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ON A SUBCLASS OF CLOSE–TO–CONVEX HARMONIC MAPPINGS

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Abstract. For $\alpha > -1$ and $\beta > 0$, let $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ denote the class of sense preserving harmonic mappings $f = h+\overline{g}$ in the open unit disk $\mathbb{D}$ satisfying $|zh''(z)+\alpha(h'(z)-1)| \leqslant \beta - |zg''(z)+\alpha g'(z)|$. First, we establish that each function belonging to this class is close-to-convex in the open unit disk if $\beta \in (0,1+\alpha]$. Next, we obtain coefficient bounds, growth estimates and convolution properties. We end the paper with applications and will construct harmonic univalent polynomials belonging to this class.

1. Introduction

A complex valued mapping $f = u+iv$ defined in a domain Ω is a planar harmonic mapping, if both u and v are real-valued harmonic functions in Ω. If the domain Ω is simply connected and $z_0 \in \Omega$, then f admits a unique canonical representation $f = h+\overline{g}$, where both h and g are analytic in Ω and $g(z_0) = 0$. The harmonic mapping f is locally univalent in Ω if and only if its Jacobian $J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2$ is non-zero in Ω (see [10]). It is sense preserving, if $J_f(z) > 0$ $(z \in \Omega)$, or equivalently, if $h'(z) \neq 0$ and the dilatation $w = g'/h'$ is analytic and satisfies $|w| < 1$ in Ω.

Let $\mathscr{H}$ denote the class of all harmonic mappings $f = h+\overline{g}$ in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $h(0) = g(0) = h'(0)-1 = 0$. Each function f in $\mathscr{H}$ can be expressed by $f = h+\overline{g}$, where h and g are analytic functions in $\mathbb{D}$, and have power series representations

$$h(z) = z+\sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1}$$

Let $\mathscr{S}_{\mathscr{H}}$ be the subclass of $\mathscr{H}$ consisting of univalent and sense-preserving harmonic mappings in $\mathbb{D}$. Also, we denote by $\mathscr{H}^0 = \{f \in \mathscr{H} : f_{\overline{z}}(0) = 0\}$ and $\mathscr{S}^0_{\mathscr{H}} = \{f \in \mathscr{S}_{\mathscr{H}} : f_{\overline{z}}(0) = 0\}$. In 1984, Clunie and Sheil-Small [3] investigated the class $\mathscr{S}_{\mathscr{H}}$ together with some of its geometric subclasses. For recent results in harmonic mappings, we refer to [2, 6, 8, 13, 19, 23] and the references therein.

In [7], Hernández and Martin introduced the concept of stable harmonic mappings. A sense preserving harmonic mapping $f = h+\overline{g}$ is said to be stable harmonic univalent (resp. stable harmonic convex, stable harmonic starlike, or stable harmonic close-to-convex) in $\mathbb{D}$, if all functions $F_\lambda = h+\lambda g$ with $|\lambda| = 1$ are univalent (resp.

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convex, starlike, or close-to-convex) in $\mathbb{D}$. They proved that for $|\lambda| = 1$, the functions $f_\lambda = h + \lambda\overline{g}$ are univalent (resp. convex, starlike, or close-to-convex) for all such λ (see [7]). We recall that, a function $f \in \mathscr{H}$ is said to be close-to-convex, if $f(\mathbb{D})$ is close-to-convex, i.e., the complement of $f(\mathbb{D})$ can be written as disjoint union of non-intersecting half lines. The following sufficient condition for the close-to-convexity of harmonic mappings is due to Clunie and Sheil-Small [3].

LEMMA 1. *If harmonic mapping $f = h + \overline{g} : \mathbb{D} \to \mathbb{C}$ satisfies $|g'(0)| < |h'(0)|$ and the function $F_\lambda = h + \lambda g$ is close-to-convex for every $|\lambda| = 1$, then f is close-to-convex function.*

An analytic function φ is said to subordinate to the analytic function ψ and written by $\varphi(z) \prec \psi(z)$, if there exists a function w analytic in $\mathbb{D}$ with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathbb{D}$, such that $\varphi(z) = \psi(w(z))$, $z \in \mathbb{D}$. Furthermore, if ψ is univalent in $\mathbb{D}$, then we have the following equivalence:

$$\varphi(z) \prec \psi(z) \Longleftrightarrow [\varphi(0) = \psi(0) \quad \text{and} \quad \varphi(\mathbb{D}) \subset \psi(\mathbb{D})].$$

In this article, we shall use the following known result of subordination.

