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DIRECT ESTIMATES FOR GUPTA TYPE GENERAL OPERATORS

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Abstract. Gupta in [6] introduced a general family of linear positive operators which produce large number of well known linear positive operators as particular cases. As the family of operators proposed by Gupta provides a unified approach this motivated us to extend the studies, and we establish some convergence estimates of these important operators. We estimate an asymptotic formula and the rate of convergence for these operators for the function having derivatives of bounded variation.

1. Introduction

In the year 1980 Mastroianni [15] suggested a discretely defined operators based on the exponential type operators, which contain three well-known operators namely Bernstein, Baskakov and Szász-Mirakyan operators as special cases. After a gap of twenty years Miheşan [16] introduced another sequence of linear positive operators based on exponential type operators. Miheşan's approach was based on substitution of the value of the parameter used in the definition, which then produce one more important operators namely the Lupaş operators. Although the additional Lupaş operators are not exponential type operators. These discretely defined operators are not possible to approximate integrable functions. In this direction Gupta and collaborators [2], [5], [10] and [11] proposed several hybrid operators of Durrmeyer type and established many interesting results concerning convergence. In this direction, some important contribution we refer to [4], [14] and [12] etc. Very recently based on unified approach concept Gupta in [6] introduced a generalized sequence of linear positive operators having different and same basis function in summation and integration. Such operators contain many well-known operators as special cases. For $x \in [0,\infty)$, the generalized operators due to Gupta [6] are defined in terms of the inner product as follows

$$V_{n,\rho,\mu}(f,x) = \sum_{i=1}^{\infty} m_{n,i}^{\rho}(x) \frac{\langle m_{n,i-1}^{\mu+1}, f\rangle}{\langle m_{n,i-1}^{\mu+1}, 1\rangle} + m_{n,0}^{\rho}(x) f(0), \tag{1}$$

where

$$m_{n,i}^{\rho}(x) = \frac{(\rho)_i \rho^{\rho}}{i!} \frac{(nx)^i}{(\rho+nx)^{\rho+i}}, m_{n,i-1}^{\mu+1}(t) = \frac{(\mu+1)_{i-1}\cdot\mu^{\mu+1}}{(i-1)!} \cdot \frac{(nt)^{i-1}}{(\mu+nt)^{\mu+i}}$$

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with the rising factorial $(\rho)_n = \prod_{k=0}^{i-1}(\rho+k)$ and $(\rho)_0 = 1$. These operators produce following well-known operators as special cases (see [6]):

1. If $\rho = \mu = n$, we obtain the Baskakov-Durrmeyer type operators defined in [8].
2. If $\rho = \mu = -n$, we obtain the Bernstein-Durrmeyer polynomial defined in [9].
3. If $\rho = \mu \to \infty$, we obtain the Phillips operators defined in [18].
4. If $\rho = \mu = n/c$, we obtain the well known Srivastav-Gupta type operators (see [19], [1] and [13]) which is generalized sequence of positive linear operators containing above three cases.
5. If $\rho = n, \mu \to \infty$, we obtain Baskakov-Szász type operators proposed in [3]
6. If $\rho \to \infty$ and $\mu = n$, we obtain the Szász-Beta type operators introduced in [17].
7. If $\rho = nx$ and $\mu = n$, we obtain the Lupaş- Beta operators defined in [10]
8. If $\rho = nx$ and $\mu \to \infty$, we obtain the Lupaş-Szász type operators proposed in [7].

The immense properties of operators (1) motivate us to extend the studies and establish some convergence estimates of these operators. In the present paper we establish an asymptotic formula and the rate of convergence for these operators for the function having derivatives of bounded variation. All the above cases except case $\rho = \mu = -n$ holds true for our results.

