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STATISTICAL CONVERGENCE IN 2–METRIC SPACES

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Abstract. In this paper, notions of statistical convergence, statistically Cauchy sequence, $\mathcal{N}_\theta$ and lacunary statistical convergence in 2-metric space will be introduced. Also some inclusion relations between these concepts will investigated.

1. Introduction

Definition of statistical convergence of number sequences was given by Fast [5]. In [22] Schoenberg obtained some fundamental properties of statistical convergence and also examined the concept as a summability method.

Let (X,d) be a metric space. If for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : \ d(x_k,x) \geqslant \varepsilon\}| = 0,$$

then we say that (x_n) is statistically convergent to x, here $|K|$ denotes the cardinality of the set K. For this case we write $st-\lim x_n = x$. $\lim x_n = x$ implies $st-\lim x_n = x$, therefore statistical convergence is a regular summability method.

f is said to be statistically continuous at x provided that whenever a sequence (x_n) is statistically convergent to x then the sequence $(f(x_n))$ is statistically convergent to $f(x)$ (see [4]).

In 1960's, Gähler has introduced the very important concept of 2-metric by generalizing the concept of metric. The concept of 2-metric gives the basic properties of the area function of a triangle determined by three vertices in Euclidean spaces. Gähler has studied the various properties of 2-metric spaces in his papers [11, 12]. Gähler first defined 2-metric space as follows:

DEFINITION 1. [11, 12] Let $X \neq \emptyset$. $d : X^3 \to \mathbb{R}$ is said to be a 2-metric on X if

(M1) given distinct elements $u,v \in X$, there exists an element $w \in X$ such that $d(u,v,w) \neq 0$

(M2) $d(u,v,w) = 0$ when at least two of $,v,w$ are equal,

(M3) $d(u,v,w) = d(u,w,v) = d(v,w,u)$ for all $u,v,w \in X$, and

(M4) $d(u,v,w) \leqslant d(u,v,z) + d(u,z,w) + d(z,v,w)$ for all $u,v,w,z \in X$.

When d is a 2-metric on X, then the ordered pair (X, d) is called a 2-metric space.

Very typical example of 2-metric $d(u,v,w)$ is the area of the triangle spanned by u,v,w.

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EXAMPLE 1. Take $X = [0,1]$. Define $d : X^3 \to \mathbb{R}$ as

$$d(u,v,w) = \min\{|u-v|, |v-u|, |w-u|\}$$

where $u,v,w \in X$. Now, (X,d) is a 2-metric space.

DEFINITION 2. [11, 12] (X,d) is said to be bounded if $sup\{d(u,v,w) : u,v,w \in X\} < \infty$.

DEFINITION 3. [11, 12] A sequence (x_n) in X is said to be a Cauchy sequence if for all $a \in X$,

$$\lim_{n,m\to\infty} d(x_n,x_m,a) = 0.$$

DEFINITION 4. [11, 12] A sequence (x_n) in X is said to be convergent to an element $x \in X$ if for all $a \in X$,

$$\lim_{n\to\infty} d(x_n,x,a) = 0.$$

DEFINITION 5. [11, 12] A complete 2-metric space is one in which every Cauchy sequence in X converges to an element of X.

A 2-metric space is not topologically equivalent to an ordinary metric. For example, every metric space is first countable, but 2-metric spaces may not first countable[16]. In this case, there is no simple relationship between the results obtained in metric spaces and the results obtained in 2-metric spaces. In a metric space a convergent sequence is a Cauchy sequence but in a 2-metric space may not be a Cauchy sequence, but if the 2-metric d is continuous on X, then each convergent sequence becomes a Cauchy sequence[19]. Although a metric is continuous on X, the 2-metric may not be continuous.

This concept has been considered by many authors (see [7], [14], [15], [17], [20], [21], [23], [24]). The reader may refer to the textbooks/monographs [2] and [18] for sequence spaces and related topics, and basic concepts of summability theory.

