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APPROXIMATION BY INTERPOLATION: THE CHEBYSHEV NODES

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Abstract. In this paper, we first revisit the well-known result stating that the Hermite interpolation polynomials of a function f continuous on $[-1,1]$, with the zeros of the Chebyshev polynomials of the first kind as nodes, converge uniformly to f on $[-1,1]$. Then we extend this result to obtain the uniform convergence of the Hermite interpolation polynomials, with the nodes taken as the zeros of the Chebyshev polynomials of the second, third and fourth kind, not on the interval $[-1,1]$ but rather on the intervals $[-\frac{2\sqrt{2}}{3},\frac{2\sqrt{2}}{3}]$, $[-\frac{\sqrt{3}}{2},1]$, $[-1,\frac{\sqrt{3}}{2}]$, respectively.

1. Introduction

Given $n+1$ distinct real numbers $\{z_{n,i}\}_{i=0}^{n}$ and $n+1$ real values $\{w_i\}_{i=0}^{n}$, there exists a unique polynomial $p_n(z)$ of degree at most n such that $p_n(z_{n,i}) = w_i$, $i = 0,\dots,n$. We can construct this interpolation polynomial using the Lagrange and Newton methods (see e.g. [1, 6, 7, 8]). The Weierstrass theorem (see e.g. [2, 10, 11]) states that if f is a continuous function on a closed interval $[a,b]$, we can find a family of polynomials which converges uniformly to f on $[a,b]$. In fact, it is shown for example in [2, 6, 10] that if $f\in\mathscr{C}[0,1]$, then the Bernstein polynomials $\Big(B_n(f;x)\Big)_n$ defined by

$$B_n(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}, n\in\mathbb{N},$$

converge uniformly to f on $[0,1]$. It follows that the polynomial family $(G_n(f,x))_n$ with

$$G_n(f,x)=\frac{1}{(b-a)^n}\sum_{k=0}^{n} f\left(a+(b-a)\frac{k}{n}\right)\binom{n}{k}(x-a)^k(b-x)^{n-k}$$

converges uniformly on $[a,b]$ to $f\in\mathscr{C}[a,b]$ for $a,b\in\mathbb{R}$. Moreover if $f\in\mathscr{C}^p[0,1]$, then the polynomials $B_{n+p}^{(p)}(f;z)$ converge uniformly to $f^{(p)}(z)$ on $[0,1]$ (see e.g. [2, 10]).

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Consider $n+1$ distinct real numbers $\{z_{n,i}\}_{i=0}^{n}$ and let $f \in \mathscr{C}^1[-1,1]$, then there exists (see e.g. [2, 10]) a unique polynomial $Q_{2n+1}(f;z)$ of degree at most $2n+1$ such that

$$\begin{cases} Q_{2n+1}(f;z_{n,i}) = f(z_{n,i}), & i=0,1,\ldots,n, \\ Q'_{2n+1}(f;z_{n,i}) = f'(z_{n,i}), & i=0,1,\ldots,n. \end{cases} \tag{1}$$

The polynomial $Q_{2n+1}(f;z)$ is called the Hermite interpolating polynomial for f (see e. g. [1, Section 3.6], [6], [10, p. 13]). There exists a basis $\{A_{n,i}(z), B_{n,i}(z),\ i = 0,1,2,\ldots,n\}$ $(A_{n,i}(z), B_{n,i}(z) \in \mathbb{R}_{2n+1}[z],$ for each $i=0,1,\ldots,n)$ such that

$$Q_{2n+1}(f;z) = \sum_{i=0}^{n} f(z_{n,i})A_{n,i}(z) + \sum_{i=0}^{n} f'(z_{n,i})B_{n,i}(z). \tag{2}$$

REMARK 1. For $f(z)=1$, $z\in[-1,1]$, (2) leads to

$$1 = Q_{2n+1}(f;z) = \sum_{i=0}^{n} f(z_{n,i})A_{n,i}(z) = \sum_{i=0}^{n} A_{n,i}(z),$$

that is

$$\sum_{i=0}^{n} A_{n,i}(z) = 1. \tag{3}$$

Looking for necessary conditions on $A_{n,i}(z)$ and $B_{n,i}(z)$ in (2) for (1) to be satisfied, we get

$$\begin{cases} A_{n,i}(z_{n,j}) = \delta_{ij} \text{ and } B_{n,i}(z_{n,j}) = 0, & i,j=0,1,\ldots,n \\ A'_{n,i}(z_{n,j}) = 0 \text{ and } B'_{n,i}(z_{n,j}) = \delta_{ij}, & i,j=0,1,\ldots,n. \end{cases} \tag{4}$$

Let $\{l_{n,i}(z),\ i=0,1,\ldots,n\}$ be the Lagrange basis polynomials in $\mathbb{R}_n[z]$ (the set of polynomials of degree at most n with real coefficients) defined by

$$l_{n,k}(z) = \prod_{\substack{i=0\\ i\neq k}}^{n} \left(\frac{z-z_{n,i}}{z_{n,k}-z_{n,i}} \right), \quad k=0,1,2,\ldots,n,$$

then the Hermite basis polynomials $\{A_{n,i}(z), B_{n,i}(z),\ i=0,1,\ldots,n\}$ in $\mathbb{R}_{2n+1}[z]$ are given by:

$$\begin{aligned} A_{n,i}(z) &= (1-2(z-z_{n,i})l'_{n,i}(z_{n,i}))l_{n,i}^2(z),\ i=0,1,\ldots,n, \\ B_{n,i}(z) &= (z-z_{n,i})l_{n,i}^2(z),\ i=0,1,\ldots,n. \end{aligned} \tag{5}$$

This means that the Hermite interpolation polynomials (2) are given in terms of the Lagrange basis polynomials as ([6, Eq. (3.50)], [10, Eq. (1.38)])

$$Q_{2n+1}(f;z) = \sum_{i=0}^{n} f(z_{n,i})\Big(1-2(z-z_{n,i})l'_{n,i}(z_{n,i})\Big)l_{n,i}^2(z) + \sum_{i=0}^{n} f'(z_{n,i})(z-z_{n,i})l_{n,i}^2(z). \tag{6}$$

We can show by direct calculus that for the Newton polynomial (see e.g. [7])

$$N_k(z) = \begin{cases} 1, & \text{if } k=0 \\ (z-z_{n+1,0})(z-z_{n+1,1})\cdots(z-z_{n+1,k-1}), & \text{if } 1 \leqslant k \leqslant n+1, \end{cases}$$

we have

$$l_{n,i}(z) = \frac{N_{n+1}(z)}{(z-z_{n+1,i})N'_{n+1}(z_{n+1,i})} \text{ and } 2l'_{n,i}(z_{n+1,i}) = \frac{N''_{n+1}(z_{n+1,i})}{N'_{n+1}(z_{n+1,i})}. \tag{7}$$

