Journal of Classical Analysis
Volume 17, Number 1 (2021), 1–11 doi:10.7153/jca-2021-17-01

A POISSON LOGARITHMIC INTEGRAL FOR INTEGER ORDER POWERS n = 0, 1, 2, AND 3

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Abstract. We give analytic expressions for the Poisson type logarithmic integral

$$\mathrm{Lp}_n(a) = \int_0^\pi \log^n(1-2a\cos x+a^2)\,dx,$$

for integer order powers $n = 0, 1, 2$, and 3. Here a is any real number. A generalisation of the integral for the $n = 2$ case is also given.

1. Introduction

In 1815 Poisson [1] introduced and evaluated for the first time the integral

$$\mathrm{Lp}_1(a) = \int_0^\pi \log(1-2a\cos x+a^2)\,dx. \tag{1}$$

Here $a \in \mathbb{R}$. Several integrals are known as Poisson integrals. In this paper we shall refer to the logarithmic integral appearing in (1) as *Poisson's log-cosine integral*. When evaluated one finds

$$\mathrm{Lp}_1(a) = \begin{cases} 0, & |a| \leqslant 1, \\ \pi\log(a^2), & |a| > 1. \end{cases} \tag{2}$$

Poisson proved his result using what is essentially a Fourier cosine series expansion for the logarithmic term [1, pp. 617–618]. It was subsequently proved by others using the definition of a Riemann sum for the definite integral [2] [3, pp. 471–472], using a functional equation [4], [5], [3, pp. 258–259], and using parametric differentiation. Modern accounts for each of these approaches can be found in [6].

In this paper we will evaluate the Poisson log-cosine integral for the next two higher order integer powers of the logarithm. More specifically, we shall obtain closed-form expressions for the integral

$$\mathrm{Lp}_n(a) = \int_0^\pi \log^n(1-2a\cos x+a^2)\,dx, \tag{3}$$

for the cases $n = 0, 1, 2$, and 3. In passing we note only the result for the case $n = 1$ can be found in well-known table of integrals [7, Entry 4.224.15], [8, Entry 322.14].

Mathematics subject classification (2010): Primary: 40C10; Secondary: 42A16.
Keywords and phrases: Poisson integral, log-sine integral, polylogarithm, Euler sums.

The approach to be used in all evaluations uses a Fourier cosine series expansion. An evaluation for the generalisation of the $n=2$ case in the form

$$\mathrm{Lp}_2(a,b) = \int_0^{\pi} \log(1-2a\cos x + a^2)\log(1-2b\cos x+b^2)\,dx, \tag{4}$$

will also be given using the same approach. Here $a,b \in \mathbb{R}$.

We begin with three elementary observations for the function $\mathrm{Lp}_n(a)$. Firstly, $\mathrm{Lp}_n(0)=0$ for $n\in\mathbb{N}$. Secondly, $\mathrm{Lp}_n(-a)=\mathrm{Lp}_n(a)$ for $n\in\mathbb{N}_0$ where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Thirdly, $\mathrm{Lp}_0(a)=\pi$. In this last result, when $a=0$ we have taken $0^0=1$. Next we collect together some results that will be needed in the ensuing analysis.

LEMMA 1. (A Fourier cosine series expansion) *For* $x\in(0,\pi)$

$$\log(1-2a\cos x+a^2) = \begin{cases} -2\displaystyle\sum_{n=1}^{\infty}\frac{a^n}{n}\cos(nx), & |a|<1,\\ \log(a^2)-2\displaystyle\sum_{n=1}^{\infty}\frac{a^{-n}}{n}\cos(nx), & |a|>1. \end{cases}$$

Proof. For $|a|<1$, from

$$\log(1-2a\cos x+a^2) = \log(1-ae^{ix})+\log(1-ae^{-ix}),$$

and

$$\log(1-z) = -\sum_{n=1}^{\infty}\frac{z^n}{n}, \qquad |z|<1,$$

we see that

$$\log(1-2a\cos x+a^2) = -\sum_{n=1}^{\infty}\frac{a^n}{n}e^{inx}-\sum_{n=1}^{\infty}\frac{a^n}{n}e^{-inx} = -2\sum_{n=1}^{\infty}\frac{a^n}{n}\cos(nx).$$

