Journal of Classical Analysis
Volume 17, Number 2 (2021), 109–118 doi:10.7153/jca-2021-17-07

AN IMPROVEMENT OF ZALCMAN'S LEMMA IN C^n

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Dedicated to Christian Pommerenke and Lawrence Zalcman

Abstract. The aim of this article is to give a proof of improving of Zalcman's lemma in C^n.

1. Introduction and main results

A family $\mathscr{F}$ of holomorphic functions on a domain $\Omega \subset C^n$ is normal in Ω if every sequence of functions $\{f_j\} \subseteq \mathscr{F}$ contains either a subsequence which converges to a limit function $f \neq \infty$ uniformly on each compact subset of Ω, or a subsequence which converges uniformly to ∞ on each compact subset.

A family $\mathscr{F}$ is said to be normal at a point $z_0 \in \Omega$ if it is normal in some neighborhood of z_0. It is routine to confirm that a family of analytic functions $\mathscr{F}$ is normal in a domain Ω if and only if $\mathscr{F}$ is normal at each point of Ω.

There are many criteria for $\mathscr{F}$ to be normal. A particularly useful one is Marty's criterion, which is in terms of the spherical derivative $f^\sharp$ of f, defined by

$$f^\sharp(z) := \max_{|v|=1} \sqrt{L_z(\log(1+|f|^2), v)},$$

where

$$L_z(\log(1+|f|^2), v) := \sum_{k,l=1}^{n} \frac{\partial^2 \log(1+|f|^2)}{\partial z_k \partial \overline{z}_l}(z) v_k \overline{v}_l \quad (z \in \Omega, v \in C^n)$$

(see [4]).

THEOREM 1. (Marty's criterion, see [4]) *A family $\mathscr{F}$ of functions holomorphic on $\Omega \subset C^n$ is normal on Ω if and only if for each compact subset $K \subset \Omega$ there exists a constant $M(K)$ such that at each point $z \in K$*

$$f^\sharp(z) \leqslant M(K)$$

for all $f \in \mathscr{F}$.

Mathematics subject classification (2010): 32A19.
Keywords and phrases: Zalcman's lemma, Zalcman-Pang's lemma, normal families, holomorphic functions of several complex variables.

Marty's criterion is one of the main ingredients of the proof that a family of holomorphic functions is not normal.

THEOREM 2. *Let $\mathscr{F}$ be a family of functions holomorphic on $\Omega \subset C^n$. Then $\mathscr{F}$ is not normal at some point $z_0 \in \Omega$ if and only if for each $\alpha \in (-1,\infty)$ there exist sequences $f_j \in \mathscr{F}$, $z_j \to z_0$, $r_j \to 0$, such that the sequence*

$$g_j(z) := r_j^{\alpha} f_j(z_j + r_j z)$$

converges locally uniformly in C^n to a non-constant entire function g satisfying $g^{\sharp}(z) \leqslant g^{\sharp}(0) = 1$.

As a corollary, we have the following important supplement to Theorem 2:

COROLLARY 1. *Let $\mathscr{F}$ be a family of zero-free holomorphic functions in a domain $\Omega \subset C^n$. The statement of Theorem 2 remains valid if $-1 < \alpha < \infty$ is replaced with $-\infty < \alpha < \infty$.*

REMARK 1. In case $n = 1$ Theorem 2 was proved in Hua [10, Lemma 6]. A similar result was proved by Chen and Gu [3, Th.2] (see also Xue and Pung [16], cf. Hua [10]). The special case $\alpha = 0$ of Theorem 2 was proved in Zalcman [18, p. 814] and is known as Zalcman's rescaling lemma. Zalcman's lemma – now upgraded to the status of theorem – was first stated at [18]; for a state-of-the-art version, see [19, Lemma 2]. The corresponding result for normal functions had been proved earlier by Lohwater and Pommerenke [14].

