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WIJSMAN LACUNARY INVARIANT STATISTICAL CONVERGENCE FOR TRIPLE SEQUENCES VIA ORLICZ FUNCTION

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Abstract. In this paper, we generalized the Wijsman lacunary invariant statistical convergence of closed sets in metric space by introducing the Wijsman lacunary invariant statistical $\widetilde{\phi}$-convergence for the sets of triple sequences. We introduce the concepts of Wijsman invariant $\widetilde{\phi}$-convergence, Wijsman invariant statistical $\widetilde{\phi}$-convergence, Wijsman lacunary invariant $\widetilde{\phi}$-convergence, Wijsman lacunary invariant statistical $\widetilde{\phi}$-convergence for the sets of triple sequences. In addition, we investigate existence of some relations among these new notations for the sets of triple sequences.

1. Introduction and background

The idea of statistical convergence was first introduced by Fast [6] and Steinhaus [28] independently in the same year 1951 and since then several generalizations and applications of this concept have been investigated by various authors, namely Fridy [7], Gürdal and Huban [10], Gürdal and Pehlivan [11, 12], Nabiev et al. [17], and many others (see [4, 8, 24]).

Statistical convergence depends on the natural density of subsets of the set $\mathbb{N}$ of positive integers. The natural density $\delta(A)$ of a subset A of $\mathbb{N}$ is defined by

$$\delta(A) = \lim_{n\to\infty} n^{-1} |\{k \leqslant n : k \in A\}|$$

where $|\{k \leqslant n : k \in A\}|$ denotes the number of elements of A not exceeding n. A sequence $(x_k) \subset \mathbb{R}$ is said to be statistically convergent to $\ell \in \mathbb{R}$ if, for each $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - \ell| \geqslant \varepsilon\}$ has the zero natural density.

The concept of convergence of sequences of points has been extended by several authors [19, 20, 29, 30] to convergence of sequences of sets. One of such extensions considered in this paper is the concept of Wijsman convergence. Nuray and Rhoades [18] extended the notion of Wijsman convergence of sequences of sets to that of Wijsman statistical convergence and introduced the notion of Wijsman strong Cesaro summability of sequences of sets and discussed its relations with Wijsman statistical convergence.

In this study, we introduce the concepts of Wijsman invariant $\widetilde{\phi}$-convergence, Wijsman invariant statistical $\widetilde{\phi}$-convergence,Wijsman lacunary invariant $\widetilde{\phi}$-convergence,

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Wijsman lacunary invariant statistical $\widetilde{\phi}$-convergence for the sets of triple sequences. Also, we investigate existence of some relations among these new $\widetilde{\phi}$-convergence concepts for the sets of triple sequences.

We now recall the following basic concepts from [2, 15, 19, 26, 27] which will be needed throughout the paper.

Let σ be a mapping of the positive integers into themselves. A continuous linear functional φ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies the following conditions:

(i) $\varphi(x) \geqslant 0$, for the sequence $x = (x_n)$ with $x_n \geqslant 0$ for all $n \in \mathbb{N}$,

(ii) $\varphi(e) = 1$, where $e = (1,1,1,\ldots)$ and

(iii) $\varphi\left(x_{\sigma(n)}\right) = \varphi(x_n)$ for all $x \in \ell_\infty$.

The mapping σ is assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all $n,m \in \mathbb{Z}^+$, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, φ extends the limit functional on c, the space of convergent sequences, in the sense that $\varphi(x_n) = \lim x_n$ for all $x \in c$. In the case σ is translation mappings $\sigma(n) = n+1$, the σ-mean is often called a Banach limit. The space V_σ of the bounded sequences whose invariant means are equal may be defined, as follows;

$$V_\sigma = \left\{ x \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \text{ uniformly in } m \right\}.$$

In [27], Schaefer proved that a bounded sequence $x = (x_n)$ of real numbers is σ-convergent to L if and only if

$$\lim_{p\to\infty} \frac{1}{p} \sum_{k=1}^{p} x_{\sigma^k(m)} = L,$$

uniformly in m.