For $|a|>1$, we write

$$\log(1-2a\cos x+a^2) = \log(a^2)+\log\left(1-\frac{2}{a}\cos x+\frac{1}{a^2}\right),$$

and proceeding as was done above we immediately find

$$\log(1-2a\cos x+a^2) = \log(a^2)-2\sum_{n=1}^{\infty}\frac{a^{-n}}{n}\cos(nx). \qquad \square$$

LEMMA 2. (A Cauchy product series expansion) *For* $|x|<1$

$$\log^2(1-x) = 2\sum_{n=2}^{\infty}\frac{H_{n-1}}{n}x^n.$$

Here H_n *is the nth harmonic number defined for* $n\in\mathbb{N}$ *by* $H_n=\sum_{k=1}^{n}\frac{1}{k}$.

Proof. From the well-known result for the generating function of the harmonic numbers [7, Entry 1.513.6], namely

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\log(1-x)}{1-x}, \quad |x|<1, \tag{5}$$

Replacing x with t before integrating from 0 to x yields

$$\sum_{n=1}^{\infty} \frac{H_n x^{n+1}}{n+1} = -\int_0^x \frac{\log(1-t)}{1-t}\,dt = \frac{1}{2}\log^2(1-x).$$

The desired result then follows on reindexing the sum $n \mapsto n-1$. □

LEMMA 3. (A harmonic number generating function) *For* $|x|<1$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} x^n = \operatorname{Li}_3(x) - \operatorname{Li}_3(1-x) + \operatorname{Li}_2(1-x)\log(1-x) + \frac{1}{2}\log(x)\log^2(1-x) + \zeta(3).$$

Here ζ *is the Riemann zeta function defined by* $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$, $s>1$, *while* Li_2 *and* Li_3 *denote the dilogarithm and trilogarithm functions respectively with the polylogarithm function* Li_s *of order* s *defined by* $\operatorname{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$, *for* $s>1$ *and* $|z| \leqslant 1$.

Proof. Dividing both sides of the generating function of the harmonic numbers given by (5) by x, replacing x with t before integrating from 0 to x gives

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n}{n} x^n &= -\int_0^x \frac{\log(1-t)}{t(1-t)}\,dt = -\int_0^x \frac{\log(1-t)}{t}\,dt - \int_0^x \frac{\log(1-t)}{1-t}\,dt \\ &= \operatorname{Li}_2(x) + \frac{1}{2}\log^2(1-x),\end{aligned} \tag{6}$$

where in the last line the integral definition of $-\int_0^x \frac{\log(1-t)}{t}\,dt$ for the dilogarithm function has been used. Dividing both sides of the equality in (6) by x, replacing x with t before integrating from 0 to x gives

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} x^n = \int_0^x \frac{\operatorname{Li}_2(t)}{t}\,dt + \frac{1}{2}\int_0^x \frac{\log^2(1-t)}{t}\,dt. \tag{7}$$

Recognising $\int_0^x \frac{\operatorname{Li}_s(t)}{t}\,dt = \operatorname{Li}_{s+1}(x)$ takes care of the first of the integrals appearing in (7) while integrating the second by parts leads to

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} x^n = \operatorname{Li}_3(x) + \frac{1}{2}\log(x)\log^2(1-x) + \int_0^x \frac{\log(t)}{1-t}\log(1-t)\,dt. \tag{8}$$

Observing that $\frac{d}{dx}\operatorname{Li}_2(1-x) = \frac{\log(x)}{1-x}$, integrating the remaining integral in (8) by parts yields

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n}{n^2} x^n &= \operatorname{Li}_3(x) + \frac{1}{2}\log(x)\log^2(1-x) + \operatorname{Li}_2(1-x)\log(1-x) + \int_0^x \frac{\operatorname{Li}_2(1-t)}{1-t}\,dt \\ &= \operatorname{Li}_3(x) - \operatorname{Li}_3(1-x) + \operatorname{Li}_2(1-x)\log(1-x) + \frac{1}{2}\log(x)\log^2(1-x) + \zeta(3),\end{aligned}$$

as desired. Here, for the lower limit of integration we have used $\mathrm{Li}_3(1) = \zeta(3)$. $\square$

2. The well-known linear case

We present an evaluation for the result given in (2) as it demonstrates the method we intend to use for the squared, $n = 2$ generalised, and cubed cases.