Zalcman's Lemma is used in the study of entire and meromorphic functions and for establishing normality criteria. It is well-known that the Rescaling Lemma of Zalcman plays an important role in Dynamic System of one complex variable. Much attention has been given to find an appropriate generalization of Zalcman's Lemma to several complex variables, and more generally to complex manifolds (see [1], [2], [11], [7], [17]). The case $\alpha = 0$ is proved in [4]. The case $-1 < \alpha \leqslant 0$ is proved in [5, Theorem 1.1]. The proof of Theorem 2 is elementary; it uses only Marty's criterion.

The Marty criterion is one of the most widely used for determining the normality of a family of holomorphic functions. Marty's criterion is in terms of the spherical derivative $f^{\sharp}$ of f. There are also criteria where it suffices to have an upper bound for $(1+|f(z)|^2)f^{\sharp}(z)$ in terms of $f(z)$.

THEOREM 3. (Schwick's Criterion, see [6]) *Let $\mathscr{F}$ be a family of holomorphic functions on a domain $\Omega \subset C^n$ with the property that for each compact set $K \subset \Omega$ there is a function $h_K : (0,\infty) \to (0,\infty)$, which is bounded on some neighborhood of each $x_0 \in (0,\infty)$, such that*

$$(1+|f(z)|^2)f^{\sharp}(z) \leqslant h_K(|f(z)|)$$

for all $f \in \mathscr{F}$ and $z \in K$. Then $\mathscr{F}$ is normal in Ω.

In combination with Nevanlinna theory on completely ramified values of an entire (holomorphic in C) function, Zalcman's Lemma immediately shows that requires even less knowledge of $(1+|f(z)|^2)f^\sharp(z)$.

THEOREM 4. *A family $\mathscr{F}$ of holomorphic functions on a domain $\Omega \subset C^n$ is normal on Ω if and only if for each compact set $K \subset \Omega$, there exists a set $E = E(K) \subset C$ containing at least three distinct values and a finite constant $M = M(K) > 0$ for which*

$$f^\sharp(z) \leqslant M, \quad z \in K, \ f(z) \in E \tag{1}$$

for all $f \in \mathscr{F}$.

We note that an analogous result for normal meromorphic functions in the unit disk was proved by Lappan in 1974 [12]; his proof is also in the spirit of Zalcman's principle.

The plan of this paper is as follows. In Section 2, we state a number of auxiliary results and prove a key lemma needed to prove Theorem 2. In Section 3, we give the proofs of Theorem 2 and Theorem 4.

2. Auxiliary results

In order to prove our theorems, we attend to a few details. Let $g(\lambda)$ be an entire (holomorphic in C) function, if the equation $g(\lambda) = a$, $a \in C$, has no simple roots then a called a totally ramified values.

Note that an omitted value trivially satisfies this definition, but that it will be useful to distinguish between omitted values and non-omitted totally ramified values.

In a proof of Theorem 4 we make use the following theorem of R. Nevanlinna.

THEOREM 5. [15, Theorem 17.3.10., p. 274] *Let g be an entire function. Then g has at most two totally ramified (finite) values.*

Let f be a holomorphic function on an open connected set Ω in C^n. Define the value set by

$$A_f(a) = \{z \in \Omega : f(z) = a\} = f^{-1}[\{a\}].$$

THEOREM 6. (Hurwitz's theorem [13, Corollary p. 80]) *Let Ω be an open connected set in C^n and let $\{f_j\}$ be a sequence of holomorphic functions on Ω, converging uniformly on compact sets to a nonconstant holomorphic function f. If $A_{f_j}(a) = \emptyset$ for all j then $A_f(a) = \emptyset$.*

Let $\zeta, v \in C^n$, $v \neq 0$. The set

$$\{\xi \in C^n : \xi = \zeta + \lambda v, \lambda \in C\}$$

is called a complex line in C^n.

The restriction of an entire (holomorphic in C^n) function g to a complex line $\{\xi = \zeta + \lambda v, \lambda \in C\}$ clearly is an entire $g(\zeta + \lambda \cdot v)$ of complex variable λ in C.