2. Moment estimation and auxiliary results

LEMMA 1. [6] *The r-th order moment $V_{n,\rho,\mu}(e_s,x)$, $e_s = t^s$, $s = 0,1,2,\cdots$ satisfy the following representation*

$$V_{n,\rho,\mu}(e_r,x) = nx\frac{\Gamma(\mu-r)\Gamma(r+1)}{\Gamma(\mu)}\left(\frac{\mu}{n}\right)^r\left(1+\frac{nx}{\rho}\right)^{-\rho-1}\sum_{k=0}^{\infty}\frac{(\rho+1)_k(r+1)_k}{(2)_k.k!}\frac{(nx)^k}{(\rho+nx)^k}.$$

REMARK 1. Using Lemma 1, few moments are given by

$$V_{n,\rho,\mu}(e_0,x) = 1$$
$$V_{n,\rho,\mu}(e_1,x) = \frac{\mu x}{\mu-1}$$
$$V_{n,\rho,\mu}(e_2,x) = \frac{\mu^2 x[2\rho+(\rho+1)nx]}{n\rho(\mu-1)(\mu-2)}$$
$$V_{n,\rho,\mu}(e_3,x) = \frac{\mu^3 x[6\rho^2+6\rho(\rho+1)nx+(\rho+1)(\rho+2)n^2x^2]}{n^2\rho^2(\mu-1)(\mu-2)(\mu-3)}$$
$$V_{n,\rho,\mu}(e_4,x) = \frac{\mu^4 x[24\rho^3+36\rho^2(\rho+1)nx+12\rho(\rho+1)(\rho+2)n^2x^2+(\rho+1)(\rho+2)(\rho+3)n^3x^3]}{n^3\rho^3(\mu-1)(\mu-2)(\mu-3)(\mu-4)}.$$

REMARK 2. Also, by Lemma 1, few central moments are given by

$$V_{n,\rho,\mu}(e_1 - xe_0, x) = \frac{x}{\mu - 1}$$
$$V_{n,\rho,\mu}((e_1 - xe_0)^2, x) = \frac{x^2[n\rho\mu + 2n\rho + n\mu^2] + 2\rho\mu^2 x}{n\rho(\mu - 1)(\mu - 2)}.$$

Consequently for each $x \in [0, \infty)$, we have

$$V_{n,\rho,\mu}((e_1 - xe_0)^m, x) = O_x(n^{-[(m+1)/2]}).$$

COROLLARY 1. *From Lemma 1 and using Cauchy-Schwarz inequality, we have*

$$V_{n,\rho,\mu}(|t-x|^r, x) \leqslant \sqrt{V_{n,\rho,\mu}((t-x)^{2r}, x)} = O(n^{-r/2}).$$

Also we have

$$V_{n,\rho,\mu}(|t-x|, x) \leqslant \sqrt{V_{n,\rho,\mu}((t-x)^2, x)}$$
$$= \sqrt{\frac{x^2[n\rho\mu + 2n\rho + n\mu^2] + 2\rho\mu^2 x}{n\rho(\mu - 1)(\mu - 2)}}.$$

Now operators (1) can be redefined as

$$V_{n,\rho,\mu}(f, x) = \langle k_n^{\rho,\mu}(x, .), f \rangle,$$

where

$$k_n^{\rho,\mu}(x,t) = n \sum_{k=1}^{\infty} m_{n,k}^{\rho}(x) m_{n,k-1}^{\mu+1}(t) + m_{n,k}^{\rho}(x)\delta(t).$$

LEMMA 2. *For fixed $x \geqslant 0$ and for sufficiently large n, we have*

$$\beta_n^{\rho,\mu}(x,y) = \int_0^y k_n^{\rho,\mu}(x,t)dt$$
$$\leqslant \frac{x^2[n\rho\mu + 2n\rho + n\mu^2] + 2\rho\mu^2 x}{n\rho(\mu - 1)(\mu - 2)} \cdot \frac{1}{(x-y)^2}, \quad 0 < y < x$$

and

$$1 - \beta_n^{\rho,\mu}(x,z) = \int_z^{\infty} k_n^{\rho,\mu}(x,t)dt$$
$$\leqslant \frac{x^2[n\rho\mu + 2n\rho + n\mu^2] + 2\rho\mu^2 x}{n\rho(\mu - 1)(\mu - 2)} \frac{1}{(z-x)^2}, \quad x < z < \infty.$$

The proof of the above lemma follows along the lines of Remark 2.

3. Direct theorems

THEOREM 1. *Let f be bounded and integrable function on the interval $[0,\infty)$ such that the second derivative of f exists at a fixed point $x \in [0,\infty)$, then we have*

$$\lim_{n\to\infty} n[V_{n,\rho,\mu}(f,x) - f(x)] = A(x)f'(x) + \frac{x(B(x)+C)}{2}f''(x),$$

where $A(x), B(x)$ are functions of x and C is certain constant.