LEMMA 2. (see [17, Eq. 16]) *Let $\mathscr{P}$ be an analytic function such that $\mathscr{P}(0) = 1$. Then for real α such that $\alpha > -1$, we have*

$$\mathscr{P}(z) + \alpha z\mathscr{P}'(z) \prec 1 + \lambda z \Rightarrow \mathscr{P}(z) \prec 1 + \frac{\lambda}{\alpha+1}z, \qquad z \in \mathbb{D}.$$

For two analytic functions ψ_1 and ψ_2 in $\mathbb{D}$, given by $\psi_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\psi_2(z) = \sum_{n=0}^{\infty} b_n z^n$, the *convolution (or Hardamard product)* is defined by $(\psi_1 * \psi_2)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$, $z \in \mathbb{D}$. Analogously, for harmonic functions $f_1 = h_1 + \overline{g_1}$ and $f_2 = h_2 + \overline{g_2}$ in $\mathscr{H}$, the convolution of f_1 and f_2 is defined as $f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2}$. Clunie and Sheil-Small [3] proved that, if f is harmonic convex function, and ϕ is an analytic convex function, then $f * \left(\phi + \alpha\overline{\phi}\right)$ is harmonic close-to-convex function for all α such that $|\alpha| < 1$. Clearly the space $\mathscr{H}$ is closed under the convolution, *i.e.* $\mathscr{H} * \mathscr{H} \subset \mathscr{H}$. We refer [4, 9, 11, 14] for more information concerning convolution of harmonic mappings and in the case of analytic functions, we refer for examples [16, 24] and the references therein.

Let $\mathscr{A}$ denote the class of analytic functions in $\mathbb{D}$ normalized by $f(0) = f'(0) - 1 = 0$ and $\mathscr{S}$ denote the subclass of $\mathscr{A}$ containing univalent functions. Let $\mathscr{S}^*$ and $\mathscr{K}$ denote the classes starlike and convex functions in $\mathbb{D}$, respectively. A function $f \in \mathscr{A}$ is close-to-convex in $\mathbb{D}$, if there exists a convex analytic function ϕ in $\mathbb{D}$, not necessarily normalized, such that $\Re(f'(z)/\phi'(z)) > 0$ in $\mathbb{D}$. Ponnusamy and Singh [21] have studied a subclass $\mathscr{B}(\alpha, \beta)$ of close-to-convex functions $f \in \mathscr{A}$ which satisfy the condition

$$|zf''(z) + \alpha(f'(z) - 1)| < \beta, \qquad z \in \mathbb{D},$$

where $\alpha > -1$ and $\beta > 0$. Also, they proved that functions in the class $\mathscr{B}(\alpha,\beta)$ are convex in $\mathbb{D}$, if $\alpha > -1$ and

$$0 < \beta \leqslant \begin{cases} \dfrac{1-\alpha}{2+\alpha} & \text{for} \quad -1 < \alpha \leqslant \sqrt{5}-2, \\ \dfrac{1+\alpha}{\sqrt{5}} & \text{for} \quad \sqrt{5}-2 \leqslant \alpha \leqslant 1, \\ \dfrac{1+\alpha}{\alpha\sqrt{5}} & \text{for} \quad 1 \leqslant \alpha \leqslant \dfrac{2}{\sqrt{5}-1}, \\ \dfrac{1+\alpha}{2+\alpha} & \text{for} \quad \dfrac{2}{\sqrt{5}-1} \leqslant \alpha \leqslant 2, \\ \dfrac{1+\alpha}{2\alpha} & \text{for} \quad \alpha \geqslant 2, \end{cases} \tag{2}$$

(see [21, Corollary 4]); and stralike in $\mathbb{D}$, if $\alpha > -1$ and

$$0 < \beta \leqslant \begin{cases} \dfrac{2(1+\alpha)}{2+\alpha^{2/(1-\alpha)}} & \text{for} \quad -1 < \alpha \neq 1 < \infty, \\ \dfrac{4e^2}{1+e^2} & \text{for} \quad \alpha = 1, \end{cases} \tag{3}$$

(see [21, Theorem 1.14]). Further, we deduce the conditions for the univalency of functions in the class $\mathscr{B}(\alpha,\beta)$ by taking $p(z) = f'(z)-1$, $k = 1/\alpha$ $(\alpha > -1)$ and $J = \beta/|\alpha|$ in [17, Eq. 16]. This provides, if $f \in \mathscr{A}$ and $|zf''(z)+\alpha(f'(z)-1)| < \beta$ $(z \in \mathbb{D})$, then $|f'(z)-1| < \beta/(1+\alpha)$ $(z \in \mathbb{D})$. Therefore, the functions in the $\mathscr{B}(\alpha,\beta)$ are close-to-convex (hence univalent) in $\mathbb{D}$, if $\alpha > -1$ and $\beta \in (0, 1+\alpha]$.