Note that

$$L_z(\log(1+|f(z)|^2),v) = \frac{|(Df(z),v)|^2}{(1+|f(z)|^2)^2}$$

on Ω. Appealing to the Cauchy-Schwarz inequality it is easy to show that

$$(1+|f(z)|^2)f^\sharp(z) = |Df(z)|.$$

The key lemma needed to prove our main result, Theorem 2, is the following one.

LEMMA 1. *Let f be a holomorphic function on the closed unit ball $\overline{B(0,1)}$, and α be a real number with $-1<\alpha<\infty$. Suppose*

$$\max_{|z|\leqslant 1/j} \frac{(1-j|z|)^{1+\alpha}(1+|f(z)|^2)f^\sharp(z)}{1+(1-j|z|)^{2\alpha}|f(z)|^2} > 1.$$

Then there exists a point ξ^*, $|\xi^*|<1/j$, and a real number ρ, $0<\rho<1$, such that

$$\max_{|z|\leqslant 1/j} \frac{(1-j|z|)^{1+\alpha}\rho^{1+\alpha}(1+|f(z)|^2)f^\sharp(z)}{1+(1-j|z|)^{2\alpha}\rho^{2\alpha}|f(z)|^2}$$
$$= \frac{(1-j|\xi^*|)^{1+\alpha}\rho^{1+\alpha}(1+|f(\xi^*)|^2)f^\sharp(\xi^*)}{1+(1-j|\xi^*|)^{2\alpha}\rho^{2\alpha}|f(\xi^*)|^2} = 1.$$

Proof. Set

$$\varphi(\rho,z) := \frac{(1-j|z|)^{1+\alpha}\rho^{1+\alpha}(1+|f(z)|^2)f^\sharp(z)}{1+(1-j|z|)^{2\alpha}\rho^{2\alpha}|f(z)|^2}.$$

We note that the denominator of the above fraction does not vanish on $[0,1]\times\{z\in C^n : |z|\leqslant 1/j\}$. Since $(1+|f(z)|^2)f^\sharp(z)$ is bounded continuous function on $\{|z|\leqslant 1/j\}$ there exists a constant $M<\infty$ with the property that

$$\varphi(\rho,z) \leqslant (1-j|z|)^{1+\alpha}\rho^{1+\alpha}M \tag{2}$$

for all $(\rho,z)\in[0,1]\times\{z\in C^n : |z|\leqslant 1/j\}$. It follows $\varphi(\rho,z)$ is continuous on $[0,1]\times\{z\in C^n : |z|\leqslant 1/j\}$ and $\varphi(0,z)\equiv 0$ on $\{z\in C^n : |z|\leqslant 1/j\}$.

Suppose that $\varphi(1,z_1^*) := \max_{|z|\leqslant 1/j}\varphi(1,z) > 1$. It is evident that $|z_1^*|<1/j$. Hence $\varphi(0,z_1^*)=0$ and $\varphi(1,z_1^*)>1$. By continuity of $\varphi(\rho,z)$ on $[0,1]\times\{z\in C^n : |z|\leqslant 1/j\}$, there exists ρ_1, $0<\rho_1<1$, such that $\varphi(\rho_1,z_1^*)=1$.

Repeating this procedure we can find ρ_m, $0<\rho_m<1$, and z_m^*, $|z_m^*|<1/j$, such that

$$\max_{|z|\leqslant 1/j}\varphi(\rho_1\ldots\rho_m,z) = \varphi(\rho_1\ldots\rho_m,z_m^*) > 1, \quad \varphi(\rho_1\ldots\rho_m\rho_{m+1},z_m^*) = 1. \tag{3}$$

The sequence $\{x_m := \rho_1\ldots\rho_m\}$ is a bounded and decreasing sequence. Then the greatest lower bound of the set $\{x_m : m\in N\}$, say ρ, is the limit of $\{x_m\}$. The sequence $\{z_m^*\}$

contains a subsequence, again denoted by $\{z_m^*\}$, such that $\lim_{m\to\infty} z_m^* = \xi^*$. From (2) follows that $0 < \rho < 1$ and $|\xi^*| < 1/j$.

