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ON THE GENERALIZED HURWITZ–LERCH ZETA FUNCTION AND GENERALIZED LAMBERT TRANSFORM

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Abstract. Raina and Srivastava [20] introduced a generalized Lambert transform. Goyal and Laddha [8] have introduced generalizations of the Riemann zeta function and generalized Lambert transform. In the present paper, we introduce generalizations of the Hurwitz-Lerch zeta function and Lambert transform in a diverse direction. We derive generating functions involving generalized Hurwitz-Lerch zeta function. Connections between the generalized Lambert transform and generalized Hurwitz-Lerch zeta function are established. An inversion formula for the generalized Lambert transform is obtained. Some examples and special cases to illustrate our results are also mentioned.

1. Introduction

Several scholars including Bhonsle [1, 2], Gupta and Agrawal [5], Goyal and Vasishta [6], Goyal and Jain [7], Kumar [13, 14, 15], Srivastava [21, 22, 23], Srivastava and Tuan [25], Srivastava and Yürekli [26] and Yakubovich and Martins [32] have studied and explored Laplace, Meijer, Stieltjes, Hankel and H-function transforms at large in the form of generalizations, convolution and connecting theorems. In this paper we study the generalized Hurwitz-Lerch zeta function and generalized Lambert transform, and establish connections between them.

Now, we mention some relevant definitions.

DEFINITION 1. The generalized (Hurwitz's) zeta function is defined by [3, p. 24, Eq. (1)]:

$$\zeta(s,\ a) = \sum_{n=0}^{\infty} (a+n)^{-s}, \tag{1}$$

where $\mathrm{Re}\,(s) > 0$ and $a \neq 0,\ -1,\ -2,\ldots,$ so that, evidently,

$$\zeta(s,\ 1) = \sum_{n=1}^{\infty} n^{-s} = \zeta(s), \tag{2}$$

where $\zeta(s)$ is the Riemann zeta function [3, p. 32, Eq. (1)].

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DEFINITION 2. The Hurwitz-Lerch zeta function $\phi(z,\ s,\ a)$ extends (1) further. This is defined by [3, p. 27, Eq. (1)]:

$$\phi(z,\ s,\ a) = \sum_{n=0}^{\infty} (a+n)^{-s}\, z^n, \tag{3}$$

where $|z| < 1$, $a \neq 0,\ -1,\ -2, \ldots$.

Equivalently, the function $\phi(z,\ s,\ a)$ has the integral representation

$$\phi(z,\ s,\ a) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}\, e^{-at}\, (1 - ze^{-t})^{-1}\, dt,$$

provided that $\mathrm{Re}\,(a) > 0$ and either $|z| \leqslant 1,\ z \neq 1,\ \mathrm{Re}\,(s) > 0$ or $z = 1,\ \mathrm{Re}\,(s) > 1$.

DEFINITION 3. Goyal and Laddha introduced the generalized Riemann zeta function in the following manner [8, p. 100, Eq. (1.5)]:

$$\phi_{\mu}^{*}(z,\ s,\ a) = \sum_{n=0}^{\infty} (a+n)^{-s}\, (\mu)_n\, \frac{z^n}{n!}, \tag{4}$$

where $\mu \in \mathbb{C}$, $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $s \in \mathbb{C}$ when $|z| < 1$; $\mathrm{Re}\,(s - \mu) > 1$ when $|z| = 1$.

Equivalently, the function $\phi_{\mu}^{*}(z,\ s,\ a)$ has the integral representation

$$\phi_{\mu}^{*}(z,\ s,\ a) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}\, e^{-at}\, (1 - ze^{-t})^{-\mu}\, dt,$$

provided that $\mathrm{Re}\,(a) > 0$; $\mathrm{Re}\,(s) > 0$ when $|z| \leqslant 1\ (z \neq 1)$; $\mathrm{Re}\,(s) > 1$ when $z = 1$.

Obviously when $\mu = 1$ in (4), it reduces to (3) which reduces to (1) when $z = 1$ and (2) when $z = 1$ and $a = 1$.

DEFINITION 4. The generalized Hurwitz-Lerch zeta function is hereby introduced and defined in the following manner:

$$\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a) = \sum_{k=0}^{\infty} (a + \alpha k z^{\beta})^{-s}\, (\mu)_k\, \frac{z^k}{k!}, \tag{5}$$

where $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$, $\mathrm{Re}\,(\mu) \geqslant 1$, $\mathrm{Re}\,(\alpha) > 0$ and either $|z| \leqslant 1,\ z \neq 1,\ \beta \geqslant 0$, $\mathrm{Re}\,(s) > 0$ or $z = 1$, $\mathrm{Re}\,(s - \mu) > 0$.

The function $\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a)$ defined by (5) is a new generalization of (3).

Substituting $\alpha = 1$ and $\beta = 0$ in (5), we obtain (4).