In Section 2, notions of statistical convergence and statistically Cauchy sequence in 2-metric space will be introduced. In Section 3, notions of N_θ and lacunary statistical convergence in 2-metric spaces will be introduced. In Section 3 we will investigate some inclusion relations between these concepts.

2. Cesàro and statistical convergence

In this section, we present the definitions of Cesàro and statistical convergence in 2-metric spaces and the relations between them.

DEFINITION 6. A sequence (x_n) in X is said to be Cesàro convergent to an element $x \in X$ if for all $a \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(x_n,x,a) = 0.$$

Let $\mathscr{C}$ be the set of Cesàro convergent sequences, that is,

$$\mathscr{C} = \{(x_n) \subseteq X : \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(x_n, x, a) = 0, \quad \text{for some } x\}.$$

DEFINITION 7. A sequence (x_n) in X is said to be a statistically Cauchy sequence if for all $a \in X$ and for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} |\{k, \ell \leqslant n : \ d(x_k, x_\ell, a) \geqslant \varepsilon\}| = 0$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{n} |\{k, \ell \leqslant n : \ d(x_k, x_\ell, a) < \varepsilon\}| = 1.$$

In this case, we write

$$st - \lim_{n,m\to\infty} d(x_n, x_m, a) = 0.$$

DEFINITION 8. A sequence (x_n) in X is said to be statistically convergent to an element $x \in X$ if for all $a \in X$ and for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : \ d(x_k, x, a) \geqslant \varepsilon\}| = 0$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : \ d(x_k, x, a) < \varepsilon\}| = 1.$$

In this case, we write

$$st - \lim_{n,m\to\infty} d(x_n, x, a) = 0.$$

Let $\mathscr{S}$ be the set of statistically convergent sequences, i.e.,

$$\mathscr{S} = \{(x_n) \subseteq X : \quad \lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : \ d(x_k, x, a) \geqslant \varepsilon\}| = 0, \quad \text{for some } x\}.$$

In a metric space a statistically convergent sequence is a statistically Cauchy sequence but in a 2-metric space a statistically convergent sequence may not be a statistically Cauchy sequence.

EXAMPLE 2. Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Define $d : X^3 \to [0, \infty)$ by

$$d(x,y,z) = \begin{cases} 1, & \text{if } x \neq y \neq z \text{ and } \{\frac{1}{n}, \frac{1}{n+1}\} \subset \{x,y,z\} \text{ for } n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

Then (X,d) is a complete 2-metric space. Let define the sequence (x_n) by

$$x_n = \begin{cases} 1, & n \text{ is square integer} \\ \frac{1}{n}, & \text{otherwise} \end{cases}$$

The sequence (x_n) is statistically convergent to 0 but is not a statistically Cauchy sequence.

THEOREM 1. *Let (X,d) be 2-metric space and d be statistically continuous on X. If the sequence (x_n) is statistically convergent then (x_n) is statistically Cauchy.*

Proof. If $st-\lim x_n = x$, then by statistical continuity of d, we have

$$st-\lim_{n,m\to\infty} d(x_n,x_m,x) = st-\lim_{n\to\infty} d(x_n,x,x). \tag{1}$$

By using property (iv) of 2-metric, we can write

$$d(x_n,x_m,a) \leqslant d(x_n,x,a)+d(x,x_m,a)+d(x_n,x_m,x).$$

From this inequality and (2.1), we get that

$$st-\lim_{n,m\to\infty} d(x_n,x_m,a) = 0$$

for all $a\in X$, that is, (x_n) is statistically Cauchy. $\square$

THEOREM 2. *Let (x_n) be a sequence in 2-metric space (X,d). Then*

1. *if (x_n) is Cesàro convergent to x then (x_n) statistically convergent to x.*
2. *If (X,d) is bounded and (x_n) statistically convergent to x then (x_n) Cesàro convergent to x.*

Proof. i) Let (x_n) be Cesàro convergent to x. For $\varepsilon>0$ and every $a\in X$, we get