We claim that

$$\max_{|z|\leqslant 1/j} \lim_{m\to\infty} \varphi(\rho_1 \dots \rho_m, z) = \lim_{m\to\infty} \max_{|z|\leqslant 1/j} \varphi(\rho_1 \dots \rho_m, z). \tag{4}$$

Since φ is continuous function on $[0,1] \times \overline{B(0,1/j)}$ by the Weierstrass theorem (see [8, Theorem (Weierstrass) p. 565]) we can find $|\eta| < 1/j$ and $|w_m| < 1/j$ such that

$$\max_{|z|\leqslant 1/j} \lim_{m\to\infty} \varphi(\rho_1 \dots \rho_m, z) = \max_{|z|\leqslant 1/j} \varphi(\rho, z) = \varphi(\rho, \eta); \tag{5}$$

$$\varphi(\rho_1 \dots \rho_m, \eta) \leqslant \max_{|z|\leqslant 1/j} \varphi(\rho_1 \dots \rho_m, z) = \varphi(\rho_1 \dots \rho_m, w_m), \ \ m = 1, 2, \dots. \tag{6}$$

By the Bolzano-Weierstrass theorem there is an infinite subsequence of $\{w_m\}$, again denoted by $\{w_m\}$, and ς, $|\varsigma| \leqslant 1/j$, such that $w_m \to \varsigma$ as $m \to \infty$. Because $w_m \to \varsigma$ and $\rho_1 \dots \rho_m \to \rho$ as $m \to \infty$ and φ is continuous function on $[0,1] \times \overline{B(0,1/j)}$ from (5) and (6) we see

$$\begin{aligned}\varphi(\rho, \eta) &\leqslant \lim_{m\to\infty} \max_{|z|\leqslant 1/j} \varphi(\rho_1 \dots \rho_m, z) = \varphi(\rho, \varsigma) \\ &\leqslant \max_{|z|\leqslant 1/j} \varphi(\rho, z) = \max_{|z|\leqslant 1/j} \lim_{m\to\infty} \varphi(\rho_1 \dots \rho_m, z) = \varphi(\rho, \eta).\end{aligned}$$

That is, the claim (4) is proved. Combining (4) and (5) we obtain

$$\max_{|z|\leqslant 1/j} \varphi(\rho, z) = \varphi(\rho, \xi^*) = 1 \ \ (|\xi^*| < 1/j).$$

The proof of the lemma is complete. □

3. Proofs of main theorems

Proof of Theorem 2. "⇒" The proof is basically that was used by author in [5] with minor modifications. To simplify matters we assume that $z_0 = 0$ and all functions under consideration are holomorphic on the closed unit ball $\overline{B(0,1)}$. By Marty's criterion (Theorem 1) $\mathscr{F}$ contains functions f_j, $j \in N$, satisfying $\max_{|z|<1/(2j)} f_j^\sharp(z) > 2^{1+|\alpha|} j^{3(1+|\alpha|)}$. Since $1 - j|z| > 1/2$ if $|z| < 1/(2j)$ there exists a ξ_j with $|\xi_j| < 1/j$ such that

$$\begin{aligned}&\max_{|z|\leqslant 1/j} (1 - j|z|)^{1+|\alpha|} f_j^\sharp(z) = (1 - j|\xi_j|)^{1+|\alpha|} f_j^\sharp(\xi_j) \\ &\geqslant \max_{|z|\leqslant 1/2j} (1 - j|z|)^{1+|\alpha|} f_j^\sharp(z) \geqslant j^{3(1+|\alpha|)}.\end{aligned}$$