Let (X,ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, the distance $d(x,A)$ from x to A is defined by

$$d(x,A) = \inf_{a\in A} \rho(x,a).$$

DEFINITION 1. Let (X,ρ) be a metric space. For any non-empty closed subsets A; $A_k \subseteq X$; we say that the sequence (A_k) is Wijsman convergent to A if

$$\lim_{k\to\infty} d(x,A_k) = d(x,A)$$

We now recall that the concept of statistical convergence for triple sequences was presented by Şahiner, Gürdal and Düden [23] as follows:

A function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) is called a real (complex) triple sequence. A triple sequence $\left(x_{jkl}\right)$ is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $|x_{jkl} - L| < \varepsilon$ whenever $j,k,l \geqslant n_0$.

DEFINITION 2. A subset K of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ is said to have natural density $\delta_3(K)$ if

$$\delta_3(K) = P - \lim_{n,k,l\to\infty} \frac{|K_{nkl}|}{nkl}$$

exists, where the vertical bars denote the number of (n,k,l) in K such that $p \leqslant n$, $q \leqslant k$, $r \leqslant l$. Then, a real triple sequence $x = (x_{pqr})$ is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\delta_3\left(\left\{(n,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N} : p\leqslant n, q\leqslant k, r\leqslant l,\ \left|x_{pqr}-L\right|\geqslant\varepsilon\right\}\right) = 0.$$

Recently, Mursaleen and Edely [16] presented the idea of statistical convergence for multiple sequences, and there are several papers dealing with double and triple statistical and ideal convergence (see literature [1, 5, 9, 13, 21]). Also, the readers should refer to the monographs [3] and [14] for the background on the sequence spaces and related topics.

In several literary works, statistical convergence of any real sequence is identified relatively to absolute value. While we have known that the absolute value of real numbers is special of an Orlicz function [22], that is, a function $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ in such a way that it is even, non-decreasing on $\mathbb{R}^+$, continuous on $\mathbb{R}$, and satisfying

$$\widetilde{\phi}(x) = 0 \text{ if and only if } x = 0 \text{ and } \widetilde{\phi}(x)\to\infty \text{ as } x\to\infty.$$

Further, an Orlicz function $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ is said to satisfy the $\triangle_2$ condition, if there exists an positive real number M such that $\widetilde{\phi}(2x)\leqslant M.\widetilde{\phi}(x)$ for every $x\in\mathbb{R}^+$.

DEFINITION 3. ([25]) Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. A sequence $x=(x_n)$ is said to be statistically $\widetilde{\phi}$-convergent to L if for each $\varepsilon>0$,

$$\lim_n \frac{1}{n}\left|\left\{k\leqslant n : \widetilde{\phi}\left(x_k - L\right)\geqslant\varepsilon\right\}\right| = 0.$$

Furthermore, a new type of sequence called triple lacunary sequence was introduced in Esi and Savaş [5]. The triple sequence $\theta_3 = \theta_{p,q,r} = \left\{(j_p,k_q,l_r)\right\}$ is called triple lacunary sequence if there exist three increasing sequences of integers such that

$$j_0 = 0,\ h_p = j_p - j_{p-1}\to\infty \text{ as } p\to\infty,$$

$$k_0 = 0,\ h_q = k_q - k_{q-1}\to\infty \text{ as } q\to\infty,$$

and

$$l_0 = 0,\ h_r = l_r - l_{r-1}\to\infty \text{ as } r\to\infty.$$

Let $k_{p,q,r} = j_pk_ql_r$, $h_{p,q,r} = h_ph_qh_r$ and $\theta_{p,q,r}$ is determined by

$$I_{p,q,r} = \left\{(j,k,l) : j_{p-1} < j \leqslant j_p, k_{q-1} < k \leqslant k_q \text{ and } l_{r-1} < l \leqslant l_r\right\},$$

$$s_p = \frac{j_p}{j_{p-1}}, s_q = \frac{k_q}{k_{q-1}},\ s_r = \frac{l_r}{l_{r-1}} \text{ and } s_{p,q,r} = s_ps_qs_r.$$

Let $D\subset\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. The number

$$\delta_3^{\theta_3}(D) = \lim_{p,q,r}\frac{1}{h_{p,q,r}}\left|\left\{(j,k,l)\in I_{p,q,r} : (j,k,l)\in D\right\}\right|$$

is said to be the $\theta_{p,q,r}$-density of D, provided the limit exists.