Let $0<\theta<\pi$, set $z=\cos\theta$ and define for $n=0,1,\ldots$

$$T_n(z) = \cos(n\theta).$$

For $n=1,2,\ldots,$ $T_n(z)$ is a polynomial of degree n in the variable z with leading coefficient 2^{n-1} and is called the Chebyshev polynomial of first kind. The roots $z_{n,k}$ of $T_n(z)$ in increasing order are given by

$$z_{n,k} = \cos\theta_{n,k}, \text{ with } \theta_{n,k} = \frac{(2(n-k)-1)\pi}{2n}, \; k=0,1,\ldots,n-1. \tag{8}$$

The zeros of orthogonal polynomials play a very important role in interpolation theory, quadrature formulas, etc. (see e. g. [3, 8, 9]). One can show that for a given function $f \in \mathscr{C}[-1,1]$, the Hermite interpolation polynomials $Q_{2n+1}(f;z)$ which satisfy

$$\begin{cases} Q_{2n+1}(f;z_{n+1,i}) = f(z_{n+1,i}), & i=0,1,\ldots,n, \\ Q'_{2n+1}(f;z_{n+1,i}) = 0, & i=0,1,\ldots,n, \end{cases} \tag{9}$$

converge uniformly on $[-1,1]$ to f (see e.g. [3]). We will revisit the proof of this very well known result in the second section of this work. The main question we are going to answer in section 3 (and which is our main contribution in this work) is whether this result is still valid if we consider now the zeros of the Chebyshev polynomials of the second, third and fourth kind defined, respectively, for $z=\cos\theta$, $0<\theta<\pi$ by (see e.g. [8, p. 123], [13])

$$U_n(z) = \frac{\sin((n+1)\theta)}{\sin\theta}, \quad V_n(z) = \frac{\cos((n+\frac{1}{2})\theta)}{\cos(\frac{\theta}{2})}, \quad W_n(z) = \frac{\sin((n+\frac{1}{2})\theta)}{\sin(\frac{\theta}{2})}? \tag{10}$$

The zeros of $U_n(z)$, $V_n(z)$ and $W_n(z)$ in increasing order are given, respectively, by

$$z_{n,k} = \cos\theta_{n,k}, \text{ with } \theta_{n,k} = \frac{n-k}{n+1}\pi, \; k=0,1,\ldots,n-1, \tag{11}$$

$$z_{n,k} = \cos\theta_{n,k}, \text{ with } \theta_{n,k} = \frac{2(n-k)-1}{2n+1}\pi, \; k=0,1,\ldots,n-1, \tag{12}$$

$$z_{n,k} = \cos\theta_{n,k}, \text{ with } \theta_{n,k} = \frac{n-k}{n+\frac{1}{2}}\pi, \; k=0,1,2,\ldots,n-1. \tag{13}$$

In the sequel, we will denote by $A_{j,n,i}(z)$ and $Q_{j,2n+1}(f;z)$, $j=1,2,3,4$, the polynomial $A_{n,i}(z)$ (see (5)) and the Hermite interpolation polynomials $Q_{2n+1}(f;z)$ (see (6)) with the zeros of $T_{n+1}(z)$, $U_{n+1}(z)$, $V_{n+1}(z)$ and $W_{n+1}(z)$ as nodes, respectively.

As Atkinson [1] wrote, the Chebyshev polynomials are extremely important in approximation theory and they also arise in many other areas of applied mathematics. For a more complete discussion of them, see e. g. [4, 12]. One area above all in which the Chebyshev polynomials have a pivotal role is the minimax approximation of functions by polynomials [4, 7]. The monic Chebyshev polynomials of the first kind of degree n is the polynomial deviating less from zero on $[-1,1]$ among monic polynomials of degree n:

$$\min\left\{\max_{-1\leqslant x\leqslant 1}|q_n(x)|, q_n\in\mathbb{R}[x],\ q_n=x^n+\dots\right\}=\max_{-1\leqslant x\leqslant 1}\left|2^{1-n}T_n(x)\right|=2^{1-n}.$$

2. Approximation using the zeros of the first kind Chebyshev polynomials as nodes (see e.g. [3])

For this polynomial family, we have

LEMMA 1. (see e.g. [3]) *The polynomial $A_{1,n,i}(z)$ for the Chebyshev polynomials of the first kind is given by*

$$A_{1,n,i}(z)=(1-zz_{n+1,i})\left(\frac{T_{n+1}(z)}{(n+1)(z-z_{n+1,i})}\right)^2. \tag{14}$$

Proof.

$$T'_{n+1}(z_{n+1,i})=(n+1)\frac{\sin(n+1)\theta_{n+1,i}}{\sin\theta_{n+1,i}},$$
$$T''_{n+1}(z_{n+1,i})=(n+1)\frac{z_{n+1,i}\sin(n+1)\theta_{n+1,i}}{\sin^3\theta_{n+1,i}}.$$

In fact,

$$\begin{aligned}T'_{n+1}(z)&=\frac{d}{d\theta}\cos(n+1)\theta\cdot\frac{d\theta}{dz}\\&=-(n+1)\frac{\sin(n+1)\theta}{-\sin\theta}\\&=(n+1)\frac{\sin(n+1)\theta}{\sin\theta},\end{aligned}$$

and for $z=z_{n+1,i}=\cos\theta_{n+1,i}$, we get

$$T'_{n+1}(z_{n+1,i})=(n+1)\frac{\sin(n+1)\theta_{n+1,i}}{\sin\theta_{n+1,i}}.$$

$$\begin{aligned}
T''_{n+1}(z) &= \frac{d}{d\theta}(T'_{n+1}(z)) \cdot \frac{d\theta}{dz} \\
&= \frac{d}{d\theta}\Big((n+1)\frac{\sin(n+1)\theta}{\sin\theta}\Big) \cdot \frac{d\theta}{dz} \\
&= \Big(\frac{(n+1)^2 \sin\theta \cos(n+1)\theta - (n+1)\sin(n+1)\theta\cos\theta}{\sin^2\theta}\Big)\Big(\frac{-1}{\sin\theta}\Big) \\
&= -\frac{(n+1)^2 \sin\theta\cos(n+1)\theta}{\sin^2\theta} + \frac{(n+1)\sin(n+1)\theta\cos\theta}{\sin^3\theta}.
\end{aligned}$$

For $z = z_{n+1,i} = \cos\theta_{n+1,i}$, we get

$$T''_{n+1}(z_{n+1,i}) = \frac{(n+1)z_{n+1,i}\sin(n+1)\theta_{n+1,i}}{\sin^3\theta_{n+1,i}}.$$