The power function $t^{2\alpha}, t > 0$, is continuous, monotone (increasing when $\alpha > 0$, decreasing when $\alpha < 0$), hence

$$(1 - j|z|)^{2\alpha}(1 + |f(z)|^2) \geqslant 1 + (1 - j|z|)^{2\alpha}|f(z)|^2 \ \ (-1 < \alpha \leqslant 0 \ \ \text{arbitrary})$$

and

$$1+(1-j|z|)^{2\alpha}|f(z)|^2 \leqslant [1+|f(z)|^2] \ \ (0<\alpha<\infty \ \ \text{arbitrary})$$

we have

$$\frac{(1-j|\xi_j|)^{1+\alpha}(1+|f_j(\xi_j)|^2)f_j^\sharp(\xi_j)}{1+(1-j|\xi_j|)^{2\alpha}|f_j(\xi_j)|^2} > (1-j|\xi_j|)^{1+|\alpha|}f_j^\sharp(\xi_j) > j^{3(1+|\alpha|)}. \tag{7}$$

Hence

$$\max_{|z|\leqslant 1/j} \frac{(1-j|z|)^{1+\alpha}(1+|f_j(z)|^2)f_j^\sharp(z)}{1+(1-j|z|)^{2\alpha}|f_j(z)|^2} > 1.$$

According to Lemma 1, there exists $\xi_j^*, \ |\xi_j^*|<1/j$, and $\rho_j, \ 0<\rho_j<1$, such that

$$\max_{|z|\leqslant 1/j} \frac{(1-j|z|)^{1+\alpha}\rho_j^{1+\alpha}(1+|f_j(z)|^2)f_j^\sharp(z)}{1+(1-j|z|)^{2\alpha}\rho_j^{2\alpha}|f_j(z)|^2}$$
$$= \frac{(1-j|\xi_j^*|)^{1+\alpha}\rho_j^{1+\alpha}(1+|f_j(\xi_j^*)|^2)f_j^\sharp(\xi_j^*)}{1+(1-j|\xi_j^*|)^{2\alpha}\rho_j^{2\alpha}|f_j(\xi_j^*)|^2} = 1.$$

Therefore inequality (7) shows that

$$\begin{aligned} 1 &= \frac{(1-j|\xi_j^*|)^{1+\alpha}\rho_j^{1+\alpha}(1+|f_j(\xi_j^*)|^2)f_j^\sharp(\xi_j^*)}{1+(1-j|\xi_j^*|)^{2\alpha}\rho_j^{2\alpha}|f_j(\xi_j^*)|^2} \\ &\geqslant \frac{(1-j|\xi_j|)^{1+\alpha}\rho_j^{1+\alpha}(1+|f_j(\xi_j)|^2)f_j^\sharp(\xi_j)}{1+(1-j|\xi_j|)^{2\alpha}\rho_j^{2\alpha}|f_j(\xi_j)|^2} \\ &\geqslant (1-j|\xi_j|)^{1+|\alpha|}\rho_j^{1+|\alpha|}f_j^\sharp(\xi_j) \geqslant \rho_j^{1+|\alpha|}j^{3(1+|\alpha|)} \ \ (|\xi_j|<1/j). \end{aligned}$$

It follows

$$\left(\frac{1}{j}\right)^3 \geqslant \rho_j \to 0. \tag{8}$$

Put

$$r_j = (1-j|\xi_j^*|)\rho_j \to 0.$$

Set

$$h_j(z) = r_j^\alpha f_j(\xi_j^*+r_jz).$$

We claim that appropriately chosen subsequences $z_k=\xi_{j_k}, \ \rho_k=r_{j_k}$, and $g_k=h_{j_k}$ will do. First of all, $h_j(z)$ is defined on $|z|<\frac{1}{j\rho_j}$, hence on $|z|<j$, since

$$|\xi_j^*+r_jz| \leqslant |\xi_j^*|+r_j|z| < |\xi_j^*|+r_j\frac{1-j|\xi_j^*|}{jr_j} = \frac{1}{j}.$$