Proof. By the Taylor's expansion of f, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + R(t,x)(t-x)^2,$$

where $\lim_{t\to x} r(t,x) = 0$. Operating $V_{n,\rho,\mu}$ to the above identity, we obtain

$$\begin{aligned} V_{n,\rho,\mu}(f,x) - f(x) &= V_{n,\rho,\mu}((e_1 - xe_0),x)f'(x) + V_{n,\rho,\mu}((e_2 - xe_0)^2,x)\frac{f''(x)}{2} \\ &\quad + V_{n,\rho,\mu}\left(R(t,x)(e_2 - xe_0)^2,x\right). \end{aligned}$$

Using the Cauchy-Schwarz inequality, we have

$$V_{n,\rho,\mu}\left(R(t,x)(e_2 - xe_0)^2,x\right) \leqslant \sqrt{V_{n,\rho,\mu}\left(R^2(t,x),x\right)}\sqrt{V_{n,\rho,\mu}((e_2 - xe_0)^4,x)}.$$

In view of Remark 2, we have

$$\lim_{n\to\infty} V_{n,\rho,\mu}\left(R^2(t,x),x\right) = R^2(x,x) = 0. \tag{2}$$

Thus, we get

$$\lim_{n\to\infty} nV_{n,\rho,\mu}\left(R(t,x)(e_2 - xe_0)^2,x\right) = 0.$$

Thus by Remark 2, we get

$$\begin{aligned} &\lim_{n\to\infty} n\left(V_{n,\rho,\mu}(f,x) - f(x)\right) \\ &= \lim_{n\to\infty} n\Big[V_{n,\rho,\mu}((e_1 - xe_0),x)f'(x) + \frac{1}{2}f''(x)V_{n,\rho,\mu}((e_2 - xe_0)^2,x) \\ &\qquad + V_{n,\rho,\mu}(R(t,x)(e_2 - xe_0)^2,x)\Big] \\ &= A(x)f'(x) + \frac{x(B(x)+C)}{2}f''(x). \quad \square \end{aligned}$$

COROLLARY 2. *Under the assumptions of Theorem 1, the conclusions of the asymptotic formulae will be as follows:*

1. *For the Baskakov-Durrmeyer type operators (see* [8]*) and* $x \geqslant 0$*:*

$$\lim_{n\to\infty} n[V_{n,\rho,\mu}(f,x)-f(x)]_{\rho=\mu=n} = xf'(x)+x(1+x)f''(x).$$

2. *For the Bernstein-Durrmeyer polynomial (see* [9]*), and* $x \in [0,1]$*:*

$$\lim_{n\to\infty} n[V_{n,\rho,\mu}(f,x)-f(x)]_{\rho=\mu=-n} = -xf'(x)+x(1-x)f''(x).$$

3. *For the well-known Phillips operators (see* [18]*) and* $x \geqslant 0$*:*

$$\lim_{n\to\infty} n[V_{n,\rho,\mu}(f,x)-f(x)]_{\rho\to\infty,\mu\to\infty} = xf''(x).$$

4. *For the general Srivastav-Gupta type operators (see* [19], [1] *and* [13]*) and* $x \geqslant 0$ *if* $c \in N\bigcup\{0\}$*;* $x \in [0,1]$ *if* $c=-1$*:*

$$\lim_{n\to\infty} n[V_{n,\rho,\mu}(f,x)-f(x)]_{\rho=\mu=n/c} = cxf'(x)+x(1+cx)f''(x).$$

5. *In case of the Baskakov-Szász type operators (see* [3]*) and* $x \geqslant 0$*:*

$$\lim_{n\to\infty} n[V_{n,\rho,\mu}(f,x)-f(x)]_{\rho=n,\mu\to\infty} = \frac{x(x+2)}{2}f''(x).$$

6. *For the Szász-Beta type operators introduced in* [17] *and* $x \geqslant 0$*:*

$$\lim_{n\to\infty} n[V_{n,\rho,\mu}(f,x)-f(x)]_{\rho=\infty,\mu=n} = xf'(x)+\frac{x(x+2)}{2}f''(x).$$

7. *In case of the Lupaş- Beta operators defined in* [10] *and* $x \geqslant 0$*:*

$$\lim_{n\to\infty} n[V_{n,\rho,\mu}(f,x)-f(x)]_{\rho=nx,\mu=n} = xf'(x)+\frac{x(x+3)}{2}f''(x).$$