Since $z_{n+1,i}$, $i = 0,1,\dots,n$ are the zeros of $T_{n+1}(z)$ which is a polynomial of degree $n+1$ with leading coefficient 2^n, then $T_{n+1}(z) = 2^n N_{n+1}(z)$. It follows that $T'_{n+1}(z) = 2^n N'_{n+1}(z)$, $T''_{n+1}(z) = 2^n N''_{n+1}(z)$ which together with (7) give

$$\begin{aligned}
A_{1,n,i}(z) &= \Big(1 - 2(z - z_{n+1,i})l'_{n,i}(z_{n+1,i})\Big)l^2_{n,i}(z) \\
&= \Big(1 - (z - z_{n+1,i})(\frac{T''_{n+1}(z_{n+1,i})}{T'_{n+1}(z_{n+1,i})})\Big)\Big(\frac{T_{n+1}(z)}{(z - z_{n+1,i})T'_{n+1}(z_{n+1,i})}\Big)^2 \\
&= \Bigg(1 - (z - z_{n+1,i})\frac{\dfrac{(n+1)z_{n+1,i}\sin(n+1)\theta_{n+1,i}}{\sin^3\theta_{n+1,i}}}{(n+1)\dfrac{\sin(n+1)\theta_{n+1,i}}{\sin\theta_{n+1,i}}}\Bigg)\Big(\frac{T_{n+1}(z)}{(z - z_{n+1,i})T'_{n+1}(z_{n+1,i})}\Big)^2 \\
&= \Big(1 - (z - z_{n+1,i})\frac{z_{n+1,i}}{\sin^2\theta_{n+1,i}}\Big)\Big(\frac{T_{n+1}(z)\sin\theta_{n+1,i}}{(n+1)(z - z_{n+1,i})\sin(n+1)\theta_{n+1,i}}\Big)^2.
\end{aligned}$$

Since $(\sin(n+1)\theta_{n+1,i})^2 = (\sin(2(n-i)+1)\frac{\pi}{2})^2 = ((-1)^{n-i})^2 = 1$ and $\sin^2\theta_{n+1,i} = 1 - z^2_{n+1,i}$, we obtain

$$\begin{aligned}
A_{1,n,i}(z) &= \Big(1 - z^2_{n+1,i} - (z - z_{n+1,i})z_{n+1,i}\Big)\Big(\frac{T_{n+1}(z)}{(n+1)(z - z_{n+1,i})}\Big)^2 \\
&= (1 - zz_{n,i})\Big(\frac{T_{n+1}(z)}{(n+1)(z - z_{n+1,i})}\Big)^2. \quad \Box
\end{aligned}$$

Let us now state and prove the well known interpolation and approximation result for the first kind Chebyshev polynomials.

THEOREM 1. (see e.g. [3]) *Let $f \in \mathscr{C}[-1,1]$, the Hermite interpolation polynomials $Q_{1,2n+1}(f;z)$ (at the zeros $z_{n+1,k}$, $k = 0,1,\dots,n$, of the Chebyshev polynomials $T_{n+1}(z)$ given by* (8)*) which satisfies* (9) *converge uniformly on $[-1,1]$ to f.*

Proof. $z, z_{n+1,i} \in [-1,1] \Rightarrow zz_{n+1,i} \in [-1,1]$ and then $1 - zz_{n+1,i} \geqslant 0$, thus $A_{1,n,i}(z) \geqslant 0$. From (3), we have $f(z) = \sum_{i=0}^{n} f(z) A_{1,n,i}(z)$.

Let $\varepsilon > 0$ and $z \in [-1,1]$. We want to show that

$$\exists\, N_\varepsilon \in \mathbb{N} \text{ such that } \forall n \geqslant N_\varepsilon,\ |f(z) - Q_{1,2n+1}(f;z)| < \varepsilon.$$

$$\begin{aligned} |f(z) - Q_{1,2n+1}(f;z)| &= \Big| \sum_{i=0}^{n} (f(z) - f(z_{n+1,i})) A_{1,n,i}(z) \Big| \\ &\leqslant \sum_{i=0}^{n} |f(z) - f(z_{n+1,i})| A_{1,n,i}(z). \end{aligned}$$

f continuous on $[-1,1]$ and $[-1,1]$ is compact implies f is uniformly continuous on $[-1,1]$. That is $\exists \delta_\varepsilon > 0$ such that $\forall\, x,y \in [-1,1]$, $|x-y| < \delta_\varepsilon$ implies $|f(x) - f(y)| < \varepsilon$. Let

$$I_{n,\varepsilon,z} := \{ i \in \{0,1,\ldots,n\} \ :\ |z - z_{n+1,i}| < \delta_\varepsilon \}$$

and

$$J_{n,\varepsilon,z} := \{ i \in \{0,1,\ldots,n\} \ :\ |z - z_{n+1,i}| \geqslant \delta_\varepsilon \}.$$

Then $I_{n,\varepsilon,z} \cup J_{n,\varepsilon,z} = \{0,1,\ldots,n\}$ and $I_{n,\varepsilon,z} \cap J_{n,\varepsilon,z} = \emptyset$, thus

$$\begin{aligned} \sum_{i=0}^{n} |f(z) - f(z_{n+1,i})| A_{1,n,i}(z) &= \sum_{i \in I_{n,\varepsilon,z}} |f(z) - f(z_{n+1,i})| A_{1,n,i}(z) \\ &\quad + \sum_{i \in J_{n,\varepsilon,z}} |f(z) - f(z_{n+1,i})| A_{1,n,i}(z). \end{aligned}$$

If $i \in I_{n,\varepsilon,z}$, $|z - z_{n+1,i}| < \delta_\varepsilon$ and then $|f(z) - f(z_{n+1,i})| < \frac{\varepsilon}{2}$ such that

$$\sum_{i \in I_{n,\varepsilon,z}} |f(z) - f(z_{n+1,i})| A_{1,n,i}(z) < \frac{\varepsilon}{2} \sum_{i \in I_{n,\varepsilon,z}} A_{1,n,i}(z) < \frac{\varepsilon}{2} \sum_{i=0}^{n} A_{1,n,i}(z) = \frac{\varepsilon}{2},$$

where we use respectively the fact that $A_{1,n,i}(z) \geqslant 0$, $I_{n,\varepsilon,z} \subset \{0,1,\ldots,n\}$ and $\sum_{i=0}^{n} A_{1,n,i}(z) = 1$.