2. Main results

Following the above definitions and results, we aim in this section to introduce some new notions of Wijsman invariant statistical convergence with the use of Orlicz function, lacunary and triple sequences and obtain some analogous results from the new definitions point of views.

DEFINITION 4. A triple sequence $x = (x_{jkl})$ of real numbers is said to be σ_3-convergence to L if

$$\lim_{p,q,r\to\infty} \frac{1}{pqr} \sum_{j=1}^{p} \sum_{k=1}^{q} \sum_{l=1}^{r} x_{\sigma^j(m),\sigma^k(n),\sigma^l(o)} = L,$$

uniformly in m,n and o.

DEFINITION 5. A triple sequence $\{A_{jkl}\}$ is Wijsman invariant convergent to A if for each $x \in X$

$$\lim_{p,q,r\to\infty} \frac{1}{pqr} \sum_{j,k,l=1,1,1}^{p,q,r} d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) = d(x,A),$$

uniformly in m,n and o, and is written $A_{jkl} \to A\,(W_3V_\sigma)$.

DEFINITION 6. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A triple sequences $\{A_{jkl}\}$ is Wijsman strongly invariant $\widetilde{\phi}$-convergent to A if for each $x \in X$

$$\lim_{p,q,r\to\infty} \frac{1}{pqr} \sum_{j,k,l=1,1,1}^{p,q,r} \widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) = 0,$$

uniformly in m,n and o, and is written $A_{jkl} \to A\left(W_3^{\widetilde{\phi}} V_\sigma\right)$.

If double sequence $\{A_{jk}\}$ is taken instead of a triple sequence $\{A_{jkl}\}$ and $\widetilde{\phi}(x) = |x|$, then the concept Wijsman strongly invariant $\widetilde{\phi}$-convergence is reduced to Wijsman strongly invariant convergence.

DEFINITION 7. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A triple sequence $\{A_{jkl}\}$ is Wijsman invariant statistically $\widetilde{\phi}$-convergent to A if for every $\varepsilon > 0$ and for each $x \in X$

$$\lim_{p,q,r\to\infty} \frac{1}{pqr} \left| \left\{ j \leqslant p, k \leqslant q, l \leqslant r : \ \widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) \geqslant \varepsilon \right\} \right| = 0$$

or

$$\delta_3\left(\left\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) \geqslant \varepsilon \right\}\right) = 0$$

uniformly in m,n and o, and is written $A_{jkl} \to A\left(W_3^{\widetilde{\phi}} S_\sigma\right)$.

DEFINITION 8. $\theta_3 = \theta_{r,s,t}$ be a lacunary triple sequence. A triple sequence $\{A_{jkl}\}$ is Wijsman lacunary invariant convergent to A if for each $x \in X$

$$\lim_{p,q,r\to\infty} \frac{1}{h_{p,q,r}} \sum_{j,k,l\in I_{p,q,r}} d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) = d(x,A)$$

uniformly in m,n and o, and is written $A_{jkl} \to A\left(W_3 V_\sigma^\theta\right)$.

DEFINITION 9. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,s,t}$ be a lacunary triple sequence. A triple sequence $\{A_{jkl}\}$ is Wijsman strongly lacunary invariant $\widetilde{\phi}$-convergence to A if for each $x \in X$

$$\lim_{p,q,r\to\infty} \frac{1}{h_{p,q,r}} \sum_{j,k,l\in I_{p,q,r}} \widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) = 0,$$

uniformly in m,n and o, and is written $A_{jkl} \to A\left(W_3^{\widetilde{\phi}} V_\sigma^\theta\right)$.