By the invariance of the Levi form under biholomorphic mappings, we have

$$L_z(\log(1+|h_j|^2),v) = L_{\xi_j^*+r_jz}(\log(1+|h_j|^2),r_jv)$$

and hence

$$h_j^{\sharp}(z) = r_j h_j^{\sharp}(\xi_j^* + r_j z).$$

Since $r_j = (1 - j|\xi_j^*|)\rho_j$ a simple computations shows that

$$\begin{aligned} h_j^{\sharp}(z) &= \frac{r_j r_j^{\alpha}(1+|f_j(\xi_j^*+r_jz)|^2) f_j^{\sharp}(\xi_j^*+r_jz)}{1+r_j^{2\alpha}|f_j(\xi_j^*+r_jz)|^2} \\ &= \frac{(1-j|\xi_j^*|)^{1+\alpha}\rho_j^{1+\alpha}(1+|f_j(\xi_j^*+r_jz)|^2) f_j^{\sharp}(\xi_j^*+r_jz)}{1+[(1-j|\xi_j^*|)/(1-j|\xi_j^*+r_jz|)]^{2\alpha}(1-j|\xi_j^*+r_jz|)^{2\alpha}\rho_j^{2\alpha}|f_j(\xi_j^*+r_jz)|^2} \\ &= \frac{(1-j|\xi_j^*|)^{1+\alpha}}{(1-j|\xi_j^*+r_jz|)^{1+\alpha}} \cdot \frac{(1-j|\xi_j^*+r_jz|)^{1+\alpha}\rho_j^{1+\alpha}(1+|f_j(\xi_j^*+r_jz)|^2) f_j^{\sharp}(\xi_j^*+r_jz)}{\left[1+\left(\frac{1-j|\xi_j^*|}{1-j|\xi_j^*+r_jz|}\right)^{2\alpha} \cdot (1-j|\xi_j^*+r_jz|)^{2\alpha}\rho_j^{2\alpha}|f_j(\xi_j^*+r_jz)|^2\right]}. \end{aligned}$$

Bearing in mind Lemma 1 it is easy to see that $h_j^{\sharp}(0) = 1$. Since

$$\frac{1}{1+1/j} \leqslant \frac{1-j|\xi_j^*|}{1-j|\xi_j^*+r_jz|} \leqslant \frac{1}{1-1/j}$$

we have

$$\begin{aligned} &1+\left(\frac{1-j|\xi_j^*|}{1-j|\xi_j^*+r_jz|}\right)^{2\alpha} \cdot (1-j|\xi_j^*+r_jz|)^{2\alpha}\rho_j^{2\alpha}|f_j(\xi_j^*+r_jz)|^2 \\ &\geqslant \left(\frac{1}{1-1/j}\right)^{2\alpha} \cdot \left[1+(1-j|\xi_j^*+r_jz|)^{2\alpha}\rho_j^{2\alpha}|f_j(\xi_j^*+r_jz)|^2\right] \ \ (-1<\alpha\leqslant 0 \ \text{ arbitrary}) \end{aligned}$$

and

$$\begin{aligned} &1+\left(\frac{1-j|\xi_j^*|}{1-j|\xi_j^*+r_jz|}\right)^{2\alpha} \cdot (1-j|\xi_j^*+r_jz|)^{2\alpha}\rho_j^{2\alpha}|f_j(\xi_j^*+r_jz)|^2 \\ &\geqslant \left(\frac{1}{1+1/j}\right)^{2\alpha} \cdot \left[1+(1-j|\xi_j^*+r_jz|)^{2\alpha}\rho_j^{2\alpha}|f_j(\xi_j^*+r_jz)|^2\right] \ \ (0<\alpha<\infty \ \text{ arbitrary}). \end{aligned}$$

From the above inequalities and Lemma 1 we infer that[1]

$$\begin{aligned} h_j^{\sharp}(z) &\leqslant \left(1+\frac{sgn(\alpha)}{j}\right)^{2\alpha} \cdot \left(\frac{1-|\xi_j^*|}{1-j|\xi_j^*+r_jz|}\right)^{1+\alpha} \\ &\quad \cdot \frac{(1-j|\xi_j^*+r_jz|)^{1+\alpha}\rho_j^{1+\alpha}(1+|f_j(\xi_j^*+r_jz)|^2) f_j^{\sharp}(\xi_j^*+r_jz)}{1+(1-j|\xi_j^*+r_jz|)^{2\alpha}\rho_j^{2\alpha}|f_j(\xi_j^*+r_jz)|^2} \\ &\leqslant \left(1+\frac{sgn(\alpha)}{j}\right)^{2\alpha} \cdot \left(\frac{1-j|\xi_j^*|}{1-j|\xi_j^*+r_jz|}\right)^{1+\alpha} \cdot 1 \\ &\leqslant \left(1+\frac{sgn(\alpha)}{j}\right)^{2\alpha} \cdot \left(\frac{1}{1-1/j}\right)^{1+\alpha} \end{aligned}$$