From Lemma 1, when $|a| < 1$ replacing the logarithmic term in the integral for $\mathrm{Lp}_1(a)$ with its corresponding Fourier cosine series expansion, one has

$$\mathrm{Lp}_1(a) = -2\int_0^{\pi} \sum_{n=1}^{\infty} \frac{a^n}{n} \cos(nx)\, dx.$$

As

$$\sum_{n=1}^{\infty} \left| \frac{a^n}{n} \cos(nx) \right| \leqslant \sum_{n=1}^{\infty} \frac{|a|^n}{n} = -\log(1-|a|) < \infty, \tag{9}$$

Fubini's theorem applies so the order of integration with the summation may be interchanged allowing the integration to be performed termwise. Doing so yields

$$\mathrm{Lp}_1(a) = -2\sum_{n=1}^{\infty} \frac{a^n}{n} \int_0^{\pi} \cos(nx)\, dx = 0,$$

where the elementary result of $\int_0^{\pi} \cos(nx)\, dx = 0$, $n \in \mathbb{N}$, has been used.

In a similar manner, when $|a| > 1$, as

$$\sum_{n=1}^{\infty} \left| \frac{a^{-n}}{n} \cos(nx) \right| \leqslant \sum_{n=1}^{\infty} \frac{|a|^{-n}}{n} = -\log(1-|a|^{-1}) < \infty, \tag{10}$$

it is again clear Fubini's theorem applies justifying term-by-term integration. Thus

$$\begin{aligned}
\mathrm{Lp}_1(a) &= \int_0^{\pi} \left[\log(a^2) - 2\sum_{n=1}^{\infty} \frac{a^{-n}}{n} \cos(nx) \right] dx \\
&= \pi \log(a^2) - 2\sum_{n=1}^{\infty} \frac{a^{-n}}{n} \int_0^{\pi} \cos(nx)\, dx = \pi \log(a^2),
\end{aligned}$$

as required to show.

3. The squared case and a generalisation

We first show how, when the logarithmic term in Poisson's log-cosine integral is squared, it can be evaluated in terms of the dilogarithm function. This result is given in the following theorem.

THEOREM 1. *For $a \in \mathbb{R}$*

$$\mathrm{Lp}_2(a) = \begin{cases} 2\pi \,\mathrm{Li}_2(a^2), & |a| \leqslant 1, \\ \pi \log^2(a^2) + 2\pi \,\mathrm{Li}_2\left(\dfrac{1}{a^2}\right), & |a| > 1. \end{cases}$$

Proof. For the case $|a| < 1$, writing the squared logarithmic term in the integrand for $\mathrm{Lp}_2(a)$ as the product between two linear logarithmic terms before replacing each of these terms with their respective Fourier cosine series expansions, one has

$$\mathrm{Lp}_2(a) = 4\int_0^{\pi} \sum_{m=1}^{\infty} \frac{a^m}{m}\cos(mx) \sum_{k=1}^{\infty} \frac{a^k}{k}\cos(kx)\,dx, \quad |a| < 1.$$

Due to (9), Fubini's theorem applies so the order of integration with the summations may be interchanged allowing the integration to be performed termwise. Doing so yields

$$\begin{aligned}\mathrm{Lp}_2(a) &= 4\sum_{m=1}^{\infty} \frac{a^m}{m} \sum_{k=1}^{\infty} \frac{a^k}{k} \int_0^{\pi} \cos(mx)\cos(kx)\,dx \\ &= 4\sum_{m=1}^{\infty} \frac{a^m}{m} \sum_{k=1}^{\infty} \frac{a^k}{k} \cdot \frac{\pi}{2}\delta_{mk} \\ &= 2\pi \sum_{n=1}^{\infty} \frac{a^{2n}}{n^2} = 2\pi\,\mathrm{Li}_2(a^2).\end{aligned}$$

Here the elementary result of $\int_0^{\pi} \cos(mx)\cos(kx)\,dx = \frac{\pi}{2}\delta_{mk}$ where δ_{mk} is the Kronecker delta has been used.