LEMMA 1. *The function* $\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a)$ *has the integral representation*

$$\phi_{\mu}^{\alpha,\ \beta}(z,\ s,\ a) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}\, e^{-at}\, (1 - ze^{-\alpha z^{\beta} t})^{-\mu}\, dt, \tag{6}$$

provided that $Re\,(a) > 0$, $Re\,(\alpha) > 0$, $Re\,(\mu) \geqslant 1$ *and either* $|z| \leqslant 1,\ z \neq 1,\ \beta \geqslant 0$, $Re\,(s) > 0$ *or* $z = 1$, $Re\,(s - \mu) > 0$.

Proof. Substituting $c = (a+\alpha k z^{\beta})$ in the following known result [3, p. 1, Eq. (5)]:

$$c^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-ct}\, t^{s-1} dt, \;\; Re(s) > 0,$$

we get

$$(a+\alpha k z^{\beta})^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-(a+\alpha k z^{\beta})t}\, t^{s-1} dt, \;\; Re(s) > 0.$$

Substituting such form of $(a+\alpha k z^{\beta})^{-s}$ in (5), we get

$$\phi_{\mu}^{\alpha,\,\beta}(z,\ s,\ a) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}\, e^{-at} \left\{ \sum_{k=0}^{\infty} \frac{(\mu)_k (z e^{-\alpha z^{\beta} t})^k}{k!} \right\} dt$$

and we easily arrive at (6) by using the following binomial expansion:

$$\sum_{k=0}^{\infty} \frac{(\mu)_k\, z^k}{k!} = (1-z)^{-\mu}, \quad |z| < 1. \quad \square \tag{7}$$

DEFINITION 5. The generalized Lambert transform is hereby introduced and defined in the following manner:

$$F(\rho) = GLM^{*}\{f(t)\} = \int_0^{\infty} \frac{\rho t^{\xi}}{(e^{\zeta \rho t x^{\sigma}} - x)^{\mu}} f(t) dt, \tag{8}$$

provided that $\mathrm{Re}(\mu) \geqslant 1$, $\mathrm{Re}(\rho) > 0$, $\sigma \geqslant 0$, $\mathrm{Re}(\zeta) > 0$, $|x| \leqslant 1$, $f(t) \in \Omega$ and $\mathrm{Re}(\gamma+\xi) > -1$, where Ω denotes the class of functions $f(t)$ which are continuous for $t > 0$ and satisfy the order estimates:

$$\begin{cases} O(t^{\gamma}) & (t \to 0+) \\ O(t^{\delta}) & (t \to \infty). \end{cases}$$

The integral transform defined by (8) is a new generalization of the Lambert transform [31].

Obviously for $\zeta = 1$ and $\sigma = 0$, (8) reduces to the generalized Lambert transform introduced and defined by Goyal and Laddha [8] in the following manner:

$$F(\rho) = GLM\{f(t)\} = \int_0^{\infty} \frac{\rho t^{\xi}}{(e^{\rho t} - x)^{\mu}} f(t) dt. \tag{9}$$

Substituting $\mu = \xi = 1$ in (9), it reduces to the generalized Lambert transform introduced and defined by Raina and Srivastava [20] which further reduces to the well known Lambert transform [31] when $x = 1$.

DEFINITION 6. The author introduced a general class of functions defined in the following manner [10] (see also [11, 12, 16, 17]):

$$V_n(x) = V_n^{h_m,\, d,\, g_j}[p,\, \tau,\, k,\, w,\, q,\, k_m,\, a_j,\, b_r,\, \alpha,\, \beta,\, \delta;\, x]$$
$$= \lambda \sum_{n=0}^{\infty} \frac{(-p)^n \prod_{m=1}^{t} \left[(h_m)_{n+k_m}\right] (d+\alpha n+\beta)^{-\tau} (x/2)^{nk+dw+q}}{\prod_{j=1}^{s} \left[(g_j)_{n+a_j}\right] \prod_{r=1}^{u} \left[(d)_{\alpha n\delta+b_r}\right]}, \tag{10}$$

where

(i) $p,\ k,\ w,\ q,\ \beta,\ \delta,\ k_m,\ a_j,\ b_r$ $(m=1,\ldots,t;\ j=1,\ldots,s;\ r=1,\ldots,u)$ are real numbers.

(ii) t, s and u are natural numbers.

(iii) $h_m,\ g_j \geqslant 1$ $(m=1,\ldots,t;\ j=1,\ldots,s)$, d may be real or complex.

(iv) $\alpha > 0$, $Re(\tau) > 0$, $Re(d) > 0$, x is a variable and λ is an arbitrary constant.

(v) The series on the right hand side of (10) converges absolutely if $t < s$ or $t = s$ with $|p(x/2)^k| \leqslant 1$.

For details of convergence conditions of the series on the right hand side of (10) one may refer to the paper [11].