Now we define harmonic analogue of the class $\mathscr{B}(\alpha,\beta)$. For $\alpha > -1$ and $\beta > 0$, let $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ be a subclass of $\mathscr{H}^0$ which is defined by

$$\begin{aligned} &\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta) \\ &= \left\{ f = h+\overline{g} \in \mathscr{H}^0 : |zh''(z)+\alpha(h'(z)-1)| \leqslant \beta - |zg''(z)+\alpha g'(z)|,\ z \in \mathbb{D} \right\}. \end{aligned}$$

Note that, for $\alpha = 0$, the class $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ reduces to the class $\mathscr{B}^0_{\mathscr{H}}(\beta)$, which was studied recently by Ghosh and Vasudevaro [5]. Further $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ reduces to $\mathscr{B}(\alpha,\beta)$, if the co-analytic part of f in $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ is zero.

In this article, we prove that the functions in $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ are close-to-convex in $\mathbb{D}$. Also, we prove that functions in $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ are stable harmonic univalent, stable harmonic convex and stable harmonic starlike in $\mathbb{D}$ for different values of its parameters. Further, the coefficient estimates, growth results, area theorem, boundary behaviour, convolution and convex combination properties of the class $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ of harmonic mapping are obtained. Finally, we consider the harmonic mappings which involve hypergeometric functions and obtain conditions on its parameters such that it belongs to the class $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$.

2. Main results

The first result provides a one-to-one correspondence between the classes $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ and $\mathscr{B}(\alpha,\beta)$.

THEOREM 1. *For $\alpha>-1$ and $\beta>0$, the harmonic mapping $f=h+\overline{g}\in\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ if and only if $F_\lambda=h+\lambda g\in\mathscr{B}(\alpha,\beta)$ for all $\lambda\,(|\lambda|=1|)$.*

Proof. We follow the method of proof of [22], for example, let $f=h+\overline{g}\in\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$. Then for all $\lambda\,(|\lambda|=1|)$, we have

$$\begin{aligned}\left|zF_\lambda''(z)+\alpha\left(F_\lambda'(z)-1\right)\right| &= \left|z(h+\lambda g)''(z)+\alpha((h+\lambda g)'(z)-1)\right|\\ &\leqslant \left|zh''(z)+\alpha(h'(z)-1)\right|+\left|zg''(z)+\alpha g'(z)\right|\\ &\leqslant \beta,\end{aligned}$$

and hence $F_\lambda\in\mathscr{B}(\alpha,\beta)$. Conversely, for all $\lambda\,(|\lambda|=1)$, let $F_\lambda=h+\lambda g\in\mathscr{B}(\alpha,\beta)$. Then

$$\begin{aligned}\left|zF_\lambda''(z)+\alpha\left(F_\lambda'(z)-1\right)\right| &= \left|zh''(z)+\alpha(h'(z)-1)+\lambda(zg''(z)+\alpha g'(z))\right|\\ &< \beta, \qquad z\in\mathbb{D}.\end{aligned}$$

For an appropriate choice of λ, we obtain from the last inequality

$$\left|zh''(z)+\alpha(h'(z)-1)\right|+\left|zg''(z)+\alpha g'(z)\right|<\beta, \qquad z\in\mathbb{D},$$

and hence $f\in\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$. □

REMARK 1. We observe that functions in $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ are stable harmonic close-to-convex if $\alpha>-1$ and $\beta\in(0,1+\alpha]$, stable harmonic convex in $\mathbb{D}$ if $\alpha>-1$ and β satisfies the conditions (2), stable harmonic starlike in $\mathbb{D}$ if $\alpha>-1$ and β satisfies the conditions (3). Further, the following result shows that functions in $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$.

THEOREM 2. *For $\alpha>-1$ and $\beta\in(0,1+\alpha]$, the harmonic mappings in $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ are close-to-convex in $\mathbb{D}$.*

Proof. If $f\in\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$, then by Theorem 1, we have $F_\lambda=h+\lambda g\in\mathscr{B}(\alpha,\beta)$ for all $\lambda\,(|\lambda|=1)$. Hence, F_λ are close-to-convex in $\mathbb{D}$ for $\alpha>-1$ and $\beta\in(0,1+\alpha]$. Now using Lemma 1, we conclude that functions in $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ are close-to-convex in $\mathbb{D}$. □

THEOREM 3. *Let $\alpha>-1$ and $\beta\in(0,1+\alpha]$. If $f=h+\overline{g}\in\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$, then*

$$|z|-\frac{\beta}{2(1+\alpha)}|z|^2\leqslant|f(z)|\leqslant|z|+\frac{\beta}{2(1+\alpha)}|z|^2, \qquad z\in\mathbb{D}.$$

Both the inequalities are sharp for the functions

$$f_1(z) = z + \frac{\beta}{2(1+\alpha)} z^2 \quad \text{and} \quad f_2(z) = z + \frac{\beta}{2(1+\alpha)} \overline{z}^2,$$

and their rotations.