For the case $|a| > 1$, we can write

$$\begin{aligned}\mathrm{Lp}_2(a) &= \int_0^{\pi} \log^2\left[a^2\left(1 - \frac{2}{a}\cos x + \frac{1}{a^2}\right)\right] dx \\ &= \pi\log^2(a^2) + 2\log(a^2)\int_0^{\pi} \log\left(1 - \frac{2}{a}\cos x + \frac{1}{a^2}\right) dx \\ &\quad + \int_0^{\pi} \log^2\left(1 - \frac{2}{a}\cos x + \frac{1}{a^2}\right) dx \\ &= \pi\log^2(a^2) + 2\log(a^2)\,\mathrm{Lp}_1\left(\frac{1}{a}\right) + \mathrm{Lp}_2\left(\frac{1}{a}\right).\end{aligned}$$

As the results for $\mathrm{Lp}_1(1/a)$ and $\mathrm{Lp}_2(1/a)$ follow from the results for $\mathrm{Lp}_1(a)$ and $\mathrm{Lp}_2(a)$ for the case $|a| < 1$, $a \neq 0$, with a replaced with $1/a$, the result for the case $|a| > 1$ is immediate.

Turning to the end-points $a = \pm 1$, at the upper end-point, on setting $a = 1$ the integral reduces to

$$\mathrm{Lp}_2(1) = \int_0^{\pi} \log^2(2 - 2\cos x)\,dx.$$

From the identity $1 - \cos x = 2\sin^2\frac{x}{2}$ we can rewrite the integral as

$$\mathrm{Lp}_2(1) = \int_0^{\pi} \log^2\left(4\sin^2\frac{x}{2}\right) dx = 4\int_0^{\pi} \log^2\left(2\sin\frac{x}{2}\right) dx. \tag{11}$$

Integrals of the type appearing in (11) have been well studied in the literature [9, 10, 11] so we need only quote the final result. It is

$$\int_0^{\pi} \log^2\left(2\sin\frac{x}{2}\right) dx = \frac{\pi^3}{12}. \tag{12}$$

Thus $\mathrm{Lp}_2(1) = \pi^3/3$. Noting that $\mathrm{Li}_2(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \pi^2/6$, as $\mathrm{Lp}_2(1) = \mathrm{Lp}_2(-1)$ the two particular cases of $a = \pm 1$ at either end-point can be combined into the case for $|a| < 1$ and completes the proof. $\square$

REMARK 1. The result given in Theorem 1 is not new, an evaluation for

$$\int_{-\pi}^{\pi} \log^2(1+2a\cos x + a^2)\, dx = 2\,\mathrm{Lp}_2(-a),$$

having been given in [12]. There the result was obtained more succinctly by applying Parseval's relation after the Fourier cosine series expansion for the logarithmic term had been found. The application of Parseval's relation, while well suited for the squared case, cannot be used once we move to the generalisation we are above to give for the squared case, nor for the higher order cubic case, and is the reason why we have chosen to follow the method we have.

A generalisation in the form of (4) that reduces to the squared case can also be given. We present this result in the next theorem.

THEOREM 2. *For* $a, b \in \mathbb{R}$

$$\mathrm{Lp}_2(a,b) = \begin{cases} 2\pi\,\mathrm{Li}_2(ab), & |ab| \leqslant 1 \text{ with } |a| \leqslant 1,\ |b| \leqslant 1, \\ \pi\log(a^2)\log(b^2) + 2\pi\,\mathrm{Li}_2\left(\dfrac{1}{ab}\right), & |ab| > 1 \text{ with } |a| > 1, |b| > 1, \\ 2\pi\,\mathrm{Li}_2\left(\dfrac{a}{b}\right), & |a| < 1 < |b|, \\ 2\pi\,\mathrm{Li}_2\left(\dfrac{b}{a}\right), & |b| < 1 < |a|. \end{cases}$$

Proof. We break the proof up into a number of different cases.