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[1] sgn denotes the signum function (i.e., $sgn(0) = 0$, $sgn(\alpha) = 1$ if $\alpha > 0$ and -1 if $\alpha < 0$).

for all $|z| < j$. For every $m \in N$ the sequence $\{h_j\}_{j>m}$ is normal in $B(0,m)$ by Marty's criterion (Theorem 1). The well-known Cantor diagonal process yields a subsequence $\{g_k = h_{j_k}\}$ which converges uniformly on every ball $B(0,R)$. The limit function g satisfies $g^\sharp(z) \leqslant \limsup_{j\to\infty} h_j^\sharp(z) \leqslant 1 = g^\sharp(0)$. Clearly, g is non-constant because $g^\sharp(0) \neq 0$.

"$\Leftarrow$" The reverse implication is essentially a backwards glance at the above, but for future considerations we go in some details. Take $\alpha = 0$. Suppose that there exist sequences $f_j \in \mathscr{F}$, $z_j \to 0$, $\rho_j \to 0$, such that the sequence

$$g_j(z) = f_j(z_j + \rho_j z)$$

converges locally uniformly in C^n to a non-constant entire function g satisfying $g^\sharp(z) \leqslant g^\sharp(0) = 1$, but $\mathscr{F}$ is normal. By Marty's criterion (Theorem 1) there exists a constant $M > 0$ such that

$$\max_{|z|\leqslant 1/2} f_j^\sharp(z) < M$$

for all j. Since $z_j \to 0$, $\rho_j \to 0$, then for $|z| < 1/2$ and j sufficiently large, we have

$$|z_j + \rho_j z| \leqslant |z_j| + \rho_j|z| \leqslant |z_j| + \rho_j/2 < 1/2.$$

Thus

$$g_j^\sharp(z) = f_j^\sharp(z_j + \rho_j z)\rho_j \leqslant M\rho_j \to 0 \;\; (|z| < 1/2).$$

This implies that $g^\sharp(0) = 0$, which is a contradiction to $g^\sharp(0) = 1$. □

Proof of Corollary 1. Since a family $\{1/f, f \in \mathscr{F}\}$ conforms to the hypotheses of Theorem 1 the earlier argument shows that there exist sequences $1/f_j$, $z_j \to z_0$, $r_j \to 0$, such that the sequence

$$g_j(z) := \frac{r_j^\alpha}{f_j(z_j + r_j z)} \;\; (-1 < \alpha < \infty \;\; arbitrary)$$

converges locally uniformly in C^n to a non-constant entire function g satisfying $g^\sharp(z) \leqslant g^\sharp(0) = 1$.

By Hurwitz's theorem either $g \equiv 0$ or g never vanishes. Since $g^\sharp(0) = 1$ it is easy to see that g never vanishes then $1/g$ is entire function in C^n. It follows $r_j^{-\alpha} f_j \to 1/g$ uniformly in C^n. Since Levi form vanishes for any pluriharmonic function,

$$L_z(\log(1 + |1/g|^2), v) = L_z(\log(1 + |g|^2), v) - 2L_z(\log|g|, v) = L_z(\log(1 + |g|^2), v).$$

Therefore,

$$g^\sharp(z) = (1/g)^\sharp(z).$$

For every $z \in C^n$ we have $g^\sharp(z) \leqslant g^\sharp(0) = 1$, hence

$$(1/g)^\sharp(z) \leqslant (1/g)^\sharp(0) = 1.$$

The case $-\infty < \alpha < 1$ is proved. This completes the proof of the theorem. □

Proof of Theorem 4. Marty's Theorem shows that (1) is necessary with $E = C$.

