_{k=0}^{n}\frac{(-1)^k}{(2n+1)!}\mathscr{F}\left\{\frac{1}{x^{2n-2k+1}}\right\}. \tag{16}$$

Consider the first of the Fourier transforms appearing to the right of the equality in (16). It can be found by applying the modulation property for the Fourier transform [3, p. A-14], namely

$$\mathscr{F}\{f(x)\sin x\} = \frac{\hat{f}(\omega-1)-\hat{f}(\omega+1)}{2i}. \tag{17}$$

Here we will set $f_n(x) = 1/x^{2n+2}$. Setting $m = 2n+2$ in (12), on combining this result with (17) we find

$$\mathscr{F}\left\{\frac{\sin x}{x^{2n+2}}\right\} = \frac{i}{2}\sqrt{\frac{\pi}{2}}\frac{(-1)^n}{(2n+1)!}\Big[(\omega-1)^{2n+1}\operatorname{sgn}(\omega-1)-(\omega+1)^{2n+1}\operatorname{sgn}(\omega+1)\Big]. \tag{18}$$

The second of the Fourier transforms appearing to the right of the equality in (16) has previously been found in Theorem 1. It is just (14). Thus

$$\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}\mathscr{F}\left\{\frac{1}{x^{2n-2k+1}}\right\} = -i\sqrt{\frac{\pi}{2}}\frac{(-1)^n\omega^{2n}}{(2n)!}\operatorname{sgn}(\omega)\sum_{k=0}^{n}\binom{2n}{2k}\frac{1}{(2k+1)\omega^{2k}}.$$

Setting $x = 1/\omega$ in the second of the binomial identities given in Lemma 2 leads to

$$\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} \mathscr{F}\left\{\frac{1}{x^{2n-2k+1}}\right\} = -\frac{i}{2}\sqrt{\frac{\pi}{2}} \frac{(-1)^n}{(2n+1)!} \left[(\omega+1)^{2n+1} - (\omega-1)^{2n+1}\right] \operatorname{sgn}(\omega). \tag{19}$$

Combining the results found in (18) and (19) into (16) one has

$$\hat{g}_n(\omega) = \frac{i}{2}\sqrt{\frac{\pi}{2}} \frac{(-1)^n}{(2n+1)!} \left[(\omega-1)^{2n+1} (\operatorname{sgn}(\omega-1) - \operatorname{sgn}(\omega)) \right.$$
$$\left. -(\omega+1)^{2n+1} (\operatorname{sgn}(\omega+1) - \operatorname{sgn}(\omega))\right],$$

giving

$$|\hat{g}_n(\omega)|^2 = \frac{\pi}{2[(2n+1)!]^2} (1-|\omega|)^{4n+2} \chi_{[-1,1]}(\omega).$$

We are now in a position to apply Plancherel's theorem to the function $g_n(x)$. Doing so yields

$$\int_{-\infty}^{\infty} \left(\frac{\sin x - \sin_n(x)}{x^{2n+2}}\right)^2 dx = \frac{\pi}{2[(2n+1)!]^2} \int_{-1}^{1} (1-|\omega|)^{4n+2} d\omega = \frac{\pi}{(4n+3)[(2n+1)!]^2}.$$

As the integrand of the integral appearing to the left of the equality is even between symmetric limits, the desired result for Λ_n then follows. □

Finally, as the proof for Theorem 3 follows in a manner similar to the proofs given in Theorems 1 and 2, they will not be given here.

3. An open problem

From Lemma 1 there is one other positive integer value in the index for x appearing in the denominator for which the improper integrals given in (a) and (b) of the lemma converge. They are

$$\int_0^{\infty} \frac{(\cos x - \cos_n(x))^2}{x^{4n+3}} dx, \tag{20}$$

and

$$\int_0^{\infty} \frac{(\sin x - \sin_n(x))^2}{x^{4n+5}} dx. \tag{21}$$

The method of Fourier transforms cannot be applied to either of these improper integrals in order to find their general forms as Fourier transforms for

$$\mathscr{F}\left\{\frac{1}{x^{2n+\frac{3}{2}}}\right\} \quad \text{and} \quad \mathscr{F}\left\{\frac{1}{x^{2n+\frac{5}{2}}}\right\},$$

are required but are not known. The evaluation of improper integrals (20) and (21) can at least be made for low orders in n. One way to achieve this is by using a result from the theory of Laplace transforms. Under suitable conditions on the regularity of the functions f and g as $x \to 0^+$ and their rate of decay as $x \to \infty$ one has

$$\int_0^\infty f(x)g(x)\,dx = \int_0^\infty \mathscr{L}\{f(x)\}\mathscr{L}^{-1}\{g(x)\}\,ds. \tag{22}$$