DEFINITION 10. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,s,t}$ be a lacunary triple sequence. A triple sequence $\{A_{jkl}\}$ is Wijsman lacunary invariant statistically $\widetilde{\phi}$-convergent to A if for every $\varepsilon > 0$ and for each $x \in X$

$$\lim_{p,q,r\to\infty} \frac{1}{h_{p,q,r}} \left| \left\{ (j,k,l) \in I_{p,q,r} : \widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) \geqslant \varepsilon \right\} \right| = 0$$

or

$$\delta_{\theta_3}\left(\left\{ (j,k,l) \in I_{p,q,r} : \widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) \geqslant \varepsilon \right\}\right) = 0$$

uniformly in m,n and o, and is written $A_{jkl} \to A\left(W_3^{\widetilde{\phi}} S_\sigma^{\theta_3}\right)$.

DEFINITION 11. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A triple sequence $\{A_{jkl}\}$ is said to be bounded if there exists $M > 0$ such that $\widetilde{\phi}\left(A_{jkl}\right) \leqslant M$ for all $j,k,l \in \mathbb{N}$.

We denote the space of all bounded triple sequences by ℓ_∞^3.

THEOREM 1. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ be a lacunary triple sequence. Then, the following statements hold:*

(i) If $\{A_{jkl}\}$ is Wijsman strongly lacunary invariant $\widetilde{\phi}$-convergent to A, then $\{A_{jkl}\}$ is Wijsman lacunary invariant statistically $\widetilde{\phi}$-convergent to A.

(ii) If $\left(A_{jkl}\right) \in \ell_\infty^3$ and $\{A_{jkl}\}$ is Wijsman lacunary invariant statistically $\widetilde{\phi}$-convergent to A, then $\{A_{jkl}\}$ is Wijsman strongly lacunary invariant $\widetilde{\phi}$-convergent to A.

Proof. (i) $A_{jkl} \to A\left(W_3^{\widetilde{\phi}} V_\sigma^\theta\right)$. For every $\varepsilon > 0$ and for each $x \in X$, then we have

$$\begin{aligned}
&\sum_{(j,k,l)\in I_{p,q,r}} \widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) \\
&\geqslant \sum_{\substack{j,k,l\in I_{p,q,r} \\ \widetilde{\phi}\left(d\left(x,A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right)-d(x,A)\right)\geqslant\varepsilon}} \widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) \\
&\geqslant \varepsilon.\left|\left\{(j,k,l) \in I_{p,q,r} : \widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) \geqslant \varepsilon\right\}\right|.
\end{aligned}$$

This shows that $A_{jkl} \to A\left(W_3^{\widetilde{\phi}} S_\sigma^{\theta_3}\right)$.

(ii) Suppose that $\{A_{jkl}\}$ belongs to the space ℓ_∞^3 and $A_{jkl} \to A\left(W_3^{\widetilde{\phi}} S_\sigma^{\theta_3}\right)$. Then we can assume that

$$\widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) \leqslant M$$

for each $x \in X$ and all j,k and l. Given every $\varepsilon > 0$ and for each $x \in X$, we have

$$\begin{aligned}
&\frac{1}{h_{p,q,r}} \sum_{(j,k,l)\in I_{p,q,r}} \widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) \\
&= \frac{1}{h_{p,q,r}} \sum_{\substack{(j,k,l)\in I_{p,q,r} \\ \widetilde{\phi}\left(d\left(x,A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right)-d(x,A)\right)\geqslant\varepsilon}} \widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) \\
&\quad + \frac{1}{h_{p,q,r}} \sum_{\substack{(j,k,l)\in I_{p,q,r} \\ \widetilde{\phi}\left(d\left(x,A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right)-d(x,A)\right)<\varepsilon}} \widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) \\
&\leqslant \frac{M}{h_{p,q,r}}\left|\left\{(j,k,l) \in I_{p,q,r} : \widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) \geqslant \varepsilon\right\}\right| + \varepsilon.
\end{aligned}$$