To prove sufficiency, suppose that (1) holds but $\mathscr{F}$ is not normal at point $z_0 \in \Omega$. Since all features of the theorem are local, and translation and scale-change invariant, it suffices to consider the case when $z_0 = 0$ and $\Omega = B(0,1)$. If $\mathscr{F}$ is not normal at 0, it follows from Zalcman's lemma [4, Theorem 3.1] that there exist $f_j \in \mathscr{F},\ z_j \to 0$, $\rho_j \to 0$, such that the sequence

$$g_j(z) := f_j(z_j + \rho_j z)$$

converges uniformly on compact subsets of C^n to a non-constant entire function g satisfying $g^\sharp(z) \leqslant g^\sharp(0) = 1$.

Let $K \subset B(0,1)$ be a closed ball in C^n about 0. Let $E(K) \supset \{a_1, a_2, a_3\}$, where a_1, a_2, a_3 are distinct values in C.

Suppose that $\zeta^l \in A_g(a_l)$. Choose $R > 0$ such that $\zeta^l \in S_R = \{\xi \in C^n : |\xi| < R\}$. Since normality is a local property, the restriction of a family $\{g_j - a_l\}$ to any open ball $s_n(\zeta^l) := \{\xi \in C^n : |\xi - \zeta^l| < 1/n\} \subset S_R$ is a normal family. By Hurwitz' theorem to $A_{g_j}(a_l) \cap s_n(\zeta^l) \neq \emptyset$ for j sufficiently large since g is not a constant function. It is routine to show that there exist a sequence $\{p_j^l\} \subset S_K$, such that $g_j(p_j^l) \to a_l$.

Since $\rho_j \to 0$ we see that $z_j + \rho_j p_j^l \in K$ for k sufficiently large. Now by (1), $f_j^\sharp(z_j + \rho_j p_j^l) \leqslant M$ for k sufficiently large, so that

$$g^\sharp(\zeta^l) = \lim_{k \to \infty} g_j^\sharp(p_j^l) = \lim_{k \to \infty} \rho_j f_j^\sharp(z_j + \rho_j p_j^l) \leqslant \lim_{k \to \infty} \rho_j M = 0.$$

Thus $g^\sharp(\zeta^l) = 0$.

Since

$$g^\sharp(\zeta^l) = \max_{\{v \in C^n : |v| = 1\}} \frac{|dg(\zeta^l + \lambda \cdot v)|_{\lambda=0}/d\lambda|^2}{1 + |g(\zeta^l)|^2} = 0$$

it follows a_l is a finite totally ramifies value for $g(\zeta^l + \lambda \cdot v)$ ($v \in C^n$ be arbitrary (but fixed) and $|v| = 1$).

If $\{\xi \in C^n : \xi = \zeta^l + \lambda \cdot v\} \cap A_g(a^k) \ni \zeta^k$ then $\zeta^k = \zeta^l + \lambda_k \cdot v$ for some $\lambda_k \in C$. Arguing as above we have

$$g^\sharp(\zeta^k) = g^\sharp(\zeta^l + \lambda_k \cdot v) = 0.$$

Hence a_k is a totally ramified value for $g(\zeta^l + \lambda \cdot v)$.

If $\{\xi \in C^n : \xi = \zeta^l + \lambda \cdot v\} \cap A_g(a^k) = \emptyset$ then a^k is omitted value for $g(\zeta^l + \lambda \cdot v)$ and hence a totally ramified (finite) value of the function $g(\zeta^l + \lambda \cdot v)$.

Thus a_1, a_2, a_3 are three totally ramified (finite) values for the entire function $g\Big(\zeta^l + \lambda \cdot v\Big)$. By Nevanlinna's theorem $g\Big(\zeta^l + \lambda \cdot v\Big)$ is constant. Since v was arbitrary the function g is constant, a contradiction. $\square$

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(Received January 2, 2021)

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