$f \in \mathscr{C}[-1,1]$ implies f is bounded. That is, $\exists\, M > 0$ suct that $|f(z)| < M$, $\forall\, z \in [-1,1]$. Therefore $|f(z) - f(z_{n+1,i})| \leqslant |f(z)| + |f(z_{n+1,i})| \leqslant 2M$. So

$$\sum_{i \in J_{n,\varepsilon,z}} |f(z) - f(z_{n+1,i})| A_{1,n,i}(z) \leqslant 2M \sum_{i \in J_{n,\varepsilon,z}} A_{1,n,i}(z).$$

We have

$$i \in J_{n,\varepsilon,z} \Rightarrow |z - z_{n+1,i}| \geqslant \delta_\varepsilon \Leftrightarrow \frac{1}{|z - z_{n+1,i}|^2} \leqslant \frac{1}{\delta_\varepsilon^2},$$

$|z| \leqslant 1,\ |z_{n+1,i}| \leqslant 1 \Rightarrow |zz_{n+1,i}| \leqslant 1 \Leftrightarrow -1 \leqslant zz_{n+1,i} \leqslant 1 \Leftrightarrow 0 \leqslant 1 - zz_{n+1,i} \leqslant 2 \Rightarrow 1 - zz_{n+1,i} \leqslant 2$, and

$$|T_{n+1}(z)| = |\cos(n+1)\theta| \leqslant 1.$$

This leads to

$$\sum_{i\in J_{n,\varepsilon,z}} |f(z)-f(z_{n+1,i})|A_{1,n,i}(z) \leqslant 2M \sum_{i\in J_{n,\varepsilon,z}} (1-zz_{n+1,i})\left(\frac{T_{n+1}(z)}{(n+1)(z-z_{n+1,i})}\right)^2$$
$$\leqslant \frac{4M}{\delta_\varepsilon^2(n+1)^2}\sum_{i=0}^{n} 1 = \frac{4M}{\delta_\varepsilon^2(n+1)}.$$

Since

$$\lim_{n\longrightarrow\infty}\left(\frac{4M}{\delta_\varepsilon^2(n+1)}\right) = 0,$$

then for $\frac{\varepsilon}{2}>0$ $\exists\, N_\varepsilon \in \mathbb{N}$ such that $\forall\, n \geqslant N_\varepsilon$, $\frac{4M}{\delta_\varepsilon^2(n+1)} < \frac{\varepsilon}{2}$ implies

$$\sum_{i\in J_{n,\varepsilon,z}} |f(z)-f(z_{n+1,i})|A_{1,n,i}(z) < \frac{\varepsilon}{2}.$$

In conclusion, we have $\forall\, n \geqslant N_\varepsilon$,

$$\sum_{i=0}^{n} |f(z)-f(z_{n+1,i})|A_{1,n,i}(z) \leqslant \sum_{i\in I_{n,\varepsilon,z}} |f(z)-f(z_{n+1,i})|A_{1,n,i}(z)$$
$$+\sum_{i\in J_{n,\varepsilon,z}} |f(z)-f(z_{n+1,i})|A_{n,i}(z)$$
$$\leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Hence $(Q_{1,2n+1}(f;z))_n$ converges uniformly to f on $[-1,1]$. $\square$

As written in [5], this kind of interpolation is referred to as Hermite-Fejér interpolation.

Is this result still valid if we consider the Chebyshev polynomials of second, third or fourth kind defined in (10) instead of the first kind? The answer is that this result is no more valid on the interval $[-1,1]$ for the Chebyshev polynomials of the second, third and fourth kind but on the interval $[-\frac{2\sqrt{2}}{3},\frac{2\sqrt{2}}{3}]$, $[-\frac{\sqrt{3}}{2},1]$, $[-1,\frac{\sqrt{3}}{2}]$, respectively. The fact is that contrary to the Chebyshev polynomials of the first kind for which we have a bound on $[-1,1]\times\mathbb{N}$ ($|T_n(z)|\leqslant 1$, $\forall n\in\mathbb{N}$, $\forall z\in[-1,1]$), this is not the case for the other families U_n, V_n, W_n. $|U_n(z)|$ is not bounded on $[-1,1]\times\mathbb{N}$ around $z=-1^+$ and $z=1^-$ as

$$\lim_{z\to-1^+,1^-}|U_n(z)| = n+1,$$

$|V_n(z)|$ is bounded on $[-1,1]\times\mathbb{N}$ around $z=1^-$ but not around $z=-1^+$ as

$$\lim_{z\to1^-}|V_n(z)| = 1,\quad \lim_{z\to-1^+}|V_n(z)| = 2n+1,$$

and $|W_n(z)|$ is bounded on $[-1,1]\times\mathbb{N}$ around $z=-1^+$ but not around $z=1^-$ as

$$\lim_{z\to-1^+}|W_n(z)| = 1,\quad \lim_{z\to1^-}|W_n(z)| = 2n+1.$$

3. Main contributions: Approximation using the zeros of the 2^{nd}, 3^{rd} and 4^{th} kind Chebyshev polynomials as nodes

For all $n = 0,1,2,\ldots$, $U_n(z)$, $V_n(z)$ and $W_n(z)$ defined in (10) are polynomials of degree n with leading coefficient 2^n. We state and prove the following.

LEMMA 2. *The polynomial $A_{2,n,i}(z)$, $A_{3,n,i}(z)$, $A_{4,n,i}(z)$ for the Chebyshev polynomials of the second, third and fourth kind are given, respectively, by*

$$A_{2,n,i}(z) = (1+2z_{n+1,i}^2 - 3zz_{n+1,i})\left(\frac{U_{n+1}(z)\sin\theta_{n+1,i}}{(n+2)(z-z_{n+1,i})}\right)^2, \tag{15}$$

where $\theta_{n+1,i}$ and $z_{n+1,i}$ are given by (11),

$$A_{3,n,i}(z) = \frac{(1+z_{n+1,i})(1-z_{n+1,i}^2 - (z-z_{n+1,i})(2z_{n+1,i}-1))V_{n+1}^2(z)}{2(n+\frac{3}{2})^2(z-z_{n+1,i})^2}, \tag{16}$$

where $z_{n+1,i}$ is given by (12),

$$A_{4,n,i}(z) = \frac{(1-z_{n+1,i})(1-z_{n+1,i}^2 - (z-z_{n+1,i})(2z_{n+1,i}+1))W_{n+1}^2(z)}{2(n+\frac{3}{2})^2(z-z_{n+1,i})^2}, \tag{17}$$

where $z_{n+1,i}$ is given by (13).