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n} d(x_k,x,a) &= \left(\frac{1}{n}\sum_{\substack{k=1\\ d(x_k,x,a)\geqslant\varepsilon}}^{n} d(x_k,x,a)+\frac{1}{n}\sum_{\substack{k=1\\ d(x_k,x,a)<\varepsilon}}^{n} d(x_k,x,a)\right)\\ &\geqslant \frac{1}{n}\sum_{\substack{k=1\\ d(x_k,x,a)\geqslant\varepsilon}}^{n} d(x_k,x,a)\\ &\geqslant \frac{1}{n}|\{1\leqslant k\leqslant n : d(x_k,x,a)\geqslant\varepsilon\}|\varepsilon.\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty}\frac{1}{n}|\{1\leqslant k\leqslant n : d(x_k,x,a)\geqslant\varepsilon\}| = 0$$

that is, (x_n) statistically convergent to x.

ii) Now suppose that (x_n) is bounded and statistically convergent to x, since (X,d) is bounded, say $d(x_k,x,a)\leqslant K$ for all k and every $a\in X$. For $\varepsilon>0$ and every $a\in X$

we get

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} d(x_k,x,a) &= \frac{1}{n}\left(\sum_{\substack{k=1\\ d(x_k,x,a)\geqslant\varepsilon}}^{n} d(x_k,x,a) + \sum_{\substack{k=1\\ d(x_k,x,a)<\varepsilon}}^{n} d(x_k,x,a)\right)\\ &\leqslant \frac{1}{n}\left(K\sum_{\substack{k=1\\ d(x_k,x,a)\geqslant\varepsilon}}^{n} 1 + \sum_{\substack{k=1\\ d(x_k,x,a)<\varepsilon}}^{n} d(x_k,x,a)\right)\\ &\leqslant K\frac{1}{n}\left|\{1\leqslant k\leqslant n : d(x_k,x,a)\geqslant \varepsilon_k\}\right| + \frac{1}{n}\sum_{k=1}^{n}\varepsilon.\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} d(x_k,x,a) = 0,$$

that is (x_n) is Cesàro convergent to x. □

3. $\mathscr{N}_\theta$ and lacunary statistical convergence

First, we recall the concept of lacunary sequences. A lacunary sequence [6] is an increasing integer sequence $\theta = (p_r)$ such that $p_0 = 0$ and $t_r = p_r - p_{r-1} \to \infty$ as $r \to \infty$. The intervals $(p_{r-1}, p_r]$ determined by $\theta = (p_r)$ will be denoted by J_r

DEFINITION 9. Let $\theta = (p_r)$ be any lacunary sequence. A sequence (x_n) in X is said to be $\mathscr{N}_\theta$-convergent to an element $x \in X$ if for all $a \in X$ and for every $\varepsilon > 0$,

$$\lim_{r\to\infty}\frac{1}{t_r}\sum_{k\in J_r} d(x_k,x,a) = 0.$$

The set of $\mathscr{N}_\theta$-convergent sequences will be denoted $\mathscr{N}_\theta$.

DEFINITION 10. Let $\theta = (p_r)$ be any lacunary sequence. A sequence (x_n) in X is said to be lacunary statistically convergent to an element $x \in X$ if for all $a \in X$ and for every $\varepsilon > 0$,

$$\lim_{r\to\infty}\frac{1}{t_r}\left|\{k\in J_r :\ d(x_k,x,a)\geqslant\varepsilon\}\right| = 0$$

or equivalently

$$\lim_{r\to\infty}\frac{1}{t_r}\left|\{k\in J_r :\ d(x_k,x,a)<\varepsilon\}\right| = 1.$$

Let $\mathscr{S}_\theta$ be the set of lacunary statistically convergent sequences, that is,

$$\mathscr{S}_\theta = \{(x_n)\subseteq X :\quad \lim_{r\to\infty}\frac{1}{t_r}\left|\{k\in J_r :\ d(x_k,x,a)\geqslant\varepsilon\}\right| = 0, \quad \text{for some } x\}.$$

The proof of the following theorem similar to that of Theorem 1, so we state it without proof.