This shows that $A_{jkl} \to A\left(W_3^{\widetilde{\phi}} V_\sigma^\theta\right)$. □

THEOREM 2. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ is an Orlicz function. Suppose for given $\delta > 0$ and every $\varepsilon > 0$, there exists*

$$\begin{aligned}
\frac{1}{pqr}\Big|\Big\{&0 \leqslant j \leqslant p-1, 0 \leqslant k \leqslant q-1, 0 \leqslant l \leqslant r-1 : \\
&\widetilde{\phi}\left(d\left(x, A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right) - d(x,A)\right) \geqslant \varepsilon\Big\}\Big| < \delta
\end{aligned}$$

for all $p \geqslant p_0$, $q \geqslant q_0$, $r \geqslant r_0$, $m \geqslant m_0$, $n \geqslant n_0$, $o \geqslant o_0$, then $\{A_{jkl}\}$ is Wijsman invariant statistically $\widetilde{\phi}$-convergent to A.

Proof. Let $\delta>0$. For every $\varepsilon>0$, we choose $p_0', q_0', r_0', m_0, n_0$ and o_0 such that for all $x\in X$,

$$\frac{1}{pqr}\Big|\Big\{0\leqslant j\leqslant p-1, 0\leqslant k\leqslant q-1, 0\leqslant l\leqslant r-1: \widetilde{\phi}\left(d\left(x,A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right)-d(x,A)\right)\geqslant\varepsilon\Big\}\Big|<\frac{\delta}{2} \tag{1}$$

for all $p\geqslant p_0',\ q\geqslant q_0',\ r\geqslant r_0',\ m\geqslant m_0,\ n\geqslant n_0,\ o\geqslant o_0$. It is enough to prove that there exists $p_0'',\ q_0'',\ r_0''$ such that for each $x\in X$,

$$\frac{1}{pqr}\Big|\Big\{0\leqslant j\leqslant p-1, 0\leqslant k\leqslant q-1, 0\leqslant l\leqslant r-1: \widetilde{\phi}\left(d\left(x,A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right)-d(x,A)\right)\geqslant\varepsilon\Big\}\Big|<\delta \tag{2}$$

for $p\geqslant p_0'',\ q>q_0'',\ r>r_0'',\ 0\leqslant m\leqslant m_0,\ 0\leqslant n\leqslant n_0$ and $0\leqslant o\leqslant o_0$.

Since taking $p_0=\max\{p_0',p_0''\}$, $q_0=\max\{q_0',q_0''\}$ and $r_0=\max\{r_0',r_0''\}$, (2) holds for each $x\in X$, $p\geqslant p_0,\ q\geqslant q_0,\ r\geqslant r_0$ and for all m,n and o, which gives the result. Once m_0,n_0 and o_0 have been chosen $0\leqslant m\leqslant m_0,\ 0\leqslant n\leqslant n_0,\ 0\leqslant o\leqslant o_0$, m_0,n_0 and o_0 fixed. So, suppose that $x\overset{S^L_{\theta_3}(\phi)}{\sim}y$, and let

$$F=\Big|\Big\{0\leqslant j\leqslant m_0-1, 0\leqslant k\leqslant n_0-1, 0\leqslant l\leqslant o_0-1: \widetilde{\phi}\left(d\left(x,A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right)-d(x,A)\right)\geqslant\varepsilon\Big\}\Big|.$$

Now taking $0\leqslant m\leqslant m_0,\ 0\leqslant n\leqslant n_0,\ 0\leqslant o\leqslant o_0$ and $p\geqslant m_0,\ q\geqslant n_0,\ r\geqslant o_0$, by (1) for each $x\in X$, we get