Proof. If $f \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$, then $F_\lambda = h + \lambda g \in \mathscr{B}(\alpha,\beta)$ for all $\lambda\,(|\lambda| = 1)$. Hence

$$zF''_\lambda(z) + \alpha F'_\lambda(z) \prec \alpha + \beta z, \quad z \in \mathbb{D}.$$

Using Lemma 2, we obtain

$$F'_\lambda(z) \prec 1 + \frac{\beta}{1+\alpha} z, \quad z \in \mathbb{D}.$$

Therefore

$$1 - \frac{\beta}{1+\alpha}|z| \leqslant \left|F'_\lambda(z)\right| = \left|h'(z) + \lambda g'(z)\right| \leqslant 1 + \frac{\beta}{1+\alpha}|z|.$$

Since $\lambda\,(|\lambda| = 1)$ is arbitrary, it follows that

$$|h'(z)| + |g'(z)| \leqslant 1 + \frac{\beta}{1+\alpha}|z|$$

and

$$|h'(z)| - |g'(z)| \geqslant 1 - \frac{\beta}{1+\alpha}|z|.$$

If Γ is the radial segment from 0 to z, then

$$|f(z)| = \left| \int_\Gamma \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \overline{\xi}} d\overline{\xi} \right| \leqslant \int_\Gamma \left(|h'(\xi)| + |g'(\xi)|\right) |d\xi|$$

$$\leqslant \int_0^{|z|} \left(1 + \frac{\beta}{1+\alpha} t\right) dt = |z| + \frac{\beta}{2(1+\alpha)}|z|^2,$$

and

$$|f(z)| = \left| \int_\Gamma \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \overline{\xi}} d\overline{\xi} \right| \geqslant \int_\Gamma \left(|h'(\xi)| - |g'(\xi)|\right) |d\xi|$$

$$\geqslant \int_0^{|z|} \left(1 - \frac{\beta}{1+\alpha} t\right) dt = |z| - \frac{\beta}{2(1+\alpha)}|z|^2,$$

which completes the proof of the theorem. □

The following theorem provides sharp coefficient bounds for functions in $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$.

THEOREM 4. *Let* $f = h + \overline{g} \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ *be given by* (1), *then for* $n \geqslant 2$,

$$|a_n| \leqslant \frac{\beta}{n(n+\alpha-1)} \quad \text{and} \quad |b_n| \leqslant \frac{\beta}{n(n+\alpha-1)}.$$

Both the inequalities are sharp.

Proof. The proof follows from the method of [12, Proof of theorem 2], but for the sake of completeness we include it here. Let $f = h + \overline{g} \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ be given by (1), then $|zh''(z) + \alpha(h'(z) - 1)| < \beta$. Now using Cauchy's theorem, we have

$$n\,(n+\alpha-1)\,a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{zh''(z) + \alpha(h'(z)-1)}{z^n}dz, \qquad |z| = r < 1,$$

and hence the bound for $|a_n|$ follows. Also, using the trivial bound $|a_n| + |b_n| \leqslant \dfrac{\beta}{n(n+\alpha-1)}$ $(n \geqslant 2)$, the bounds for $|b_n|$ follows. The sharpness of result can be shown by taking

$$f_1(z) = z + \frac{\beta}{n\,(n+\alpha-1)}z^n \quad \text{and} \quad f_2(z) = z + \frac{\beta}{n\,(n+\alpha-1)}\overline{z}^n.$$

This completes the proof of the theorem. □

The following result gives a sufficient condition for functions belonging to the class $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$.

THEOREM 5. *Let $\alpha > -1$ and $\beta > 0$. If $f = h + \overline{g} \in \mathscr{H}^0$ be given by (1) and*

$$\sum_{n=2}^{\infty} n(n+\alpha-1)(|a_n| + |b_n|) \leqslant \beta, \tag{4}$$

then $f \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$.

Proof. Let $f = h + \overline{g} \in \mathscr{H}^0$. Then from (1) and (4), we obtain

$$\begin{aligned}
\left|zh''(z) + \alpha\left(h'(z) - 1\right)\right| &= \left|\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1} + \alpha \sum_{n=2}^{\infty} na_n z^{n-1}\right| \\
&\leqslant \sum_{n=2}^{\infty} n(n+\alpha-1)|a_n||z|^{n-1} \\
&\leqslant \beta - \sum_{n=2}^{\infty} n(n+\alpha-1)|b_n| \\
&\leqslant \beta - \left|\sum_{n=2}^{\infty} n(n+\alpha-1)b_n z^{n-1}\right| \\
&= \beta - \left|zg''(z) + \alpha g'(z)\right|,
\end{aligned}$$

and hence $f \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$. □

In view of Remark 1 and Theorem 5, we obtain following interesting corollaries.