8. *For the Lupaş-Szász type operators proposed in* [7] *and for* $x \geqslant 0$*:*

$$\lim_{n\to\infty} n[V_{n,\rho,\mu}(f,x)-f(x)]_{\rho=nx,\mu\to\infty} = \frac{3x}{2}f''(x).$$

We describe the class DB_φ of entirely continuous function f having a derivative of bounded variation on the interval $[0,\infty)$ as

$$DB_\varphi = \left\{ f : f(x) = f(c) + \int_c^x \varphi(t)dt; f(t) = O(t^r), t \to \infty \right\} \tag{3}$$

where φ is a function of bounded variation on every finite sub interval of $[0,\infty)$ and $|f(t)| \leqslant Mt^r$ for $r \geqslant 0$.

THEOREM 2. *Let $f \in DB_{\varphi}$ for all $x \in [0,\infty)$ with the condition $\rho \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} n/\rho = l$ and $\mu \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} n/\mu = m$, then for adequately large n, we have*

$$\begin{aligned}
&\left|V_{n,\rho,\mu}(f,x) - f(x)\right| \\
&\leqslant \frac{\mu}{\mu+1}\left|\varphi(x+) - \varphi(x-)\right| \sqrt{\frac{x^2[n\rho\mu + 2n\rho + n\mu^2] + 2\rho\mu^2 x}{n\rho(\mu-1)(\mu-2)}} \\
&\quad + \frac{\mu}{\mu+1}\left|\varphi(x+) - \varphi(x-)\right| \frac{x^2[n\rho\mu + 2n\rho + n\mu^2] + 2\rho\mu^2 x}{n\rho(\mu-1)(\mu-2)} \sum_{i=1}^{[\sqrt{n}]} \vee_{x-x/i}^{x+x/i} \varphi_x \\
&\quad + \frac{x}{\sqrt{n}} \vee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} + \frac{|f(x)|}{x} \frac{\mu}{\mu+1}\left|\varphi(x+) - \varphi(x-)\right| \left\{\frac{x[n\rho\mu + 2n\rho + n\mu^2] + 2\rho\mu^2}{n\rho(\mu-1)(\mu-2)}\right\} \\
&\quad + \frac{x^2[n\rho\mu + 2n\rho + n\mu^2] + 2\rho\mu^2 x}{n\rho(\mu-1)(\mu-2)} \{|f(2x) - f(x) - x\varphi(x+)| + |f(x)|\} \\
&\quad + |\varphi(x+)| \sqrt{\frac{x^2[n\rho\mu + 2n\rho + n\mu^2] + 2\rho\mu^2 x}{n\rho(\mu-1)(\mu-2)}} + M2^r O(n^{-r/2}),
\end{aligned}$$

where the secondary function φ_x is given by

$$\varphi_x(t) = \begin{cases} \varphi(t) - \varphi(x-) & 0 \leqslant < x \\ 0 & t = x \\ \varphi(t) - \varphi(x+) & x < t < \infty \end{cases}$$

and $V_a^b \varphi_x$ denotes the total varitions of φ_x on $[a,b]$

Proof. From the definition of operators (1) and (3), we have

$$\begin{aligned}
V_{n,\rho,\mu}(f,x) - f(x) &= \int_0^{\infty} k_n^{\rho,\mu}(x,t)(f(t) - f(x))dt \\
&= \int_0^{\infty} k_n^{\rho,\mu}(x,t)\left(\int_x^t \varphi(u)du\right)dt.
\end{aligned} \tag{4}$$

For $\varphi \in DB_{\varphi}$, using (4) and applying the identity

$$\begin{aligned}
\varphi(u) &= \varphi_x(u) + \frac{\varphi(x+) + \mu\varphi(x-)}{\mu+1} \\
&\quad + \frac{\varphi(x+) - \varphi(x-)}{2}\left(sgn(u-x) + \frac{\mu-1}{\mu+1}\right) \\
&\quad + \left(\varphi(u) - \frac{\varphi(x+) - \mu\varphi(x-)}{2}\right)\chi_x(u),
\end{aligned} \tag{5}$$