REMARK 1. The general class of functions defined by (10) is quite general in nature as it unifies and extends a number of useful functions such as unified Riemann-zeta function [8], generalized hypergeometric function [3], Bessel function [4], Wright's generalized Bessel function [24, 30], Struve's function [4], Lommel's function [4], generalized Mittag-Leffler function [27], exponential function, sine function, cosine function and MacRobert's E-function [3] etc. (see, e.g. [10, 12]).

2. Main theorems

In this section, we prove the convergence conditions of the generalized Hurwitz-Lerch zeta function defined by (5). We further derive new generating functions involving the generalized Hurwitz-Lerch zeta function and establish relations between generalized Lambert transform defined by (8) and the generalized Hurwitz-Lerch zeta function defined by (5). We obtain also an inversion formula for the generalized Lambert transform.

THEOREM 1. *If*

(i) $a \neq 0,\ -1,\ -2,\ldots,$ $Re(\mu) \geqslant 1$ *and* $Re(\alpha) > 0$ *and either*

(ii) $|z| \leqslant 1,\ z \neq 1,\ \beta \geqslant 0$ *and* $Re(s) > 0$ *or*

(iii) $z = 1$ *and* $Re(s-\mu) > 0$,

then the series (5) *is absolutely convergent.*

Proof. We apply D' Alembert's ratio test to prove the theorem 1. Let

$$U_k(z) = (a+\alpha k z^{\beta})^{-s}\, (\mu)_k\, \frac{z^k}{k!}.$$

Then

$$\left|\frac{U_{k+1}(z)}{U_k(z)}\right| = \left|\frac{\left(\frac{a}{k}+\alpha z^{\beta}\right)^{s}\left(\frac{\mu}{k}+1\right)z}{\left(\frac{a}{k}+\alpha\left(1+\frac{1}{k}\right)z^{\beta}\right)^{s}\left(1+\frac{1}{k}\right)}\right|.$$

Now, we observe that

$$\lim_{k\to\infty}\left|\frac{U_{k+1}(z)}{U_k(z)}\right| = |z|.$$

Thus, the series (5) is absolutely convergent if

$$|z| < 1$$

with $a \neq 0,\ -1,\ -2,\ldots,\ \mathrm{Re}(\mu) \geqslant 1,\ \mathrm{Re}(\alpha) > 0,\ \beta \geqslant 0$ and $\mathrm{Re}(s) > 0$.

To check the convergence of the series (5) when $|z| = 1$, we compare this series with the series $\sum_{k=0}^{\infty}\frac{1}{k^{1+\delta}}$, where $2\delta = \mathrm{Re}(s-\mu) > 0$.

Now, we have

$$\lim_{k\to\infty}\left|k^{1+\delta}\frac{(\mu)_k}{\Gamma(k)k^{\mu}}\frac{\Gamma(k)k^{\mu}}{(a+\alpha k)^{s}k!}\right| = \lim_{k\to\infty}\left|k^{1+\delta}\frac{1}{\Gamma(\mu)}\frac{\Gamma(\mu+k)}{\Gamma(k)k^{\mu}}\frac{k^{\mu}}{(a+\alpha k)^{s}k}\right|. \tag{11}$$

Using the following result [3, p. 47, Eq. (5)] in (11)

$$\lim_{|z|\to\infty} e^{-a\ log\ z}\,\frac{\Gamma(z+a)}{\Gamma(z)} = 1,$$

we get

$$\begin{aligned}\lim_{k\to\infty}\left|k^{1+\delta}\frac{(\mu)_k}{\Gamma(k)k^{\mu}}\frac{\Gamma(k)k^{\mu}}{(a+\alpha k)^{s}k!}\right| &= \lim_{k\to\infty}\left|\frac{1}{\Gamma(\mu)}k^{1+\delta}\frac{k^{\mu}}{k^{s}\left(\frac{a}{k}+\alpha\right)^{s}k}\right|\\ &= \left|\frac{1}{\Gamma(\mu)}\right|\lim_{k\to\infty}\left|\frac{1}{k^{s-\mu-\delta}\left(\frac{a}{k}+\alpha\right)^{s}}\right| = 0.\end{aligned}$$

Now, we observe that the series (5) is absolutely convergent if $\mathrm{Re}(s-\mu) > 0$ with $a \neq 0,\ -1,\ -2,\ldots,\ \mathrm{Re}(\mu) \geqslant 1$ and $\mathrm{Re}(\alpha) > 0$. □

THEOREM 2. *If* $|t| < |a|$, *Re* $(\mu) \geqslant 1$ *and* $\lambda \neq 1$, *then we have the following generating function:*

$$\sum_{n=0}^{\infty}(\lambda)_n\,\phi_{\mu}^{\alpha,\,\beta}(x,\ \lambda+n,\ a)\frac{t^n}{n!} = \phi_{\mu}^{\alpha,\,\beta}(x,\ \lambda,\ a-t). \tag{12}$$