THEOREM 3. *Let (x_n) be a sequence in 2-metric space (X,d) and $\theta = (p_r)$ be any lacunary sequence. Then, the following statements hold:*

1. *if (x_n) is $\mathscr{N}_\theta$-convergent to x then (x_n) $\mathscr{S}_\theta$-convergent to x.*
2. *If (X,d) is bounded and (x_n) $\mathscr{S}_\theta$-convergent to x then (x_n) $\mathscr{N}_\theta$- convergent to x.*

By using the similar techniques to that in Lemmas 2.1 and 2.2 of [6], we can prove following theorems. These theorems state the relationships between Cesàro convergence and $\mathscr{N}_\theta$-convergence and between statistically convergence and lacunary statistically convergence in 2-metric space.

THEOREM 4. *Let $\theta = (p_r)$ be any lacunary sequence and (X,d) be a 2-metric space. Then the following statements hold:*

1. *$\mathscr{C} \subseteq \mathscr{N}_\theta$ if and only if* $\liminf_r \frac{p_r}{p_{r-1}} > 1$.
2. *$\mathscr{N}_\theta \subseteq \mathscr{C}$ if and only if* $\limsup_r \frac{p_r}{p_{r-1}} < \infty$.
3. *$\mathscr{C} = \mathscr{N}_\theta$ if and only if*

$$1 < \liminf_r \frac{p_r}{p_{r-1}} \leqslant \limsup_r \frac{p_r}{p_{r-1}} < \infty.$$

Proof. We only prove (1). The others can be proved by the similar way used in proving Lemmas 2.1 and 2.2 of [6].

If $\liminf_r \frac{p_r}{p_{r-1}} > 1$ there exists $\alpha > 0$ such that $1+\alpha \leqslant \frac{p_r}{p_{r-1}}$ for all $r \geqslant 1$. Suppose that $(x_n) \in \mathscr{C}$, hence we can write

$$\begin{aligned} \frac{1}{t_r}\sum_{k\in J_r} d(x_k,x,a) &= \frac{1}{t_r}\sum_{k=1}^{p_r} d(x_k,x,a) - \frac{1}{t_r}\sum_{k=1}^{p_{r-1}} d(x_k,x,a) \\ &= \frac{p_r}{t_r}\Big(\frac{1}{p_r}\sum_{k=1}^{p_r} d(x_k,x,a)\Big) - \frac{p_{r-1}}{t_r}\Big(\frac{1}{p_{r-1}}\sum_{k=1}^{p_{r-1}} d(x_k,x,a)\Big) \end{aligned}$$

Since $t_r = p_r - p_{r-1}$, we have

$$\frac{p_r}{t_r} \leqslant \frac{1+\alpha}{\alpha} \text{ and } \frac{p_{r-1}}{t_r} \leqslant \frac{1}{\alpha}.$$

Therefore right side of above equality tends to zero as $r \to \infty$, it follows that $x \in \mathscr{N}_\theta$.

Suppose that $\liminf_r \frac{p_r}{p_{r-1}} = 1$. We can select a subsequence (p_{r_i}) of lacunary sequence $\theta = (p_r)$ such that

$$\frac{p_{r_i}}{p_{r_i-1}} < 1 + \frac{1}{i} \text{ and } \frac{p_{r_i-1}}{p_{r_{i-1}}} > i \text{ where } r_i \geqslant r_{i-1} + 2.$$

Now consider the 2-metric d in Example 1.1, and define (x_k) by

$$x_k = \begin{cases} 1, \; k \in J_{r_i}, & \text{for some } i = 1,2,\dots \\ 0, & \text{otherwise} \end{cases}$$