COROLLARY 1. *Let $\alpha > -1$ and $\beta \in (0, 1+\alpha]$. If $f = h + \overline{g} \in \mathscr{H}^0$ is given by (1) and satisfies the inequality (4), then f is stable harmonic close-to-convex in $\mathbb{D}$.*

COROLLARY 2. *Let $\alpha > -1$ and β satisfy the condition* (2). *If $f = h + \overline{g} \in \mathscr{H}^0$ is given by* (1) *and satisfies the inequality* (4), *then f is stable harmonic convex in $\mathbb{D}$.*

COROLLARY 3. *Let $\alpha > -1$ and β satisfy the condition* (3). *If $f = h + \overline{g} \in \mathscr{H}^0$ is given by* (1) *and satisfies the inequality* (4), *then f is stable harmonic starlike in $\mathbb{D}$.*

EXAMPLE 1. Consider the function

$$\theta_{\alpha,\beta}(z) = z + \frac{\beta}{4(1+\alpha)}(z^2 - \overline{z}^2) \qquad (\alpha > -1,\ \beta > 0,\ z \in \mathbb{D}). \tag{5}$$

Then we have

$$\sum_{n=2}^{\infty} n(n+\alpha-1)\,(|a_n| + |b_n|) = \beta.$$

Hence $\theta_{\alpha,\beta} \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$. Also, $\theta_{\alpha,\beta}$ is stable harmonic close-to-convex in $\mathbb{D}$ if $\alpha > -1, \beta \in (0, 1+\alpha]$, stable harmonic convex in $\mathbb{D}$ if $\alpha > -1$ and β satisfy the condition (2), stable harmonic starlike if $\alpha > -1$ and β satisfy the condition (3).

The following results shows that the boundary of $f(\mathbb{D})$ is a rectifiable Jordan curve for each $f \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$.

THEOREM 6. *For real α and β such that $\alpha > -1$ and $\beta \in (0, 1+\alpha]$, each function in $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ maps the $\mathbb{D}$ onto a domain which is bounded by a rectifiable Jordan curve.*

Proof. Each function $f = h + \overline{g} \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ is uniformly continuous in $\mathbb{D}$, and hence can be extended continuously onto $|z| = 1$. To see this, let z_1 and z_2 be two distinct points in $\mathbb{D}$, and $[z_1, z_2]$ be the line segment joining z_1 to z_2. We have from (4)

$$\begin{aligned}
|f(z_1) - f(z_2)| &= \left| \int_{[z_1,z_2]} \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \overline{\xi}} d\overline{\xi} \right| \\
&\leqslant \int_{|z_2|}^{|z_1|} \left(1 + \frac{\beta}{1+\alpha} t\right) dt \\
&= (|z_1| - |z_2|)\left(1 + \frac{\beta}{2(1+\alpha)}(|z_1| + |z_2|)\right) \\
&\leqslant 2(|z_1| - |z_2|) \leqslant 2|z_1 - z_2|.
\end{aligned}$$

Now, let the curve $\mathscr{C}$ be defined by $w = f(e^{i\theta}),\ 0 \leqslant \theta \leqslant 2\pi$. If $0 = \theta_0 < \theta_1 < \cdots \theta_n = 2\pi$ is a partition of $[0, 2\pi]$, then

$$\sum_{k=1}^{n} \left| f(e^{i\theta_k}) - f(e^{i\theta_{k-1}}) \right| \leqslant 2 \sum_{k=1}^{n} \left| e^{i\theta_k} - e^{i\theta_{k-1}} \right| < 4\pi,$$

which shows that $\mathscr{C}$ is a rectifiable curve. It remains to show that f is univalent on $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. In view of Theorem 1, functions $F_\lambda = h + \lambda g$ belongs to the class $\mathscr{B}(\alpha,\beta)$ for all $|\lambda|=1$. In particular each F_λ is univalent in $\partial\mathbb{D}$ by [15, Theorem 3].