Proof. Using the result (5), we have

$$\begin{aligned}\sum_{n=0}^{\infty}(\lambda)_n\,\phi_{\mu}^{\alpha,\,\beta}(x,\ \lambda+n,\ a)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}(\lambda)_n\sum_{k=0}^{\infty}(a+\alpha kx^{\beta})^{-\lambda-n}\,(\mu)_k\,\frac{x^k}{k!}\frac{t^n}{n!}\\ &= \sum_{k=0}^{\infty}(a+\alpha kx^{\beta})^{-\lambda}\left[\sum_{n=0}^{\infty}(\lambda)_n\left(\frac{t}{a+\alpha kx^{\beta}}\right)^{n}\frac{1}{n!}\right](\mu)_k\frac{x^k}{k!}.\end{aligned} \tag{13}$$

Now, applying the result (7) in (13), we get

$$\sum_{n=0}^{\infty} (\lambda)_n \, \phi_\mu^{\alpha,\,\beta}(x,\ \lambda+n,\ a)\frac{t^n}{n!} = \sum_{k=0}^{\infty}(a+\alpha k x^\beta)^{-\lambda}\left(1-\frac{t}{a+\alpha k x^\beta}\right)^{-\lambda}(\mu)_k\,\frac{x^k}{k!}.$$

After a little simplification, we easily arrive at the generating function (12), provided that $|t|<|a|$ and $\lambda \neq 1$. □

THEOREM 3. *If* $|t|<|a|$, $Re(\mu)\geqslant 1$ *and* $Re(\lambda+u) > Re(v) > 0$, *then we have the following bilateral generating function:*

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\lambda)_n(u)_n}{(v)_n}\,\phi_\mu^{\alpha,\,\beta}(x,\ \lambda+u-v+n,\ a)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}(a+\alpha n x^\beta)^{v-\lambda-u}(\mu)_n\,\frac{x^n}{n!}\,{}_2F_1\left(\lambda,\ u;\ v;\ \frac{t}{a+\alpha n x^\beta}\right),\end{aligned}$$

where ${}_2F_1(a,\ b;\ v;\ z)$ *stands for the Gauss's hypergeometric function* [3].

Proof. We have

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\lambda)_n(u)_n}{(v)_n}\,\phi_\mu^{\alpha,\,\beta}(x,\ \lambda+u-v+n,\ a)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\frac{(\lambda)_n(u)_n}{(v)_n}\sum_{k=0}^{\infty}(a+\alpha k x^\beta)^{-\lambda-u+v-n}\,(\mu)_k\,\frac{x^k}{k!}\frac{t^n}{n!}\\ &=\sum_{k=0}^{\infty}(a+\alpha k x^\beta)^{v-\lambda-u}(\mu)_k\frac{x^k}{k!}\left[\sum_{n=0}^{\infty}\frac{(\lambda)_n(u)_n}{(v)_n}\left(\frac{t}{a+\alpha k x^\beta}\right)^n\frac{1}{n!}\right]\\ &=\sum_{k=0}^{\infty}(a+\alpha k x^\beta)^{v-\lambda-u}(\mu)_k\,\frac{x^k}{k!}\,{}_2F_1\left(\lambda,\ u;\ v;\ \frac{t}{a+\alpha k x^\beta}\right)\\ &=\sum_{n=0}^{\infty}(a+\alpha n x^\beta)^{v-\lambda-u}(\mu)_n\,\frac{x^n}{n!}\,{}_2F_1\left(\lambda,\ u;\ v;\ \frac{t}{a+\alpha n x^\beta}\right),\end{aligned}$$

provided that $|t|<|a|$, $\mathrm{Re}(\mu)\geqslant 1$ and $\mathrm{Re}(\lambda+u) > \mathrm{Re}(v) > 0$. □

THEOREM 4. *If* $\rho \neq 1$, $Re(\mu)\geqslant 1$ *and the conditions mentioned with* (10) *are satisfied, then we have the following bilateral generating function:*

$$\begin{aligned}\sum_{n=0}^{\infty}(\rho)_n\,\phi_\mu^{\gamma,\,\sigma}(x,\ \rho+n,\ a)\,V_n(y)\,\frac{t^n}{n!} &= \phi_\mu^{\gamma,\,\sigma}\left(x,\ \rho,\ a+p\left(\frac{y}{2}\right)^k\right)\left(\frac{y}{2}\right)^{dw+q}\\ &\times\lambda\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{t}[(h_m)_{n+k_m}](d+\alpha n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}.\end{aligned} \tag{14}$$

Proof. Substituting the value of $V_n(y)$ in the left hand side of (14) with the help of (10), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} (\rho)_n\, \phi_{\mu}^{\gamma,\,\sigma}(x,\,\rho+n,\,a)\, V_n(y)\,\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty} (\rho)_n\, \phi_{\mu}^{\gamma,\,\sigma}(x,\,\rho+n,\,a)\frac{t^n}{n!}\\
&\quad\times\lambda\sum_{n=0}^{\infty}\frac{(-p)^n\prod_{m=1}^{t}[(h_m)_{n+k_m}](d+\alpha n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}\left(\frac{y}{2}\right)^{nk+dw+q}.
\end{aligned}\tag{15}$$