Case 1: $|ab| < 1$ with $|a|, |b| < 1$

Replacing the two logarithmic terms with their respective Fourier cosine series expansions given in Lemma 1 one has

$$\mathrm{Lp}_2(a,b) = 4\int_0^{\pi} \sum_{m=1}^{\infty} \frac{a^m}{m}\cos(mx) \sum_{k=1}^{\infty} \frac{b^k}{k}\cos(kx)\, dx.$$

The result then follows in a manner analogous to the proof given for the $|a| < 1$ case in Theorem 1.

Case 2: $|ab| > 1$ with $|a|, |b| > 1$

On replacing the two logarithmic terms with their respective Fourier cosine series expansions one obtains

$$\begin{aligned}\mathrm{Lp}_2(a,b) = \log(a^2)\log(b^2)\int_0^\pi dx - \log(a^2)\int_0^\pi \sum_{k=1}^\infty \frac{a^{-k}}{k}\cos(kx)\,dx \\ - \log(b^2)\int_0^\pi \sum_{m=1}^\infty \frac{b^{-m}}{m}\cos(mx)\,dx \\ + 4\int_0^\pi \sum_{m=1}^\infty \frac{a^{-m}}{m}\cos(mx)\sum_{k=1}^\infty \frac{b^{-k}}{k}\cos(kx)\,dx.\end{aligned}$$

Due to (10), Fubini's theorem applies so the order of integration with the summations may be interchanged allowing the integration to be performed termwise. Doing so yields the required result in a manner analogous to the proof given for the $|a| > 1$ case in Theorem 1.

Cases 3 and 4: $|a| < 1 < |b|$ and $|b| < 1 < |a|$

As the procedure is similar to that already presented, the results are immediate.

Case 5: $a = b = \pm 1$

It is elementary to show $\mathrm{Lp}_2(a,b) = \mathrm{Lp}_2(-a,-b)$. Setting $a = b = \pm 1$ the integral reduces to

$$\mathrm{Lp}_2(\pm 1, \pm 1) = \int_0^\pi \log^2(2 - 2\cos x)\,dx = 4\int_0^\pi \log^2\left(2\sin\frac{x}{2}\right)dx = \frac{\pi^3}{3},$$

where the value for the integral given in (12) has been used.

Case 6: $a = \pm 1, b = \mp 1$

On setting $a = \pm 1$ and $b = \mp 1$ the integral reduces to

$$\begin{aligned}\mathrm{Lp}_2(\pm 1, \mp 1) &= \int_0^\pi \log(2 + 2\cos x)\log(2 - 2\cos x)\,dx \\ &= 4\int_0^\pi \log\left(2\cos\frac{x}{2}\right)\log\left(2\sin\frac{x}{2}\right)dx.\end{aligned}$$

Integrals of the type appearing on the second line have been well studied in the past [10, 13], so we need only quote the final result. It is

$$\int_0^\pi \log\left(2\cos\frac{x}{2}\right)\log\left(2\sin\frac{x}{2}\right)dx = -\frac{\pi^3}{24}.$$

Thus $\mathrm{Lp}_2(\pm 1, \mp 1) = -\pi^3/6$. Noting that $\mathrm{Li}_2(1) = \sum_{n=1}^\infty \frac{1}{n^2} = \zeta(2) = \pi^2/6$ and $\mathrm{Li}_2(-1) = \sum_{n=1}^\infty \frac{(-1)^n}{n^2} = -\frac{1}{2}\sum_{n=1}^\infty \frac{1}{n^2} = -\frac{1}{2}\zeta(2) = -\pi^2/12$, the particular cases found in *Cases* 5 and 6 can be combined into *Case* 1 and completes the proof. □