Now suppose that z_1, z_2 are two distinct points on $\partial\mathbb{D}=\{z:|z|=1\}$ such that $f(z_1)=f(z_2)$. Then

$$h(z_1)-h(z_2)=\overline{g(z_2)-g(z_1)}. \tag{6}$$

If $h(z_1)=h(z_2)$, then $g(z_1)=g(z_2)$ and so $z_1=z_2$ by the univalence of F_1. Now assume that $h(z_1)\neq h(z_2)$, and let $\theta = arg\{h(z_1)-h(z_2)\}\in[0,2\pi)$. Then $e^{-i\theta}(h(z_1)-h(z_2))$ is a positive real number, now multiplying (6) by $e^{-i\theta}$ and taking the conjugate on both sides, we have

$$h(z_1)-h(z_2)=e^{2i\theta}\left(g(z_2)-g(z_1)\right),$$

which implies that $F_\lambda(z_1)=F_\lambda(z_2)$ with $\lambda=e^{2i\theta}$. Thus $z_1=z_2$, which shows that f is univalent in $\partial\mathbb{D}$. This completes the proof of Theorem 6. □

Now, we will show that the class $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ is closed under convex combinations. Also, we show that for $\phi\in\mathscr{K}$ and $f\in\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$, the function $f*\left(\phi+\beta\overline{\phi}\right)\in\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ for all $|\beta|=1$. To show that the class $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ is closed under convex combinations, we shall need following Lemma:

LEMMA 3. (see [25]) *Let p be an analytic function in $\mathbb{D}$, with $p(0)=1$ and $\Re(p(z))>1/2$ in $\mathbb{D}$. Then for any analytic function f in $\mathbb{D}$, the function $p*f$ takes values in the convex hull of the image of $\mathbb{D}$ under f.*

THEOREM 7. *The class $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ is closed under convex combination.*

Proof. Let $f_k=h_k+\overline{g_k}\in\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ for $k=1,2,\cdots n$, and $\sum_{k=1}^n t_k=1$ $(0\leqslant t_k\leqslant 1)$. The convex combination of the f_k's can be written as

$$f(z)=\sum_{k=1}^n t_k f_k(z)=h(z)+\overline{g(z)},$$

where $h(z)=\sum_{k=1}^n t_k h_k(z)$ and $g(z)=\sum_{k=1}^n t_k g_k(z)$. A computation shows that

$$\begin{aligned}\left|zh''(z)+\alpha(h'(z)-1)\right| &= \left|\sum_{k=1}^n t_k(zh_k''(z)+\alpha(h_k'(z)-1))\right| \\ &\leqslant \sum_{k=1}^n t_k\left|zh_k''(z)+\alpha(h_k'(z)-1)\right| \\ &= \sum_{n=1}^n t_k\left(\beta-|zg_k''(z)+\alpha g_k'(z)|\right) \\ &\leqslant \beta-\left|z\sum_{k=1}^n t_k g_k''(z)+\alpha\sum_{k=1}^n t_k g_k'(z)\right| \\ &= \beta-|zg''(z)+\alpha g'(z)|,\end{aligned}$$

and so $f\in\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$. □

THEOREM 8. *Let* $f \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ *and* $\phi \in \mathscr{K}$. *Then* $f * \left(\phi + \lambda\overline{\phi}\right) \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ *for all* λ $(|\lambda| = 1)$.

Proof. Let $f = h + \overline{g}$ be in $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$. Then

$$f * \left(\phi + \lambda\overline{\phi}\right) = h * \phi + \overline{\overline{\lambda}(g * \phi)}.$$

It sufficient to show that $F_\lambda = h * \phi + \overline{\lambda}\,(g * \phi) \in \mathscr{B}(\alpha,\beta)$ for all $\lambda(|\lambda| = 1)$. A computation shows that

$$zF_\varepsilon''(z) + \alpha\left(F_\varepsilon'(z) - 1\right) = \left(z\,(h + \lambda g)''\,(z) + \alpha\left((h + \lambda g)' - 1\right)\right) * \frac{\phi(z)}{z}. \qquad (7)$$

Since $f = h + \overline{g} \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$, the function $h + \overline{\lambda}g \in \mathscr{B}(\alpha,\beta)$, and so

$$\left|zh''(z) + \alpha\left(h'(z) - 1\right) + \overline{\lambda}(zg''(z) + \alpha g'(z))\right| \leqslant \beta, \qquad z \in \mathbb{D}.$$

Since $\phi \in \mathscr{K}$, implies that $\Re\left(\dfrac{\phi(z)}{z}\right) > \dfrac{1}{2}$ in $\mathbb{D}$. Now applying Lemma 3, we obtain that

$$\left|zF_\lambda''(z) + \alpha(F_\lambda'(z) - 1)\right| \leqslant \beta, \qquad z \in \mathbb{D}.$$

Hence $F_\lambda \in \mathscr{B}(\alpha,\beta)$ for all λ $(|\lambda| = 1)$, equivalently $f * \left(\phi + \lambda\overline{\phi}\right) \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$ for all λ $(|\lambda| = 1)$. □