where

$$\chi_x(u) = \begin{cases} 1 & u = x \\ 0 & u \neq x. \end{cases}$$

From (4) and (5), we have

$$\begin{aligned} V_{n,\rho,\mu}(f,x)-f(x) &= -E_1^{n,\rho,\mu}(\varphi_x,x)+E_2^{n,\rho,\mu}(\varphi_x,x)+E_3^{n,\rho,\mu}(\varphi_x,x) \\ &\quad +E_4^{n,\rho,\mu}+E_5^{n,\rho,\mu}+E_6^{n,\rho,\mu}, \end{aligned}$$

where

$$\begin{aligned} E_1^{n,\rho,\mu}(\varphi_x,x) &= \int_0^x \left(\int_t^x \varphi_x(u)du\right) k_n^{\rho,\mu}(x,t)dt, \\ E_2^{n,\rho,\mu}(\varphi_x,x) &= \int_x^{2x} \left(\int_x^t \varphi_x(u)du\right) k_n^{\rho,\mu}(x,t)dt, \\ E_3^{n,\rho,\mu}(\varphi_x,x) &= \int_{2x}^{\infty} \left(\int_x^t \varphi_x(u)du\right) k_n^{\rho,\mu}(x,t)dt, \\ E_4^{n,\rho,\mu} &= \int_0^{\infty} \left(\int_x^t \frac{\varphi(x+)+\mu\varphi(x-)}{\mu+1}du\right) k_n^{\rho,\mu}(x,t)dt, \\ E_5^{n,\rho,\mu} &= \int_0^{\infty} \left(\int_x^t \frac{\varphi(x+)-\varphi(x-)}{2}\left(sgn(u-x)+\frac{\mu-1}{\mu+1}\right)du\right) k_n^{\rho,\mu}(x,t)dt, \\ E_6^{n,\rho,\mu} &= \int_0^{\infty} \left(\int_x^t \left(\varphi(u)-\frac{\varphi(x+)-\varphi(x-)}{2}\right)\chi_x(u)du\right) k_n^{\rho,\mu}(x,t)dt. \end{aligned}$$

From Lemma 2, we obtain

$$\begin{aligned} E_4^{n,\rho,\mu} &= \frac{\varphi(x+)+\mu\varphi(x-)}{\mu+1}\int_0^{\infty}(t-x)k_n^{\rho,\mu}(x,t)dt \\ &= \frac{\varphi(x+)+\mu\varphi(x-)}{\mu+1}.V_{n,\rho,\mu}(e_1-xe_0,x) \\ &= \frac{\varphi(x+)+\mu\varphi(x-)}{\mu+1}\frac{x}{\mu-1}. \end{aligned}$$

From Lemma 2 and Corollary 1 for sufficiently large n, we have

$$\begin{aligned} E_5^{n,\rho,\mu} &= \frac{\varphi(x+)-\varphi(x-)}{2}\left[-\int_0^x\left(\int_t^x\left(sgn(u-x)+\frac{\mu-1}{\mu+1}\right)du\right)K_n^{\rho,\mu}(x,t)dt\right] \\ &\quad +\frac{\varphi(x+)-\varphi(x-)}{2}\left[-\int_x^{\infty}\left(\int_x^t\left(sgn(u-x)+\frac{\mu-1}{\mu+1}\right)du\right)K_n^{\rho,\mu}(x,t)dt\right] \\ &\leqslant \frac{\mu}{\mu+1}|\varphi(x+)-\varphi(x-)|V_{n,\rho\mu}(|t-x|,x) \\ &= \frac{\mu}{\mu+1}|\varphi(x+)-\varphi(x-)|\sqrt{\frac{x^2[n\rho\mu+2n\rho+n\mu^2]+2\rho\mu^2x}{n\rho(\mu-1)(\mu-2)}}. \end{aligned}$$

$E_6^{n,\rho,\mu}$ is obviously zero by definition of $\chi_x(u)$. Now by Stieltjes integral and integration by part, for $y = x - x\sqrt{n}$, we have

$$\begin{aligned} E_1^{n,\rho,\mu}(\varphi_x,x) &= \int_0^x \left(\int_t^x \varphi_x(u)du\right) d_t\left(\beta_n^{\rho,\mu}(x,t)\right)dt \\ &= \int_0^x \varphi_x(t)\beta_n^{\rho,\mu}(x,t)dt \\ &= \int_0^{x-x/\sqrt{n}} \varphi_x(t)\beta_n^{\rho,\mu}(x,t)dt \\ &= \int_{x-x/\sqrt{n}}^x \varphi_x(t)\beta_n^{\rho,\mu}(x,t)dt. \end{aligned}$$