4. The cubed case

For the case when $n=3$ in $\operatorname{Lp}_n(a)$ the result is given in the next theorem,

THEOREM 3. *For $a \in \mathbb{R}$*

$$\operatorname{Lp}_3(a) = \begin{cases} 12\pi\left[\operatorname{Li}_3(1-a^2) - \operatorname{Li}_2(1-a^2)\log(1-a^2) \right. \\ \left. -\frac{1}{2}\log(a^2)\log^2(1-a^2) - \zeta(3)\right], & |a| \leqslant 1, \\ \pi\log^3(a^2) + 12\pi\left[\frac{1}{2}\log(a^2)\operatorname{Li}_2\left(\frac{1}{a^2}\right) + \operatorname{Li}_3\left(1-\frac{1}{a^2}\right) \right. \\ \left. -\operatorname{Li}_2\left(1-\frac{1}{a^2}\right)\log\left(1-\frac{1}{a^2}\right) - \frac{1}{2}\log\left(\frac{1}{a^2}\right)\log^2\left(1-\frac{1}{a^2}\right) - \zeta(3)\right], & |a| > 1. \end{cases}$$

At $a=\eta$ where $\eta=\{-1,0,1\}$ the results need to be understood as a limit (one-sided where needed) where $a \to \eta$.

Proof. For the case $|a|<1$, by writing the cubed logarithmic term appearing in the integrand for $\operatorname{Lp}_3(a)$ as the product between three linear logarithmic terms before replacing each of these linear logarithmic terms with their respective Fourier cosine series expansions, one has

$$\operatorname{Lp}_3(a) = -8\int_0^\pi \sum_{n_1=1}^\infty \frac{a^{n_1}}{n_1}\cos(n_1 x) \sum_{n_2=1}^\infty \frac{a^{n_2}}{n_2}\cos(n_2 x) \sum_{n_3=1}^\infty \frac{a^{n_3}}{n_3}\cos(n_3 x)\,dx, \quad |a|<1.$$

Due to (9), Fubini's theorem applies so the order of integration with the summations may be interchanged allowing the integration to be performed termwise. Doing so yields

$$\operatorname{Lp}_3(a) = -8\sum_{n_1=1}^\infty \frac{a^{n_1}}{n_1}\sum_{n_2=1}^\infty \frac{a^{n_2}}{n_2}\sum_{n_3=1}^\infty \frac{a^{n_3}}{n_3}\int_0^\pi \cos(n_1 x)\cos(n_2 x)\cos(n_3 x)\,dx. \tag{13}$$

Now

$$\int_0^\pi \cos(n_1 x)\cos(n_2 x)\cos(n_3 x)\,dx = \frac{\pi}{4}\delta_{2\cdot\max\{n_i\}=n_1+n_2+n_3}.$$

Here $i=1,2,3$ while $\delta_{2\cdot\max\{n_i\}=n_1+n_2+n_3}$ is a place holder for 1 when $2\cdot\max\{n_i\} = n_1+n_2+n_3$ is satisfied and zero otherwise. Thus (13) reduces to

$$\begin{aligned} \operatorname{Lp}_3(a) &= -2\pi\sum_{n_1=1}^\infty\sum_{n_2=1}^\infty\sum_{n_3=1}^\infty \frac{a^{n_1+n_2+n_3}}{n_1 n_2 n_3}\delta_{2\cdot\max\{n_i\}=n_1+n_2+n_3} \\ &= -6\pi\sum_{n=1}^\infty\sum_{k=1}^\infty \frac{a^{2n+2k}}{nk(n+k)}. \end{aligned} \tag{14}$$