Proof. We prove the result for the second kind Chebyshev polynomials whereas the same procedure remains valid for the third and fourth kind Chebyshev polynomials.

$$U_{n+1}(z) = 2^{n+1}N_{n+1}(z),\ U'_{n+1}(z_{n+1,i}) = -\frac{(n+2)\cos(n+2)\theta_{n+1,i}}{\sin^2\theta_{n+1,i}},$$

$$U''_{n+1}(z_{n+1,i}) = \frac{-3z_{n+1,i}(n+2)\cos(n+2)\theta_{n+1,i}}{\sin^4\theta_{n+1,i}}.$$

Therefore

$$\frac{U''_{n+1}(z_{n+1,i})}{U'_{n+1}(z_{n+1,i})} = \frac{3z_{n+1,i}}{\sin^2\theta_{n+1,i}},$$

and then since $\cos^2(n+2)\theta_{n+1,k} = \cos^2(n-k+1)\pi = ((-1)^{n-k+1})^2 = 1$,

$$\begin{aligned}
A_{2,n,i}(z) &= \Big(1-2(z-z_{n+1,i})l'_{n,i}(z_{n+1,i})\Big)l_{n,i}^2(z)\\
&= \left(1-(z-z_{n+1,i})\frac{U''_{n+1}(z_{n+1,i})}{U'_{n+1}(z_{n+1,i})}\right)\left(\frac{U_{n+1}(z)\sin^2\theta_{n+1,i}}{(n+2)(z-z_{n+1,i})\cos(n+2)\theta_{n+1,i}}\right)^2\\
&= \left(1-(z-z_{n+1,i})\frac{3z_{n+1,i}}{\sin^2\theta_{n+1,i}}\right)\left(\frac{U_{n+1}(z)\sin^2\theta_{n+1,i}}{(n+2)(z-z_{n+1,i})\cos(n+2)\theta_{n+1,i}}\right)^2\\
&= (1+2z_{n+1,i}^2-3zz_{n+1,i})\left(\frac{U_{n+1}(z)\sin\theta_{n+1,i}}{(n+2)(z-z_{n+1,i})}\right)^2. \quad \square
\end{aligned}$$

LEMMA 3. *For* $n=0,1,2,\ldots,$

$$A_{2,n,i}(z) \geqslant 0,\ \forall z \in \left[-\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}\right], \quad A_{3,n,i}(z) \geqslant 0,\ \forall z \in \left[-\frac{\sqrt{3}}{2}, 1\right],$$
$$A_{4,n,i}(z) \geqslant 0,\ \forall z \in \left[-1, \frac{\sqrt{3}}{2}\right].$$

Proof. Case of $A_{2,n,i}$

If $z_{n+1,i}=0$, then $A_{2,n,i}(z)=\left(\frac{U_{n+1}(z)}{z\,(n+2)}\right)^2 \geqslant 0$.

Suppose $z_{n+1,i} \neq 0$. $A_{2,n,i}(z) \geqslant 0$ if $1+2z_{n+1,i}^2-3zz_{n+1,i} \geqslant 0$. $1+2z_{n+1,i}^2-3zz_{n+1,i}=0$ if $z=\eta_{1,n,i}=\frac{1+2z_{n+1,i}^2}{3z_{n+1,i}}$. We consider the function $g_2(z)=\frac{1+2z^2}{3z}$. If $z\in[-1,0[$ then $g_2(z)\leqslant -\frac{2\sqrt{2}}{3}$ and $z\in]0,1]$ implies $g_2(z)\geqslant \frac{2\sqrt{2}}{3}$. Therefore, we deduce, taking into account that $z_{n+1,i}\in[-1,1]$ and $\eta_{1,n,i}=g_2(z_{n+1,i})$, that $\eta_{1,n,i}\notin]-\frac{2\sqrt{2}}{3},\frac{2\sqrt{2}}{3}[$. Hence the linear term $1+2z_{n+1,i}^2-3zz_{n+1,i}=-3z_{n+1,i}(z-\eta_{1,n,i})$ is positive on the interval $]-\frac{2\sqrt{2}}{3},\frac{2\sqrt{2}}{3}[$. In fact, if $z_{n+1,i}>0$, therefore $\eta_{1,n,i}>0$, that is, $\eta_{1,n,i}>\frac{2\sqrt{2}}{3}$ and then $z-\eta_{1,n,i}\leqslant\frac{2\sqrt{2}}{3}-\eta_{1,n,i}<0$ which yields $-3z_{n+1,i}(z-\eta_{1,n,i})>0$. Similarly, if $z_{n+1,i}<0$, therefore $\eta_{1,n,i}<0$, that is, $\eta_{1,n,i}<-\frac{2\sqrt{2}}{3}$ and then $0<-\frac{2\sqrt{2}}{3}-\eta_{1,n,i}<z-\eta_{1,n,i}$ which yields $-3z_{n+1,i}(z-\eta_{1,n,i})>0$. As conclusion, $A_{2,n,i}(z)\geqslant 0,\ \forall z\in[-\frac{2\sqrt{2}}{3},\frac{2\sqrt{2}}{3}]$.

Case of $A_{3,n,i}$

If $z_{n+1,i}=\frac{1}{2}$, then $A_{3,n,i}(z)=\left(\frac{3V_{n+1}(z)}{(2n+3)(2z-1)}\right)^2\geqslant 0$.

Suppose $z_{n+1,i}\neq\frac{1}{2}$, then $1-z_{n+1,i}^2-(z-z_{n+1,i})(2z_{n+1,i}-1)=0$ if $z=\eta_{2,n,i}=\frac{z_{n+1,i}^2-z_{n+1,i}+1}{2z_{n+1,i}-1}$. The study of the variations of the function $g_3(z)=\frac{z^2-z+1}{2z-1}$ yields $g_3(z)\leqslant -\frac{\sqrt{3}}{2}$ for $z\in[-1,1/2[$ and $g_3(z)\geqslant 1$ for $z\in]1/2,1]$. Therefore we deduce, taking into account that $z_{n+1,i}\in[-1,1]$ and $\eta_{2,n,i}=g_3(z_{n+1,i})$, that $\eta_{2,n,i}\notin]-\frac{\sqrt{3}}{2},1[$. Hence the linear term $1-z_{n+1,i}^2-(z-z_{n+1,i})(2z_{n+1,i}-1)=-(2z_{n+1,i}-1)(z-\eta_{2,n,i})$ is positive on the interval $]-\frac{\sqrt{3}}{2},1[$. In fact, since $z^2-z+1>0$ on $\mathbb{R}$, if $z_{n+1,i}>\frac{1}{2}$, therefore $\eta_{2,n,i}>0$, that is, $\eta_{2,n,i}>1$ and then $z-\eta_{2,n,i}\leqslant 1-\eta_{2,n,i}<0$ which yields $-(2z_{n+1,i}-1)(z-\eta_{2,n,i})>0$. Similarly, if $z_{n+1,i}<\frac{1}{2}$, therefore $\eta_{2,n,i}<0$, that is, $\eta_{1,n,i}<-\frac{\sqrt{3}}{2}$ and then $0\leqslant-\frac{\sqrt{3}}{2}-\eta_{2,n,i}<z-\eta_{2,n,i}$ which yields $-(2z_{n+1,i}-1)(z-\eta_{2,n,i})\geqslant 0$. As conclusion, $A_{3,n,i}(z)\geqslant 0,\ \forall z\in[-\frac{\sqrt{3}}{2},1]$.