Using the result (5) in (15), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} (\rho)_n\, \phi_{\mu}^{\gamma,\,\sigma}(x,\,\rho+n,\,a)\, V_n(y)\,\frac{t^n}{n!}\\
&=\lambda\sum_{n=0}^{\infty}\frac{(-p)^n\prod_{m=1}^{t}[(h_m)_{n+k_m}](d+\alpha n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}\left(\frac{y}{2}\right)^{nk+dw+q}\\
&\quad\times\left[\sum_{n=0}^{\infty}(\rho)_n\left\{\sum_{l=0}^{\infty}(a+\gamma l x^{\sigma})^{-(\rho+n)}(\mu)_l\frac{x^l}{l!}\right\}\frac{t^n}{n!}\right]\\
&=\left(\frac{y}{2}\right)^{dw+q}\lambda\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{t}[(h_m)_{n+k_m}](d+\alpha n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}\\
&\quad\times\sum_{l=0}^{\infty}(a+\gamma l x^{\sigma})^{-\rho}(\mu)_l\frac{x^l}{l!}\left[\sum_{n=0}^{\infty}(\rho)_n\left\{\frac{-p(\frac{y}{2})^k\,t}{a+\gamma l x^{\sigma}}\right\}^n\frac{1}{n!}\right].
\end{aligned}\tag{16}$$

Applying the result (7) in (16), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} (\rho)_n\, \phi_{\mu}^{\gamma,\,\sigma}(x,\,\rho+n,\,a)\, V_n(y)\,\frac{t^n}{n!}\\
&=\left(\frac{y}{2}\right)^{dw+q}\lambda\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{t}[(h_m)_{n+k_m}](d+\alpha n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}\\
&\quad\times\sum_{l=0}^{\infty}(a+\gamma l x^{\sigma})^{-\rho}(\mu)_l\frac{x^l}{l!}\left\{1+\frac{p(\frac{y}{2})^k\,t}{a+\gamma l x^{\sigma}}\right\}^{-\rho}\\
&=\left(\frac{y}{2}\right)^{dw+q}\lambda\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{t}[(h_m)_{n+k_m}](d+\alpha n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}\\
&\quad\times\sum_{l=0}^{\infty}\left[\left\{a+p\left(\frac{y}{2}\right)^k t\right\}+\gamma l x^{\sigma}\right]^{-\rho}(\mu)_l\frac{x^l}{l!}.
\end{aligned}\tag{17}$$

Now, we use the result (5) in (17) and arrive at the desired result (14). □

THEOREM 5. *If $F(\rho)$ is the generalized Lambert transform of $f(t)$, then the* *inversion formula for this transform is*

$$\frac{1}{2}\{f(t+0)+f(t-0)\}$$
$$=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{\Gamma(1-\eta)\,\phi_{\mu}^{\zeta,\,\sigma}(x,\,1-\eta,\,\zeta x^{\sigma}\mu)\right\}^{-1} t^{-(\eta+\xi)}\,g(\eta)d\eta, \tag{18}$$

provided that $f\in\Omega,\ \gamma>0,\ u^{(\eta+\xi)-1}f(u)\in L(0,\,\infty),\ f(u)$ *is of bounded variation in the neighbourhood of the point* $u=t$, $Re(\mu)\geqslant 1,\ |x|\leqslant 1,\ Re(\zeta)>0,\ Re(\rho)>0,\ \sigma\geqslant 0,\ Re(1-\eta)>0$ *and* $g(\eta)$ *is given by the equation*

$$g(\eta)=\int_0^{\infty}\rho^{-\eta-1}F(\rho)d\rho.$$

Proof. From (8), we have

$$\int_0^{\infty}\rho^{-\eta-1}\,F(\rho)d\rho=\int_0^{\infty}\rho^{-\eta-1}\left\{\int_0^{\infty}\frac{\rho t^{\xi}}{\left(e^{\zeta\rho t x^{\sigma}}-x\right)^{\mu}}f(t)dt\right\}d\rho$$
$$=\int_0^{\infty}t^{\xi}f(t)\left\{\int_0^{\infty}\rho e^{-\zeta\rho t\mu x^{\sigma}}\left(1-xe^{-\zeta\rho t x^{\sigma}}\right)^{-\mu}\rho^{-\eta-1}d\rho\right\}dt. \tag{19}$$

Substituting $\rho t=y$ in (19), we get

$$g(\eta)=\int_0^{\infty}\rho^{-\eta-1}\,F(\rho)d\rho$$
$$=\int_0^{\infty}t^{(\eta+\xi)-1}f(t)\left\{\int_0^{\infty}y^{-\eta}e^{-\zeta x^{\sigma}\mu y}\left(1-xe^{-\zeta x^{\sigma}y}\right)^{-\mu}dy\right\}dt. \tag{20}$$

Using the result (6) in (20), we get

$$g(\eta)=\Gamma(1-\eta)\,\phi_{\mu}^{\zeta,\,\sigma}(x,\,1-\eta,\,\zeta x^{\sigma}\mu)\int_0^{\infty}t^{(\eta+\xi)-1}\,f(t)dt. \tag{21}$$

Now, we apply the Mellin inversion theorem [28] in (21) and get the desired result (18). □

3. Examples

In this section, we mention two examples connecting the generalized Lambert transform and the generalized Hurwitz-Lerch zeta function.