3. Applications

In this section, we consider harmonic mappings which involve the Gaussian hypergeometric function and obtain conditions so that such harmonic mappings belongs to the class $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$. The Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \qquad (8)$$

where $a,b,c \in \mathbb{C}$, $c \neq 0,-1,-2,-3,\cdots$ and $(x)_n$ is the Pochhammer symbol defined by $(x)_0 = 1$, $(x)_{n+1} = (x+n)(x)_n = x(x+1)_n$ $(n = 0,1,2,\cdots)$. The series (8) is absolutely convergent in $\mathbb{D}$. Moreover, if $\Re(c-a-b) > 0$, then series (8) is convergent in $|z| \leqslant 1$. The well-known Gauss formula (see [26])) for hypergeometric function is given by ${}_2F_1(a,b;c;1) = \Lambda$ for $\Re(c-a-b) > 0$, and where

$$\Lambda = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

We shall use the following Lemma to prove result in this section.

LEMMA 4. (see [20]) *Let* $a,b \in \mathbb{R}\setminus\{0\}$ *and* c *is a positive real number. Then the following holds* *(i) For* $c > a+b+1$,

$$\sum_{n=0}^{\infty} \frac{(n+1)(a)_n(b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}(ab+c-a-b-1).$$

(ii) For $c > a+b+2$,

$$\sum_{n=0}^{\infty} \frac{(n+1)^2(a)_n(b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{(a)_2(b)_2}{(c-a-b-2)_2} + \frac{3ab}{c-a-b-1} + 1\right).$$

(iii) For $a \neq 1, b \neq 1$ *and* $c \neq 1$ *with* $c > max\{0, a+b+1\}$,

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(n+1)!} = \frac{1}{(a-1)(b-1)}\left(\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} - (c-1)\right).$$

Below we use the ideas used by [1, 18] for the univalency of harmonic mappings involving the Gaussian hypergeometric functions. The first result in this section is given by

THEOREM 9. *Let* $a, b \in \mathbb{R} \setminus \{0\}$ *and* c *is a positive real number. Suppose that* $f_1(z) = z + \overline{z^2 F(a,b;c;z)}$, $f_2(z) = z + \overline{z(F(a,b;c;z)-1)}$ *and* $f_3(z) = z + \overline{z\int_0^z F(a,b;c;t)dt}$, *then the following holds*

(i) If $c > a+b+2$ *and*

$$\frac{(a)_2(b)_2}{(c-a-b-2)_2} + \frac{ab(\alpha+4)}{c-a-b-1} + 2(1+\alpha) \leqslant \frac{\beta}{\Lambda}, \tag{9}$$

then $f_1 \in \mathscr{B}^0_{\mathscr{H}}(\alpha, \beta)$.

(ii) If $c > a+b+2$ *and*

$$\frac{ab(ab+c-1)}{(c-a-b-2)_2} + \frac{ab(1+\alpha)}{c-a-b-1} + \alpha \leqslant \frac{\beta-\alpha}{\Lambda}, \tag{10}$$

then $f_2 \in \mathscr{B}^0_{\mathscr{H}}(\alpha, \beta)$.

(iii) If $a \neq 1, b \neq 1$ *and* $c \neq 1$ *with* $c > max\{0, a+b+1\}$ *and*

$$\Lambda\left(\frac{ab}{c-a-b-1} + \frac{\alpha}{(a-1)(b-1)(c-a-b-1)} + \alpha\right) - \frac{\alpha(c-1)}{(a-1)(b-1)} \leqslant \beta, \tag{11}$$

then $f_3 \in \mathscr{B}^0_{\mathscr{H}}(\alpha, \beta)$.

Proof. (i) Let $f_1(z) = z + \overline{\sum_{n=2}^{\infty} C_n z^n}$, where $C_n = \frac{(a)_{n-2}(b)_{n-2}}{(c)_{n-2}(n-2)!}$ $(n \geqslant 2)$. Using Lemma 4 and Gauss formula, we have

$$\begin{aligned}\sum_{n=2}^{\infty} n(n+\alpha-1)|C_n| &= \sum_{n=2}^{\infty} n(n+\alpha-1)\frac{(a)_{n-2}(b)_{n-2}}{(c)_{n-2}(n-2)!}\\ &= \sum_{n=0}^{\infty}(n+1)^2\frac{(a)_n(b)_n}{(c)_n n!} + (1+\alpha)\sum_{n=0}^{\infty}(n+1)\frac{(a)_n(b)_n}{(c)_n n!}\\ &\quad + \alpha\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}\\ &= \Lambda\left(\frac{(a)_2(b)_2}{(c-a-b-2)_2} + \frac{ab(\alpha+4)}{c-a-b-1} + 2(1+\alpha)\right).\end{aligned} \tag{12}$$

Now if (9) holds, then $\sum_{n=2}^{\infty} n(n+\alpha-1)|C_n| \leqslant \beta$. Now using Theorem 5, we conclude that $f_1 \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$.