Case of $A_{4,n,i}$

The proof follows from the latter case. In fact, since $W_n(z)=(-1)^nV_n(-z)$ due to the uniqueness of a family of polynomials orthogonal with respect to a given weight function, if $\{z_{n+1,i},\ i=0,1,\ldots,n\}$ are the zeros of $V_{n+1}(z)$ and $\{\zeta_{n+1,i},\ i=0,1,\ldots,n\}$ the zeros of $W_{n+1}(z)$, then $\zeta_{n+1,i}=-z_{n+1,i},\ i=0,1,\ldots,n$.

Let us denote by $A_{3,n,i}(z,t)$ and $A_{4,n,i}(z,t)$ the respective expressions of $A_{3,n,i}(z)$ and $A_{4,n,i}(z)$ in which $z_{n+1,i}$ is replaced by t. Then, $A_{3,n,i}(z) := A_{3,n,i}(z, z_{n+1,i})$ and $A_{4,n,i}(z) := A_{4,n,i}(z, \zeta_{n+1,i})$. By using the equality $A_{4,n,i}(z,t) = A_{3,n,i}(-z,-t)$ which can be proven by direct computation using equations (16) and (17), we deduce that

$$A_{4,n,i}(z) = A_{4,n,i}(z,\zeta_{n+1,i}) = A_{3,n,i}(-z,-\zeta_{n+1,i}) = A_{3,n,i}(-z, z_{n+1,i}) \geqslant 0,\ \forall z \in \Big[-1, \frac{\sqrt{3}}{2}\Big]$$

since $A_{3,n,i}(z, z_{n+1,i}) \geqslant 0$, $\forall z \in [-\frac{\sqrt{3}}{2}, 1]$. $\square$

THEOREM 2.

(a) If $f \in \mathscr{C}[-\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}]$ then the Hermite interpolation polynomials $Q_{2,2n+1}(f;z)$ (at the zeros $z_{n+1,i}$, $i = 0,1,\ldots,n$, of the Chebyshev polynomial $U_{n+1}(z)$ given by (11)) which satisfy (9) converge uniformly on $[-\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}]$ to f.

(b) If $f \in \mathscr{C}[-\frac{\sqrt{3}}{2}, 1]$ then the Hermite interpolation polynomials $Q_{3,2n+1}(f;z)$ (at the zeros $z_{n+1,i}$, $i = 0,1,\ldots,n$, of the Chebyshev polynomial $V_{n+1}(z)$ given by (12)) which satisfy (9) converge uniformly on $[-\frac{\sqrt{3}}{2}, 1]$ to f.

(c) If $f \in \mathscr{C}[-1, \frac{\sqrt{3}}{2}]$ then the Hermite interpolation polynomials $Q_{4,2n+1}(f;z)$ (at the zeros $z_{n+1,i}$, $i = 0,1,\ldots,n$, of the Chebyshev polynomial $W_{n+1}(z)$ given by (13)) which satisfy (9) converge uniformly on $[-1, \frac{\sqrt{3}}{2}]$ to f.

Proof. Let $\varepsilon > 0$.

(a) Let $f \in \mathscr{C}[-\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}]$ and $z \in [-\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}]$. Proceeding as in the proof of Theorem 1, we show that,

$$\sum_{i \in I_{n,\varepsilon,z}} |f(z) - f(z_{n+1,i})| A_{2,n,i}(z) < \frac{\varepsilon}{2},$$

and for $i \in J_{n,\varepsilon,z}$, we also have $\frac{1}{(z-z_{n+1,i})^2} \leqslant \frac{1}{\delta_\varepsilon^2}$, $|1 + 2z_{n+1,i}^2 - 3zz_{n+1,i}| \leqslant 6$. Since $f(z)$, $1/(1-z^2)$ are continuous on $[-\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}]$ which is closed, there exist M_1 and M_2 positive real numbers such that $|f(z) - f(z_{n+1,i})| \leqslant M_1$ and $U_{n+1}^2(z) \leqslant 1/(1-z^2) \leqslant M_2$. It follows that

$$\begin{aligned}\sum_{i \in J_{n,\varepsilon,z}} |f(z) - f(z_{n+1,i})| A_{2,n,i}(z) &\leqslant M_1 \sum_{i \in J_{n,\varepsilon,z}} |1 + 2z_{n+1,i}^2 - 3zz_{n+1,i}| \Big(\frac{U_{n+1}(z)}{(n+2)(z - z_{n+1,i})}\Big)^2 \\ &\leqslant \frac{6M_1M_2}{\delta_\varepsilon^2 (n+2)^2} \sum_{i=0}^{n} 1 = \frac{6M_1M_2(n+1)}{\delta_\varepsilon^2(n+2)^2}.\end{aligned}$$

But

$$\lim_{n\to\infty} \frac{6M_1M_2(n+1)}{\delta_\varepsilon^2(n+2)^2} = 0,$$

then for $\frac{\varepsilon}{2} > 0$, $\exists\, N_\varepsilon \in \mathbb{N}$ such that $\forall\, n \geqslant N_\varepsilon$, $\frac{6M_1M_2(n+1)}{\delta_\varepsilon^2(n+2)^2} < \frac{\varepsilon}{2}$ implies

$$\sum_{i\in J_{n,\varepsilon,z}} |f(z)-f(z_{n+1,i})|A_{2,n,i}(z) < \frac{\varepsilon}{2}.$$

In conclusion, we have $\forall\, n \geqslant N_\varepsilon$,

$$\begin{aligned}\sum_{i=0}^{n} |f(z)-f(z_{n+1,i})|A_{2,n,i}(z) &\leqslant \sum_{i\in I_{n,\varepsilon,z}} |f(z)-f(z_{n+1,i})|A_{2,n,i}(z)\\ &\quad + \sum_{i\in J_{n,\varepsilon,z}} |f(z)-f(z_{n+1,i})|A_{2,n,i}(z)\\ &\leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence $(Q_{2,2n+1}(f;z))_n$ converges uniformly to f on $[-\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}]$.

(b) Let $f \in \mathscr{C}[-\frac{\sqrt{3}}{2}, 1]$ and $z \in [-\frac{\sqrt{3}}{2}, 1]$. We proceed as in (a) using the fact that $A_{3,n,i}(z) \geqslant 0$, $\forall z \in [-\frac{\sqrt{3}}{2}, 1]$ and $\left|V_{n+1}^2(z)\right| \leqslant \frac{2}{1+z} \leqslant \frac{2}{1-\frac{\sqrt{3}}{2}} = M_2$.