EXAMPLE 1. If we take $f(t)=t^{\gamma-1}\,e^{-v\rho t}$ in (8), then

$$GLM^{*}(t^{\gamma-1}\,e^{-v\rho t})=\frac{\Gamma(\gamma+\xi)}{\rho^{\gamma+\xi-1}}\phi_{\mu}^{\zeta,\,\sigma}(x,\,\gamma+\xi,\,v+\zeta x^{\sigma}\mu). \tag{22}$$

Proof. Using the result (8), we have

$$GLM^*(t^{\gamma-1}\, e^{-\nu\rho t}) = \int_0^\infty \frac{\rho t^\xi}{(e^{\zeta\rho t x^\sigma} - x)^\mu} t^{\gamma-1}\, e^{-\nu\rho t} dt$$
$$= \int_0^\infty t^{\gamma+\xi-1} \rho e^{-\zeta\rho t\mu x^\sigma} (1 - x e^{-\zeta\rho t x^\sigma})^{-\mu}\, e^{-\nu\rho t} dt. \qquad (23)$$

Substituting $\rho t = y$ in (23), we get

$$GLM^*(t^{\gamma-1}\, e^{-\nu\rho t}) = \frac{1}{\rho^{\gamma+\xi-1}} \int_0^\infty y^{(\gamma+\xi)-1} e^{-(\nu+\zeta x^\sigma \mu)y} (1 - x e^{-\zeta x^\sigma y})^{-\mu} dy. \qquad (24)$$

Now, we apply the result (6) in (24) and get the desired result (22). □

EXAMPLE 2. If we take $f(t) = e^{-\nu\rho t}\, V_n(t)$ in (8), then

$$GLM^*\{e^{-\nu\rho t}\, V_n(t)\} = \lambda \sum_{n=0}^\infty \frac{(-p)^n \prod_{m=1}^t [(h_m)_{n+k_m}] (d + \alpha n + \beta)^{-\tau}}{\prod_{j=1}^s [(g_j)_{n+a_j}] \prod_{r=1}^u [(d)_{\alpha n\delta + b_r}]}$$
$$\times \frac{\Gamma(\xi + nk + dw + q + 1)}{2^{nk+dw+q}\, \rho^{\xi+nk+dw+q}}$$
$$\times \phi_\mu^{\zeta,\,\sigma}(x,\ \xi + nk + dw + q + 1,\ \nu + \zeta x^\sigma \mu). \qquad (25)$$

Proof. Using the result (8), we have

$$GLM^*\{e^{-\nu\rho t}\, V_n(t)\} = \int_0^\infty \frac{\rho t^\xi}{(e^{\zeta\rho t x^\sigma} - x)^\mu}\, e^{-\nu\rho t}\, V_n(t) dt. \qquad (26)$$

Substituting the value of $V_n(t)$ in (26) with the help of (10), we get

$$GLM^*\{e^{-\nu\rho t}\, V_n(t)\} = \lambda \sum_{n=0}^\infty \frac{(-p)^n \prod_{m=1}^t [(h_m)_{n+k_m}] (d + \alpha n + \beta)^{-\tau}}{\prod_{j=1}^s [(g_j)_{n+a_j}] \prod_{r=1}^u [(d)_{\alpha n\delta + b_r}]}$$
$$\times \frac{1}{2^{nk+dw+q}} \int_0^\infty \rho t^{\xi+nk+dw+q} e^{-(\nu+\zeta\mu x^\sigma)\rho t} (1 - x e^{-\zeta\rho t x^\sigma})^{-\mu} dt. \qquad (27)$$

Substituting $\rho t = y$ in (27), we get

$$GLM^*\{e^{-\nu\rho t}\, V_n(t)\} = \lambda \sum_{n=0}^\infty \frac{(-p)^n \prod_{m=1}^t [(h_m)_{n+k_m}] (d + \alpha n + \beta)^{-\tau}}{\prod_{j=1}^s [(g_j)_{n+a_j}] \prod_{r=1}^u [(d)_{\alpha n\delta + b_r}]}$$
$$\times \frac{1}{2^{nk+dw+q}\, \rho^{\xi+nk+dw+q}}$$
$$\times \int_0^\infty y^{(\xi+nk+dw+q+1)-1} e^{-(\nu+\zeta\mu x^\sigma)y} (1 - x e^{-\zeta x^\sigma y})^{-\mu} dy. \qquad (28)$$

Now, we apply the result (6) in (28) and get the desired result (25). □

4. Special cases

In this section, we mention some special cases of the results (14) and (25) as the general class of functions $V_n(z)$ involved in these results is reducible to a large number of special functions due to its general nature.