Since $\beta_n^{\rho,\mu}(x,t) \leqslant 1$ and $\varphi_x(x) = 0$, we have

$$\left|\int_{x-x/\sqrt{n}}^x (\varphi_x(t) - \varphi_x(x))\beta_n^{\rho,\mu}(x,t)dt\right| \leqslant \int_{x-x/\sqrt{n}}^x V_t^x\varphi_x dt \leqslant \frac{x}{\sqrt{n}} V_{x-x\sqrt{n}}^x \varphi_x,$$

and from Lemma 2 and taking $h = \frac{x}{x-t}$, we get

$$\begin{aligned} \left|\int_0^{x-x/\sqrt{n}} \varphi_x(t)\beta_n^{\rho,\mu}(x,t)dt\right| &\leqslant \frac{x^2[n\rho\mu + 2n\rho + n\mu^2] + 2\rho\mu^2 x}{n\rho(\mu-1)(\mu-2)} \int_0^{x-x/\sqrt{n}} V_t^x \varphi_x \frac{1}{(x-t)^2} dt \\ &= \frac{x^2[n\rho\mu + 2n\rho + n\mu^2] + 2\rho\mu^2 x}{n\rho(\mu-1)(\mu-2)} \int_1^{\sqrt{n}} V_{x-x/\sqrt{n}}^x \varphi_x dh \\ &\leqslant \frac{x^2[n\rho\mu + 2n\rho + n\mu^2] + 2\rho\mu^2 x}{n\rho(\mu-1)(\mu-2)} \sum_{i=1}^{[\sqrt{n}]} V_{x-x/\sqrt{n}}^x \varphi_x. \end{aligned}$$

Thus, we have

$$\begin{aligned} \left|E_1^{n,\rho,\mu}(\varphi_x,x)\right| \leqslant{} & \frac{x^2[n\rho\mu + 2n\rho + n\mu^2] + 2\rho\mu^2 x}{n\rho(\mu-1)(\mu-2)} \sum_{i=1}^{[\sqrt{n}]} V_{x-x/\sqrt{i}}^x \varphi_x \\ & + \frac{x}{\sqrt{n}} V_{x-x\sqrt{n}}^x \varphi_x. \end{aligned}$$

Now we find estimate of $E_2^{n,\rho,\mu}(\varphi_x,x)$ using Lemma 2 and integration by part, we obtain

$$\begin{aligned} E_2^{n,\rho,\mu}(\varphi_x,x) &= \int_x^{2x} \left(\int_x^t \varphi_x(u)du\right) d_t\left(\beta_n^{\rho,\mu}(x,t)\right)dt \\ &= -\int_x^{2x} \left(\int_x^t \varphi_x(u)du\right) d_t\left(1 - \beta_n^{\rho,\mu}(x,t)\right)dt \\ &= -\int_x^{2x} \varphi_x(u)du\left(1 - \beta_n^{\rho,\mu}(x,2x)\right) \\ &\quad + \int_x^{2x} \varphi_x(t)\left(1 - \beta_n^{\rho,\mu}(x,t)\right)dt \\ &=: E_2^1 + E_2^2. \end{aligned}$$

From Lemma 1

$$
\begin{aligned}
|E_2^1| &\leqslant \left|\int_x^{2x}\varphi_x(u)du\right||1-\beta_n^{\rho,\mu}(x,2x)| \\
&\leqslant \frac{x^2[n\rho\mu+2n\rho+n\mu^2]+2\rho\mu^2 x}{n\rho(\mu-1)(\mu-2)}\left|\int_x^{2x}(\varphi(u)-\varphi(x+))du\right| \\
&= \frac{x[n\rho\mu+2n\rho+n\mu^2]+2\rho\mu^2}{n\rho x(\mu-1)(\mu-2)}|f(2x)-f(x)-x\varphi(x+)|,
\end{aligned}
$$

and

$$
\begin{aligned}
|E_2^2| &\leqslant \left|\int_x^{x+x/\sqrt{n}}\varphi_x(t)dt\right| + \frac{x^2[n\rho\mu+2n\rho+n\mu^2]+2\rho\mu^2 x}{n\rho(\mu-1)(\mu-2)}\left|\int_{x+x/\sqrt{n}}^{2x}\frac{\varphi_x(t)}{(t-x)^2}dt\right| \\
&= \frac{x}{\sqrt{n}}V_x^{x+x\sqrt{n}}\varphi_x + \frac{x^2[n\rho\mu+2n\rho+n\mu^2]+2\rho\mu^2 x}{n\rho(\mu-1)(\mu-2)}\sum_{i=1}^{[\sqrt{n}]}V_x^{x+x/\sqrt{i}}\varphi_x.
\end{aligned}
$$