The factor of three gained in the second line is the result of the triple summation over $\{n_1,n_2,n_3\}$ not depending on order. As the set of numbers $\{n_1,n_2,n_3\}$ can be permuted in three ways, this leads to three ways a non-zero term can arise when $2 \cdot \max\{n_i\} = n_1 + n_2 + n_3$ is satisfied. The double summation that results can be found by converting it first to a triple integral. Observing that

$$\frac{1}{n} = \int_0^1 x^{n-1}\,dx, \quad \frac{1}{k} = \int_0^1 y^{k-1}\,dy, \quad \text{and} \quad \frac{1}{n+k} = \int_0^1 z^{n+k-1}\,dz,$$

the double summation in the second line of (14) can be rewritten as

$$\begin{aligned}\mathrm{Lp}_3(a) &= -6\pi \int_0^1 \int_0^1 \int_0^1 \frac{1}{xyz} \sum_{n=1}^{\infty} (xza^2)^n \sum_{k=1}^{\infty} (yza^2)^k \,dx\,dy\,dz \\ &= -6\pi a^4 \int_0^1 \int_0^1 \int_0^1 \frac{z}{(1-xza^2)(1-yza^2)}\,dx\,dy\,dz,\end{aligned}$$

where in the last line we have summed the resulting geometric series. Note the change made in the order between the integral signs and the summations is justified by Tonelli's theorem since all terms involved are positive. The two inner iterated integrals for x and y are elementary. After performing each of these, one obtains

$$\mathrm{Lp}_3(a) = -6\pi \int_0^1 \frac{\log^2(1-za^2)}{z}\,dz.$$

Making use of the result given in Lemma 2, after replacing x with za^2 we have

$$\mathrm{Lp}_3(a) = -12\pi \sum_{n=2}^{\infty} \frac{H_{n-1}}{n} a^{2n} \int_0^1 z^{n-1}\,dz = -12\pi \sum_{n=2}^{\infty} \frac{H_{n-1}}{n^2} a^{2n}, \tag{15}$$

with the interchange made between the integration and summation being justified by Tonelli's theorem as all terms involved are positive. Now consider the two sums

$$\sum_{n=1}^{\infty} \frac{a^{2n}}{n^3} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{H_n}{n^2} a^{2n}.$$

Recognising the first of the sums is $\sum_{n=1}^{\infty} \frac{a^{2n}}{n^3} = \mathrm{Li}_3(a^2)$ while the second is given by Lemma 3, as both sums converge we can write

$$12\pi \sum_{n=1}^{\infty} \frac{a^{2n}}{n^3} - 12\pi \sum_{n=1}^{\infty} \frac{H_n}{n^2} a^{2n} = -12\pi \sum_{n=1}^{\infty} \left(H_n - \frac{1}{n}\right) \frac{a^{2n}}{n^2} = -12\pi \sum_{n=1}^{\infty} \frac{H_{n-1}}{n^2} a^{2n},$$

where the recurrence relation for the harmonic numbers, namely $H_n = H_{n-1} + \frac{1}{n}$, has been used. One can therefore write (15) as

$$\mathrm{Lp}_3(a) = 12\pi \sum_{n=1}^{\infty} \frac{a^{2n}}{n^3} - 12\pi \sum_{n=1}^{\infty} \frac{H_n}{n^2} a^{2n},$$

and the result for $|a| < 1$ then follows, since as was noted above the first of the sums is $\sum_{n=1}^{\infty} \frac{a^{2n}}{n^3} = \mathrm{Li}_3(a^2)$ while the second is given by Lemma 3. Notice the result is seen to contain the $a = 0$ case as a limiting value since

$$\lim_{a \to 0} \left(\mathrm{Li}_2(1-a^2) \log(1-a^2) + \frac{1}{2} \log(a^2) \log^2(1-a^2) \right) = 0,$$

a result that can be established using a single application of l'Hôptial's rule, and on noting that $\mathrm{Li}_3(1) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3)$.