(i) If we take $p=2,\ m=1,\ j=2,\ r=1,\ h_1=1,\ g_1=1,\ g_2=1,\ \tau=1,\ k=1,\ w=0,\ q=0,\ k_1=0,\ a_1=0,\ a_2=0,\ b_1=0,\ \beta=0,\ \delta=1$ and $\lambda=\frac{1}{\Gamma(d)}$ in (14) and (25), the general class of functions reduces to the Wright's generalized Bessel function [24, 30] and we get the following results respectively:

$$\sum_{n=0}^{\infty}(\rho)_n\,\phi_{\mu}^{\gamma,\,\sigma}(x,\ \rho+n,\ a)\,J_d^{\alpha}(y)\,\frac{t^n}{n!}=\phi_{\mu}^{\gamma,\,\sigma}(x,\ \rho,\ a+yt)\sum_{n=0}^{\infty}\frac{1}{\Gamma(d+\alpha n+1)\,n!}$$

and

$$GLM^*\{e^{-\nu\rho t}\,J_d^{\alpha}(t)\}=\sum_{n=0}^{\infty}\frac{(-1)^n\,\Gamma(\xi+n+1)}{\rho^{\xi+n}\,\Gamma(d+\alpha n+1)\,n!}\,\phi_{\mu}^{\zeta,\,\sigma}(x,\ \xi+n+1,\ v+\zeta x^{\sigma}\mu),$$

where $J_d^{\alpha}(z)$ stands for the Wright's generalized Bessel function [24, 30].

(ii) If we take $p=1,\ m=1,\ j=2,\ r=1,\ h_1=1,\ g_1=3/2,\ g_2=1,\ \tau=1,\ k=2,\ w=1,\ q=1,\ k_1=0,\ a_1=0,\ a_2=0,\ b_1=1/2,\ \alpha=1,\ \beta=1/2,\ \delta=1$ and $\lambda=\frac{1}{\Gamma(d)\,\Gamma(3/2)}$ in (14) and (25), the general class of functions reduces to the Struve's function [4] and we get the following results respectively:

$$\sum_{n=0}^{\infty}(\rho)_n\,\phi_{\mu}^{\gamma,\,\sigma}(x,\ \rho+n,\ a)\,H_d(y)\,\frac{t^n}{n!}$$
$$=\phi_{\mu}^{\gamma,\,\sigma}\left(x,\ \rho,\ a+\frac{y^2t}{4}\right)\left(\frac{y}{2}\right)^{d+1}\sum_{n=0}^{\infty}\frac{1}{\Gamma\left(\frac{3}{2}+n\right)\ \Gamma\left(\frac{3}{2}+d+n\right)}$$

and

$$GLM^*\{e^{-\nu\rho t}\,H_d(t)\}=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\xi+2n+d+2)}{\Gamma\left(\frac{3}{2}+n\right)\Gamma\left(\frac{3}{2}+d+n\right)\rho^{\xi+2n+d+1}\,2^{2n+d+1}}$$
$$\times\,\phi_{\mu}^{\zeta,\,\sigma}(x,\ \xi+2n+d+2,\ v+\zeta x^{\sigma}\mu),$$

where $H_d(z)$ stands for the Struve's function [4].

(iii) If we take $p=1,\ m=1,\ j=2,\ r=1,\ h_1=1,\ g_1=\frac{u'+v'+3}{2},\ g_2=\frac{u'-v'+3}{2},\ \tau=1,\ k=2,\ w=u',\ q=1,\ k_1=0,\ a_1=0,\ a_2=0,\ b_1=-1,\ d=1,\ \alpha=1,\ \beta=-1,\ \delta=1$ and $\lambda=\frac{2^{u'+1}}{(u'+v'+1)(u'-v'+1)}$ in (14) and (25), the general class of functions reduces to the Lommel's function [4] and we get the following results respectively:

$$\sum_{n=0}^{\infty}(\rho)_n\,\phi_{\mu}^{\gamma,\,\sigma}(x,\ \rho+n,\ a)\,s_{u',\,v'}(y)\,\frac{t^n}{n!}$$
$$=\phi_{\mu}^{\gamma,\,\sigma}\left(x,\ \rho,\ a+\frac{y^2t}{4}\right)\left(\frac{y^{u'+1}}{(u'\pm v'+1)}\right)\sum_{n=0}^{\infty}\frac{1}{\left(\frac{u'\pm v'+3}{2}\right)_n}$$

and

$$GLM^*\{e^{-\nu\rho t}\, s_{u',\,v'}(t)\} = \frac{2^{u'+1}}{(u'\pm v'+1)}\sum_{n=0}^{\infty}\frac{(-1)^n\,\Gamma(\xi+2n+u'+2)}{\rho^{\xi+2n+u'+1}\,2^{2n+u'+1}\left(\frac{u'\pm v'+3}{2}\right)_n}$$
$$\times\ \phi_\mu^{\zeta,\,\sigma}(x,\ \xi+2n+u'+2,\ v+\zeta x^\sigma\mu),$$

where $s_{u',\,v'}(z)$ stands for the Lommel's function [4].