(c) Let $f \in \mathscr{C}[-1, \frac{\sqrt{3}}{2}]$ and $z \in [-1, \frac{\sqrt{3}}{2}]$. We proceed as in (a) taking into consideration that $A_{4,n,i}(z) \geqslant 0$, $\forall z \in [-1, \frac{\sqrt{3}}{2}]$ and $\left|W_{n+1}^2(z)\right| \leqslant \frac{2}{1-z} \leqslant \frac{2}{1-\frac{\sqrt{3}}{2}} = M_2$. □

LEMMA 4. *Let $z_{n,k}$, $k = 0,1,\ldots,n-1$ be the zeros of $U_n(z)$. Then the set $\{z_{n,k},\ n \in \mathbb{N},\ k = 0,1,\ldots,n-1\}$ is dense in $[-1,1]$.*

Proof. Let $a, b \in [-1,1]$. We want to show that there exist two positive integers $k, n, k < n$, such that

$$a \leqslant \cos\frac{n-k}{n+1}\pi \leqslant b \Leftrightarrow \frac{\arccos b}{\pi} \leqslant \frac{n-k}{n+1} \leqslant \frac{\arccos a}{\pi},$$

since arccos is a decreasing function on $[-1,1]$. But $\mathbb{Q}$ (the set of rational numbers) is dense in $\mathbb{R}$ (the set of real numbers) and $[-1,1] \subset \mathbb{R}$. It follows that there exist two integers $p, q > 0$, $p < q$ such that

$$\frac{\arccos b}{\pi} \leqslant \frac{p}{q} \leqslant \frac{\arccos a}{\pi}.$$

Taking $n = q-1$ and $k = q-p-1$, we have the result. □

REMARK 2.

1. Using the same approach, we prove that the zeros of $V_n(z)$ and $W_n(z)$ are also dense in $[-1,1]$. The density in $[-1,1]$ implies the density in every interval $[a,b] \subset [-1,1]$.

2. We consider a function $g \in \mathscr{C}[a,b]$, $a,b \in \mathbb{R}$. $\varphi : [-1,1] \to [a,b]$, $t \longmapsto \frac{1}{2}(b-a)t + \frac{1}{2}(a+b)$ is an increasing bijection with $\varphi(-1) = a$ and $\varphi(1) = b$. Then $\varphi^{-1} : [a,b] \to [-1,1]$, $z \longmapsto \frac{2z-a-b}{b-a}$. Therefore, $g \in \mathscr{C}[a,b]$ if and only if $g \circ \varphi \in \mathscr{C}[-1,1]$. And we deduce that $g_n(t) = Q_{1,2n+1}(g \circ \varphi; t)$ converges uniformly on $[-1,1]$ to $g \circ \varphi$. We conclude that $g_n \circ \varphi^{-1}$ converges uniformly on $[a,b]$ to g. This means that our results can be extended to every continuous function on a subinterval of $\mathbb{R}$.

4. Some simulations

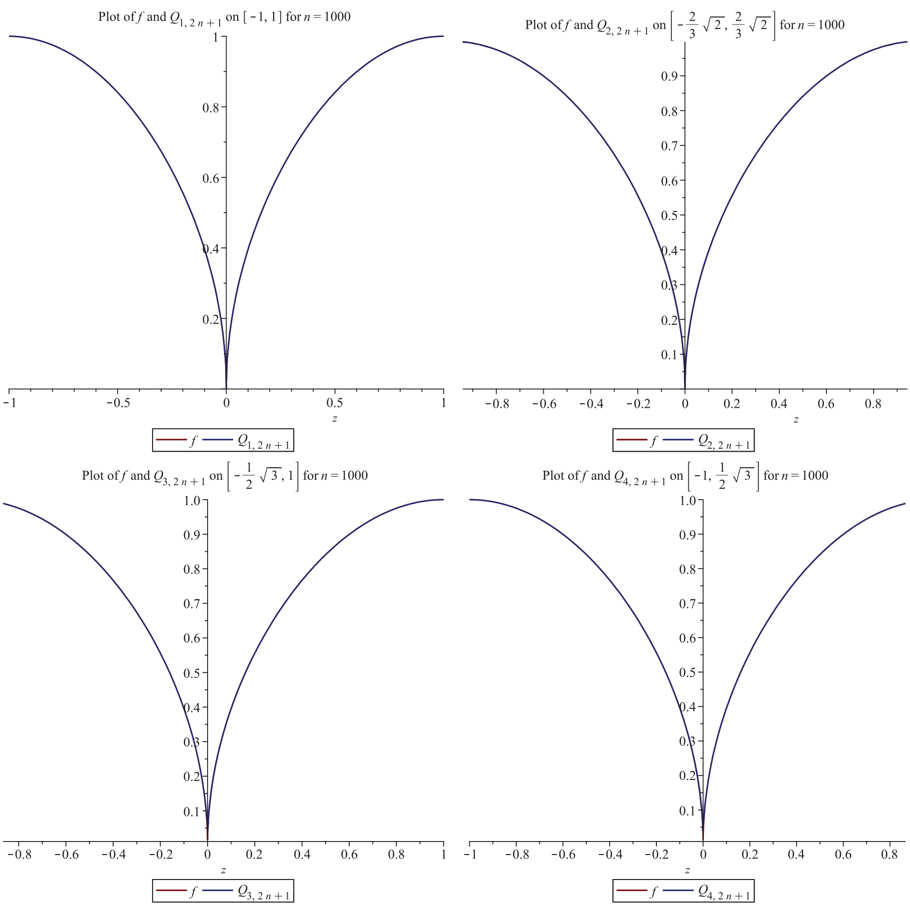

Figure 1: Plots of f, $Q_{1,2n+1}$, $Q_{2,2n+1}$, $Q_{3,2n+1}$ and $Q_{4,2n+1}$ for $n = 1000$

In this section, we simulate the results of Theorems 1, 2. Here we consider the function

$$f(x) = \sqrt{\left|\sin(\frac{\pi x}{2})\right|},\ x \in [-1,1].$$

In Figure 1, we plot $f(z)$ and $Q_{j,2n+1}(z)$, $j = 1,2,3,4$ for $n = 1000$ on the interval of convergence to visualize the results of Theorems 1, 2. In Figure 2, we plot $Q_{2,2n+1}(z)$ and $f(z)$ on $[-1, -\frac{2\sqrt{2}}{3}]$ and $[\frac{2\sqrt{2}}{3}, 1]$; $Q_{3,2n+1}(z)$ and $f(z)$ on $[-1, -0.95]$, $Q_{4,2n+1}(z)$ and $f(z)$ on $[0.95, 1]$ to see the behaviour of the Hermite interpolation polynomials outside the interval of convergence. We observe from the simulations that there is not convergence outside the interval of convergence.