For the case $|a| > 1$ we can write

$$\begin{aligned}
\mathrm{Lp}_3(a) &= \int_0^{\pi} \log^3 \left[a^2 \left(1 - \frac{2}{a} \cos x + \frac{1}{a^2} \right) \right] dx \\
&= \pi \log^3(a^2) + 3\log^2(a) \int_0^{\pi} \log\left(1 - \frac{2}{a}\cos x + \frac{1}{a^2}\right) dx \\
&\quad + 3\log(a^2) \int_0^{\pi} \log^2 \left(1 - \frac{2}{a}\cos x + \frac{1}{a^2}\right) dx + \int_0^{\pi} \log^3\left(1 - \frac{2}{a}\cos x + \frac{1}{a^2}\right) dx \\
&= \pi \log^3(a^2) + 3\log^2(a^2)\,\mathrm{Lp}_1\left(\frac{1}{a}\right) + 3\log(a^2)\,\mathrm{Lp}_2\left(\frac{1}{a}\right) + \mathrm{Lp}_3\left(\frac{1}{a}\right).
\end{aligned}$$

Observing the results for $\mathrm{Lp}_1(1/a)$, $\mathrm{Lp}_2(1/a)$, and $\mathrm{Lp}_3(1/a)$ correspond respectively to the previously found results for $\mathrm{Lp}_1(a)$, $\mathrm{Lp}_2(a)$, and $\mathrm{Lp}_3(a)$ for the case $|a| < 1$, $a \neq 0$, with a replaced with $1/a$, the result for the case $|a| > 1$ then follows.

Turning to the end-points at $a = \pm 1$, at the upper end-point, on setting $a = 1$ the integral for $\mathrm{Lp}_3(a)$ reduces to

$$\mathrm{Lp}_3(1) = \int_0^{\pi} \log^3(2 - 2\cos x)\, dx = 8 \int_0^{\pi} \log^3\left(2\sin\frac{x}{2}\right) dx.$$

Here we have made use of the identity $1 - \cos x = 2\sin^2 \frac{x}{2}$. Once again, integrals of the type appearing to the right have been well studied in the literature [9, 10, 11], so we need only quote the final result. It is

$$\int_0^{\pi} \log^3\left(2\sin\frac{x}{2}\right) dx = -\frac{3\pi}{2}\zeta(3).$$

Thus $\mathrm{Lp}_3(1) = -12\pi\zeta(3)$. Noting that

$$\lim_{a \to 1^-} \left(\mathrm{Li}_2(1 - a^2)\log(1 - a^2) + \frac{1}{2}\log(a^2)\log^2(1 - a^2) \right) = 0,$$

a result that can be established using a single application of l'Hôptial's rule, and as $\mathrm{Lp}_3(1) = \mathrm{Lp}_3(-1)$, the two particular cases of $a = \pm 1$ at either end-point can be combined into the case for $|a| < 1$ and completes the proof. □

5. Two special values for $\mathrm{Lp}_3(a)$

From the following known closed-form values for the dilogarithm [14, Eq. (1.16) and Eq. (1.20)]

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2(2) \quad \text{and} \quad \mathrm{Li}_2\left(\frac{1}{\varphi^2}\right) = \frac{\pi^2}{15} - \log^2(\varphi),$$

and the trilogarithm [14, Eq. (6.12) and Eq. (6.13)]

$$\operatorname{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) + \frac{1}{6}\log^3(2) - \frac{\pi^2}{12}\log(2),$$

and

$$\operatorname{Li}_3\left(\frac{1}{\varphi^2}\right) = \frac{4}{5}\zeta(3) + \frac{2}{3}\log^3(\varphi) - \frac{2\pi^2}{15}\log(\varphi),$$

two special values for $\operatorname{Lp}_3(a)$ can be found. Here $\varphi = (1+\sqrt{5})/2$ denotes the *golden ratio*. For the first of the special values, setting $a = 1/\sqrt{2}$ gives

$$\operatorname{Lp}_3\left(\frac{1}{\sqrt{2}}\right) = \int_0^\pi \log^3\left(\frac{3}{2} - \sqrt{2}\cos x\right) dx = 2\pi\log^3(2) - \frac{3}{2}\pi\zeta(3).$$

For the second, setting $a = 1/\sqrt{\varphi}$ gives

$$\operatorname{Lp}_3\left(\frac{1}{\sqrt{\varphi}}\right) = \int_0^\pi \log^3\left(\varphi - \frac{2}{\sqrt{\varphi}}\cos x\right) dx = 8\pi\log^3(\varphi) - \frac{12\pi}{5}\zeta(3).$$

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(Received December 16, 2019)

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