(iv) If we take $p=-2,\ m=1,\ j=1,\ r=1,\ h_1=h,\ g_1=g,\ \tau=1,\ k=1,\ w=0,\ q=0,\ k_1=0,\ a_1=0,\ b_1=-1,\ \beta=-1,\ \delta=1$ and $\lambda=\frac{1}{\Gamma(d)}$ in (14) and (25), the general class of functions reduces to the generalized Mittag-Leffler function [27] and we get the following results respectively:

$$\sum_{n=0}^{\infty}(\rho)_n\,\phi_\mu^{\gamma,\,\sigma}(x,\ \rho+n,\ a)\,E_{\alpha,\,d}^{h,\,g}(y)\,\frac{t^n}{n!} = \phi_\mu^{\gamma,\,\sigma}(x,\ \rho,\ a-yt)\sum_{n=0}^{\infty}\frac{(h)_n}{(g)_n\,\Gamma(d+\alpha n)}$$

and

$$GLM^*\{e^{-\nu\rho t}\,E_{\alpha,\,d}^{h,\,g}(t)\} = \sum_{n=0}^{\infty}\frac{(h)_n\Gamma(\xi+n+1)}{\rho^{\xi+n}\,(g)_n\,\Gamma(d+\alpha n)}\phi_\mu^{\zeta,\,\sigma}(x,\ \xi+n+1,\ v+\zeta x^\sigma\mu),$$

where $E_{\alpha,\,d}^{h,\,g}(t)$ stands for the generalized Mittag-Leffler function [27].

If we put $g=1$, the generalized Mittag-Leffler function reduces to the generalized Mittag-Leffler function $E_{\alpha,\,d}^{h}(z)$ introduced by Prabhakar [19].

If we put $h=1,\ g=1$, the generalized Mittag-Leffler function reduces to the generalized Mittag-Leffler function $E_{\alpha,\,d}(z)$ introduced by Wiman [29].

If we put $h=1,\ g=1,\ d=1$, the generalized Mittag-Leffler function reduces to the Mittag-Leffler function $E_\alpha(z)$ [9, 18].

(v) If we take $p=2,\ r=1,\ d=1,\ t=P,\ s=Q,\ \tau=1,\ k=1,\ w=0,\ q=0,\ k_m=0,\ a_j=0,\ b_1=-1,\ \alpha=1,\ \beta=-1,\ \delta=1$ and $\lambda=\frac{\prod_{m=1}^{P}\Gamma(h_m)}{\prod_{j=1}^{Q}\Gamma(g_j)}$ in (14) and (25), the general class of functions reduces to the MacRobert's E-function [3] and we get the following results respectively:

$$\sum_{n=0}^{\infty}(\rho)_n\,\phi_\mu^{\gamma,\,\sigma}(x,\ \rho+n,\ a)\,E\left[P;\ (h_P);\ Q;\ (g_Q);\ -\frac{1}{y}\right]\frac{t^n}{n!}$$
$$= \phi_\mu^{\gamma,\,\sigma}(x,\ \rho,\ a+yt)\,\frac{\prod_{m=1}^{P}\Gamma(h_m)}{\prod_{j=1}^{Q}\Gamma(g_j)}\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{P}(h_m)_n}{\prod_{j=1}^{Q}(g_j)_n\,n!}$$

and

$$GLM^*\left\{e^{-\nu\rho t}\,E\left[P;\ (h_P);\ Q;\ (g_Q);\ -\frac{1}{t}\right]\right\}$$
$$= \frac{\prod_{m=1}^{P}\Gamma(h_m)}{\prod_{j=1}^{Q}\Gamma(g_j)}\sum_{n=0}^{\infty}\frac{(-1)^n\,\Gamma(\xi+n+1)\,\prod_{m=1}^{P}(h_m)_n}{\rho^{\xi+n}\,n!\,\prod_{j=1}^{Q}(g_j)_n}\phi_\mu^{\zeta,\,\sigma}(x,\ \xi+n+1,\ v+\zeta x^\sigma\mu),$$

where $E\left[P;\ (h_P);\ Q;\ (g_Q);\ -\frac{1}{z}\right]$ stands for the MacRobert's E-function [3].

REMARK 2. In this paper, an approach has been made to develop the Hurwitz-Lerch zeta function and Lambert transform in a diverse direction. Generating functions involving the generalized Hurwitz-Lerch zeta function and several special functions have been derived. Connections between the generalized Lambert transform of special functions and the generalized Hurwitz-Lerch zeta function have been established.

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