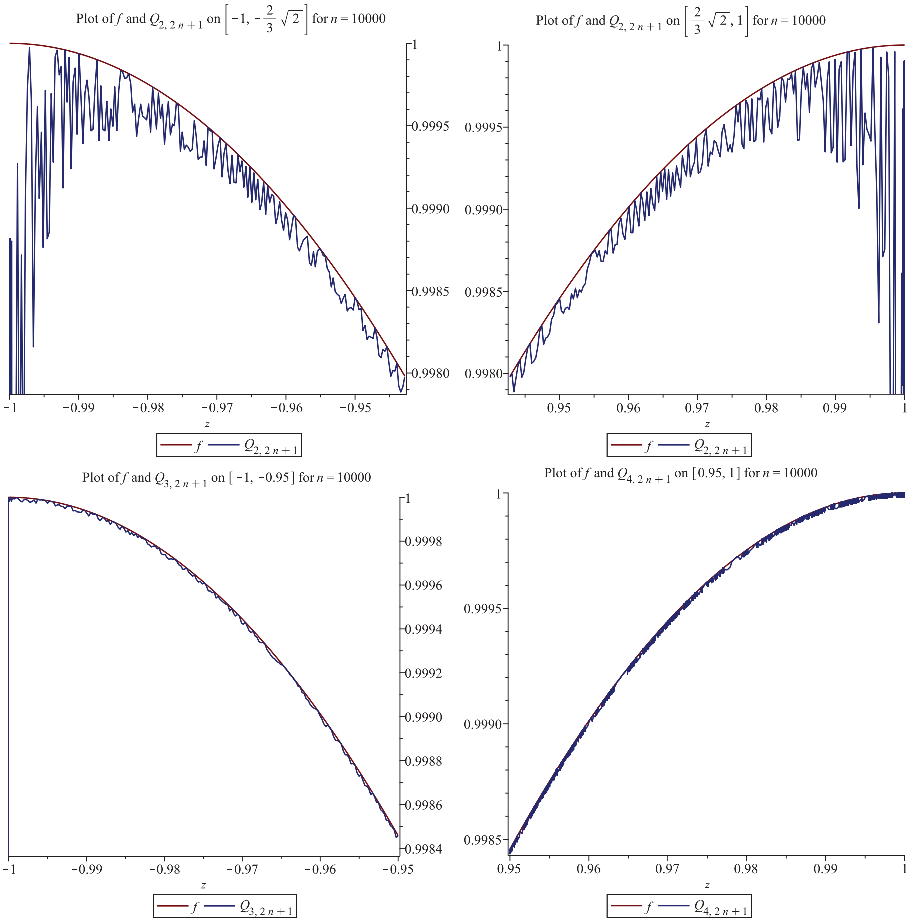

Figure 2: Plots of f, $Q_{2,2n+1}$, $Q_{3,2n+1}$ and $Q_{4,2n+1}$ outside the interval of convergence

5. Possible extension

In Sections 2 and 3, we have proved for a continuous function f, the uniform convergence towards f of the Hermite interpolation polynomials $Q_{j,2n+1}(f;z)$, $j=1,2,3,4$, satisfying (9). Now we consider a function $f\in\mathscr{C}^1[-1,1]$ and the Hermite interpolation polynomial $Q_{j,2n+1}(f;z)$, $j=1,2,3,4$, which satisfies

$$\begin{cases} Q_{2n+1}(f;z_{n+1,i})=f(z_{n+1,i}), & i=0,1,\dots,n,\\ Q'_{2n+1}(f;z_{n+1,i})=f'(z_{n+1,i}), & i=0,1,\dots,n,\end{cases}$$

defined by

$$Q_{j,2n+1}(f;z)=\sum_{i=0}^{n}f(z_{n+1,i})A_{j,n,i}(z)+\sum_{i=0}^{n}f'(z_{n+1,i})B_{j,n,i}(z),$$

where $A_{j,n,i}(z)$, $B_{j,n,i}$ and $Q_{j,2n+1}(f;z)$, $j=1,2,3,4$, are the polynomial $A_{n,i}(z)$, $B_{n,i}(z)$ and the Hermite interpolation polynomials $Q_{2n+1}(f;z)$ with the zeros of $T_{n+1}(z)$, $U_{n+1}(z)$, $V_{n+1}(z)$ and $W_{n+1}(z)$ as nodes, respectively. By direct computation, we obtain

$$B_{1,n,i}(z)=\frac{(\sin\theta_{n+1,i}T_{n+1}(z))^2}{(n+1)^2(z-z_{n+1,i})}=\frac{1-z_{n+1,i}^2}{(n+1)^2(z-z_{n+1,i})}(T_{n+1}(z))^2,$$

$$B_{2,n,i}(z)=\frac{(\sin^2\theta_{n+1,i}U_{n+1}(z))^2}{(n+2)^2(z-z_{n+1,i})}=\frac{(1-z_{n+1,i}^2)^2}{(n+2)^2(z-z_{n+1,i})}(U_{n+1}(z))^2,$$

$$B_{3,n,i}(z)=\frac{(\sin\theta_{n+1,i}\cos\frac{\theta_{n+1,i}}{2}V_{n+1}(z))^2}{(n+\frac{3}{2})^2(z-z_{n+1,i})}=\frac{(1-z_{n+1,i}^2)(1+z_{n+1,i})}{2(n+\frac{3}{2})^2(z-z_{n+1,i})}(V_{n+1}(z))^2,$$

$$B_{4,n,i}(z)=\frac{(\sin\theta_{n+1,i}\sin\frac{\theta_{n+1,i}}{2}W_{n+1}(z))^2}{(n+\frac{3}{2})^2(z-z_{n+1,i})}=\frac{(1-z_{n+1,i}^2)(1-z_{n+1,i})}{2(n+\frac{3}{2})^2(z-z_{n+1,i})}(W_{n+1}(z))^2.$$

If we take for example the functions $f(x)=\frac{1}{1+25x^2}$ or $f(x)=\sqrt{3+2x+4x^2}$ which are in $\mathscr{C}^1[-1,1]$, we remark from numerical simulations with Maple that the sequence $\{Q_{j,2n+1}(f;z)\}_n$, $j=1,2,4$ and its derivative $\{Q'_{j,2n+1}(f;z)\}_n$, $j=1,2,4$ converge uniformly to f and f' on $[-1,1]$, respectively, and the sequence $\{Q_{3,2n+1}(f;z)\}_n$ and its derivative $\{Q'_{3,2n+1}(f;z)\}$ converge uniformly to f and f' on $(-1,1]$, respectively. So one possible extension of this work could be to provide theoretical proof of this convergence guessed by numerical simulation.

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