Then for any $x \in X$ and for all $a \in X$,

$$\frac{1}{t_{r_i}} \sum_{k \in J_{r_i}} d(x_k, x, a) = |1 - x| \quad \text{for } i = 1,2,3,\dots.$$

$$\frac{1}{t_r} \sum_{k \in J_r} d(x_k, x, a) = |x| \quad \text{for } r \neq r_i.$$

It follows that $(x_k) \notin \mathscr{N}_\theta$. But (x_k) is Cesàro convergent, since if m is any sufficiently large integer we can find the unique i for which $p_{r_i - 1} < m \leqslant p_{r_{i+1}-1}$ and write

$$\frac{1}{m} \sum_{k=1}^{m} d(x_k, x, a) \leqslant \frac{p_{r_{i-1}} + t_{r_i}}{p_{r_i - 1}} \leqslant \frac{2}{i}$$

as $m \to \infty$ it follows that $(x_k) \in \mathscr{C}$. $\square$

THEOREM 5. *Let $\theta = (p_r)$ be any lacunary sequence and (X,d) be a 2-metric space. Then the following statements hold:*

1. $\mathscr{S} \subseteq \mathscr{S}_\theta$ *if and only if* $\liminf_r \frac{p_r}{p_{r-1}} > 1$.
2. $\mathscr{S}_\theta \subseteq \mathscr{S}$ *if and only if* $\limsup_r \frac{p_r}{p_{r-1}} < \infty$.
3. $\mathscr{S} = \mathscr{S}_\theta$ *if and only if*

$$1 < \liminf_r \frac{p_r}{p_{r-1}} \leqslant \limsup_r \frac{p_r}{p_{r-1}} < \infty.$$

THEOREM 6. *Let $\theta = (p_r)$ be any lacunary sequence and (X,d) be a 2-metric space. If $(x_n) \in \mathscr{S} \subseteq \mathscr{S}_\theta$, then $\mathscr{S}_\theta - \lim(x_n) = \mathscr{S} - \lim(x_n)$.*

Proof. Assume that $\mathscr{S} - \lim(x_n) = x$ and $\mathscr{S}_\theta - \lim(x_n) = y$ and $x \neq y$. Then $d(x,y,a) \neq 0$. From $(M3)$ and $(M4)$ we can write

$$d(x, y.a) \leqslant d(x_k, x, y) + d(x_k, x, a) + d(x_k, y, a) \tag{2}$$

for all $x_k, x, y, a \in X$. For all $a \in X$, taking $\varepsilon < \frac{1}{3} d(x,y,a)$ from inequality (3.1), we get

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leqslant n : \; d(x_k, y, a) \geqslant \varepsilon\}| = 1.$$

Consider the ith term of the statistical limit expression $\lim_{n\to\infty}\frac{1}{n}|\{k\leqslant n:\ d(x_k,y,a)\geqslant\varepsilon\}|$:

$$\begin{aligned}
&\frac{1}{k_i}\Big|\Big\{k\in\bigcup_{r=1}^{i}J_r:\ d(x_k,y,a)\geqslant\varepsilon\Big\}\Big| \\
&=\frac{1}{k_i}\sum_{r=1}^{i}|\{k\in J_r:\ d(x_k,y,a)\geqslant\varepsilon\}| \\
&=\frac{1}{\sum_{r=1}^{i}t_r}\sum_{r=1}^{i}t_r\frac{1}{t_r}|\{k\in J_r:\ d(x_k,y,a)\geqslant\varepsilon\}|\to 0.
\end{aligned} \tag{3}$$

Since $\theta=(p_r)$ is a lacunary sequence (3.2) is a regular weighted mean transformation of the sequence converging to zero, so that itself converges to zero as $i\to\infty$. Also since this is a subsequence of

$$\left\{\frac{1}{n}|\{k\leqslant n:\ d(x_k,y,a)\geqslant\varepsilon\}|\right\}_n,$$

we infer that

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leqslant n:\ d(x_k,y,a)\geqslant\varepsilon\}|\neq 1,$$

and this contradiction shows that we can not take $x\neq y$. $\square$

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