Next, we estimate $E_3^{n,\rho,\mu}(\varphi_x,x)$ as follows. Let there exist an integer r such that $f(t)=O(t^r)$ as $t\to\infty$ then for some positive constant M depending on f,x,r, then we get

$$
\begin{aligned}
&\left|E_3^{n,\rho,\mu}(\varphi_x,x)\right| \\
&= \left|\int_{2x}^{\infty}\left(\int_x^t(\varphi(t)-\varphi(x+))du\right)k_n^{\rho,\mu}(x,t)dt\right|, \\
&\leqslant \left|\int_{2x}^{\infty}\left(\int_x^t\varphi(t)du\right)k_n^{\rho,\mu}(x,t)dt\right| + |\varphi(x+)|\left|\int_{2x}^{\infty}k_n^{\rho,\mu}(x,t)dt\right| \\
&= \left|\int_{2x}^{\infty}(f(t)-f(x))k_n^{\rho,\mu}(x,t)dt\right| + |\varphi(x+)|\left|\int_{2x}^{\infty}k_n^{\rho,\mu}(x,t)dt\right| \\
&\leqslant \left|\int_{2x}^{\infty}f(t)k_n^{\rho,\mu}(x,t)dt\right| + |f(x)|\left|\int_{2x}^{\infty}k_n^{\rho,\mu}(x,t)dt\right| + |\varphi(x+)|\left|\int_{2x}^{\infty}k_n^{\rho,\mu}(x,t)dt\right| \\
&\leqslant M\left|\int_{2x}^{\infty}t^{2r}k_n^{\rho,\mu}(x,t)dt\right| + |f(x)|\left|\int_{2x}^{\infty}k_n^{\rho,\mu}(x,t)dt\right| + |\varphi(x+)|\left|\int_{2x}^{\infty}k_n^{\rho,\mu}(x,t)dt\right|,
\end{aligned}
$$

using the inequality $t\leqslant 2(t-x)$ for $t>2x$, we have

$$
\begin{aligned}
\left|E_3^{n,\rho,\mu}(\varphi_x,x)\right| \leqslant M&\left|\int_{2x}^{\infty}2^r(t-x)^2k_n^{\rho,\mu}(x,t)dt\right| \\
&+\frac{|f(x)|}{x^2}\left|\int_{2x}^{\infty}(t-x)^2k_n^{\rho,\mu}(x,t)dt\right| \\
&+|\varphi(x+)|\left|\int_{2x}^{\infty}k_n^{\rho,\mu}(x,t)dt\right|.
\end{aligned}
$$

Using Corollary 1 and Lemma 1 and Hölder inequality, we have following estimation

$$\left|E_3^{n,\rho,\mu}(\varphi_x,x)\right| \leqslant M2^r O(n^{-r/2}) + \frac{|f(x)|}{x}\left(\frac{x[n\rho\mu+2n\rho+n\mu^2]+2\rho\mu^2}{n\rho(\mu-1)(\mu-2)}\right)$$
$$+|\varphi(x+)|\frac{x^2[n\rho\mu+2n\rho+n\mu^2]+2\rho\mu^2 x}{n\rho(\mu-1)(\mu-2)}.$$

Collecting all the estimates $E_1^{n,\rho,\mu}(\varphi_x,x), E_2^{n,\rho,\mu}(\varphi_x,x), E_3^{n,\rho,\mu}(\varphi_x,x), E_4^{n,\rho,\mu}, E_5^{n,\rho,\mu}$ and $E_6^{n,\rho,\mu}$, we obtain the required result. □

REMARK 3. one can obtain rate of convergence of different operators mentioned in Section 1 for the different values of ρ and μ from above theorem.

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