Here $\mathscr{L}$ and $\mathscr{L}^{-1}$ denote the Laplace transform and the inverse Laplace transform respectively. As a technique for the evaluation of improper integrals it is quite old (see, for example, [2, pp. 209–212]) though appears to be not widely known.

Applying the result of (22) to the improper integral given by (20) when $n = 0$, one has

$$\begin{aligned}\int_0^\infty \frac{(\cos x - 1)^2}{x^3}\,dx &= \int_0^\infty \mathscr{L}\left\{(\cos x - 1)^2\right\}\mathscr{L}^{-1}\left\{\frac{1}{x^3}\right\}ds\\ &= \int_0^\infty \frac{3s}{(s^2+1)(s^2+4)}\,ds\\ &= \int_0^\infty \left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]ds\\ &= \frac{1}{2}\left[\log\left(\frac{s^2+1}{s^2+4}\right)\right]_0^\infty = \log(2).\end{aligned}$$

In a similar manner it can be shown that

$$n = 1: \int_0^\infty \frac{\left(\cos x - 1 + \frac{x^2}{2!}\right)^2}{x^7}\,dx = \frac{2}{45}\log(2) - \frac{11}{720},$$

$$n = 2: \int_0^\infty \frac{\left(\cos x - 1 + \frac{x^2}{2!} - \frac{x^4}{4!}\right)^2}{x^{11}}\,dx = \frac{2}{14\,175}\log(2) - \frac{1321}{21\,772\,800}.$$

Based on these low order results for n we conjure that

$$\int_0^\infty \frac{(\cos x - \cos_n(x))^2}{x^{4n+3}}\,dx = a_n\log(2) - b_n,$$

where $a_n = 2^{4n+1}/(4n+2)!$ and b_n is a non-negative rational number.

Likewise, for lower orders in n for the improper integral (21) it can be shown that

$$n = 0: \int_0^\infty \frac{(\sin x - x)^2}{x^5}\,dx = \frac{1}{3}\log(2) - \frac{1}{12},$$

$$n = 1: \int_0^\infty \frac{\left(\sin x - x + \frac{x^3}{3!}\right)^2}{x^9}\,dx = \frac{1}{315}\log(2) - \frac{19}{15\,120},$$

$$n = 2: \int_0^\infty \frac{\left(\sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!}\right)^2}{x^{13}}\,dx = \frac{2}{467\,775}\log(2) - \frac{6983}{3\,592\,512\,000}.$$

Based on these low order results for n we conjure that

$$\int_0^\infty \frac{(\sin x - \sin_n(x))^2}{x^{4n+5}}\,dx = \alpha_n \log(2) - \beta_n,$$

where $\alpha_n = 2^{4n+3}/(4n+4)!$ and β_n is a positive rational number. The problem of determining general expressions for b_n and β_n remain open.

REFERENCES

[1] W. APPEL, *Mathematics for Physics and Physicists*, Princeton University Press, Princeton, 2007.
[2] J. EDWARDS, *A Treatise on the Integral Calculus with Applications, Examples and Problems*, Macmillan and Co., London, 1922.
[3] D. W. KAMMLER, *A First Course in Fourier Analysis*, Cambridge University Press, Cambridge, 2007.
[4] C. LINDMAN, *De integralibus quibusdam definitis*, Arch. Math. Phys., **16**, (1851), 94–103.
[5] S. M. STEWART, *Problem 2092*, Math. Mag., **93**, 2 (2020), 150.
[6] B. VAN DER POL AND J. WICHERS, *Vraagstuk CIL*, Wiskundige Opgaven met de Oplossingen, **17**, (1942), 364–365.
[7] J. WOLSTENHOLME, *Mathematical Problems on the First and Second Divisions of the Schedule Subjects for the Cambridge Mathematical Tripos Examination*, Second Edition, Macmillan and Co., London, 1878.

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