(ii) Let $f_2(z) = z + \overline{\sum_{n=2}^{\infty} D_n z^n}$, where $D_n = \dfrac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}$ $(n \geqslant 2)$. Using Lemma 4 and Gauss formula, we have

$$\begin{aligned}\sum_{n=2}^{\infty} n(n+\alpha-1)|D_n| &= \sum_{n=2}^{\infty} n(n+\alpha-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}\\ &= \sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1} n!} + (1+\alpha)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1} n!}\\ &\quad + \alpha\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(n+1)!}\\ &= \Lambda\left[\frac{ab(ab+c-1)}{(c-a-b-2)_2} + \frac{ab(1+\alpha)}{c-a-b-1} + \alpha\right] - \alpha.\end{aligned}$$

Now if (10) holds, then in view of Theorem 5, we have $f_2 \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$.

(iii) Let $f_3(z) = z + \overline{\sum_{n=2}^{\infty} E_n z^n}$, where $E_n = \dfrac{(a)_{n-2}(b)_{n-2}}{(c)_{n-2}(n-1)!}$ $n \geqslant 2$. Therefore in view of Lemma 4 and Gauss formula, we have

$$\begin{aligned}\sum_{n=2}^{\infty} n(n+\alpha-1)|E_n| &= \sum_{n=2}^{\infty} n(n+\alpha-1)\frac{(a)_{n-2}(b)_{n-2}}{(c)_{n-2}(n-1)!}\\ &= \sum_{n=0}^{\infty}(n+1)\frac{(a)_n(b)_n}{(c)_n n!} + (1+\alpha)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}\\ &\quad + \alpha\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n (n+1)!}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}(ab+c-a-b-1)\\
&\quad + (1+\alpha)\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\\
&\quad + \frac{\alpha}{(a-1)(b-1)}\left(\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} - (c-1)\right).
\end{aligned}
$$

If (11) holds, then by Theorem 5, we have $f_3 \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$. $\square$

Note that for $\eta \in \mathbb{C} \setminus \{-1,-2,\cdots\}$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\frac{(-1)^n(-\eta)_n}{n!} = \binom{\eta}{n} = \frac{\Gamma(\eta+1)}{n!\Gamma(\eta-n+1)}.$$

In particular, when $\eta = m\,(m \in \mathbb{N}, m \geqslant n)$, we have

$$(-m)_n = \frac{(-1)^n m!}{(m-n)!}.$$

Using this relation in Theorem 9, we can obtain harmonic univalent polynomials that belong to the class $\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$.

COROLLARY 4. *Let $m \in \mathbb{N}, c$ be a positive real numbers. Let*

$$F_1(z) = z + \overline{\sum_{n=0}^{m}\binom{m}{n}\frac{(m-n+1)_n}{(c)_n}z^{n+2}}, \quad F_2(z) = z + \overline{\sum_{n=0}^{m}\binom{m}{n}\frac{(m-n+1)_n}{(c)_n}z^{n+1}}$$

and

$$F_3(z) = z + \overline{\sum_{n=0}^{m}\binom{m}{n}\frac{(m-n+1)_n}{(c)_n}\frac{z^{n+2}}{n+1}}.$$

Then the following holds.

(i) If

$$\frac{m^2(m-1)^2}{(c+2m-1)(c+2m-2)} + \frac{m^2(\alpha+4)}{c+2m-1} + 2(1-\alpha) \leqslant \frac{\beta\,[\Gamma(c+m)]^2}{\Gamma(c)\Gamma(c+2m)},$$

then $F_1 \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$.

(ii) If

$$\frac{m^2(c+m^2-1)}{(c+2m-2)(c+2m-1)} + \frac{m^2(1+\alpha)}{c+2m-1} + \alpha \leqslant \frac{(\beta-\alpha)\,[\Gamma(c+m)]^2}{\Gamma(c)\Gamma(c+2m)},$$

then $F_2 \in \mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$.

(iii) If

$$\frac{\Gamma(c)\Gamma(c+2m)}{(\Gamma(c+2m))^2}\left(\frac{m^2}{c+2m-1}+\frac{\alpha}{(m+1)^2(c+2m-1)}+\alpha\right)-\frac{\alpha(c-1)}{(m+1)^2}\leqslant\beta,$$

then $F_3\in\mathscr{B}^0_{\mathscr{H}}(\alpha,\beta)$.

Proof. The results follow, if we put $a=b=-m$ in Theorem 9. □

We conclude this paper by remarking that, by appropriately selecting parameters in Theorem 9 and Corollary 4, our results would lead to new results and further applications. These consideration can fruitfully be worked out and we skip the details in this regards.

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