$$\begin{aligned}
&\frac{1}{pqr}\Big|\Big\{0\leqslant j\leqslant p-1, 0\leqslant k\leqslant q-1, 0\leqslant l\leqslant r-1: \widetilde{\phi}\left(d\left(x,A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right)-d(x,A)\right)\geqslant\varepsilon\Big\}\Big|\\
&\leqslant\frac{1}{pqr}\Big|\Big\{0\leqslant j\leqslant m_0-1, 0\leqslant k\leqslant n_0-1, 0\leqslant l\leqslant o_0-1: \widetilde{\phi}\left(d\left(x,A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right)-d(x,A)\right)\geqslant\varepsilon\Big\}\Big|\\
&\quad+\frac{1}{pqr}\Big|\Big\{m_0\leqslant j\leqslant p-1, n_0\leqslant k\leqslant q-1, o_0\leqslant l\leqslant r-1: \widetilde{\phi}\left(d\left(x,A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right)-d(x,A)\right)\geqslant\varepsilon\Big\}\Big|\\
&\leqslant\frac{F}{pqr}+\frac{\delta}{2}
\end{aligned}$$

and taking p,q,r sufficiently large, we can write

$$\frac{F}{pqr}+\frac{\delta}{2}<\delta$$

which gives (2) and thus, the result follows. □

THEOREM 3. *Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function and $\theta_3=\theta_{r,s,t}=\{(j_r,k_s,l_t)\}$ be a lacunary triple sequence. Then $\{A_{jkl}\}$ is Wijsman lacunary invariant statistically $\widetilde{\phi}$-convergent to A iff $\{A_{jkl}\}$ is Wijsman invariant statistically $\widetilde{\phi}$-convergent to A.*

Proof. Let $A_{jkl}\to A\left(W_3^{\widetilde{\phi}}S_\sigma^{\theta_3}\right)$. Then, for given $\delta>0$ there exists p_0,q_0,r_0 such that for all $\varepsilon>0$ and for each $x\in X$,

$$\frac{1}{h_{p,q,r}}\Big|\Big\{0\leqslant j\leqslant h_{p-1},0\leqslant k\leqslant h_{q-1},0\leqslant l\leqslant h_{r-1}:$$
$$\widetilde{\phi}\left(d\left(x,A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right)-d(x,A)\right)\geqslant\varepsilon\Big\}\Big|<\delta$$

for $p\geqslant p_0,\ q\geqslant q_0,\ r\geqslant r_0$ and $m=j_{p-1}+1+v,\ v\geqslant 0,\ n=k_{q-1}+1+w,\ w\geqslant 0,$ $o=l_{r-1}+1+z,\ z\geqslant 0$. Let $s\geqslant h_p,\ t\geqslant h_q$ and $u\geqslant h_r$. Write $s=\alpha h_p+e,\ t=\beta h_q+f$ and $u=\gamma h_r+g$ where $0\leqslant e\leqslant h_p,\ 0\leqslant f\leqslant h_q$ and $0\leqslant g\leqslant h_r,\ \alpha,\beta$ and γ are integers. Since $s\geqslant h_p,\ t\geqslant h_q$ and $u\geqslant h_r$, we can write

$$\begin{aligned}
&\frac{1}{stu}\Big|\Big\{0\leqslant j\leqslant s-1,0\leqslant k\leqslant t-1,0\leqslant l\leqslant u-1:\\
&\qquad\widetilde{\phi}\left(d\left(x,A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right)-d(x,A)\right)\geqslant\varepsilon\Big\}\Big|\\
&\leqslant\frac{1}{stu}\Big|\Big\{0\leqslant j\leqslant(\alpha+1)h_p-1,0\leqslant k\leqslant(\beta+1)h_q-1,0\leqslant l\leqslant(\gamma+1)h_r-1:\\
&\qquad\widetilde{\phi}\left(d\left(x,A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right)-d(x,A)\right)\geqslant\varepsilon\Big\}\Big|\\
&=\frac{1}{stu}\sum_{f=0}^{\alpha}\sum_{g=0}^{\beta}\sum_{h=0}^{\gamma}\Big|\Big\{eh_p\leqslant j\leqslant(e+1)h_p-1,fh_q\leqslant k\leqslant(f+1)h_q-1,\\
&\qquad gh_r\leqslant l\leqslant(g+1)h_r-1:\widetilde{\phi}\left(d\left(x,A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right)-d(x,A)\right)\geqslant\varepsilon\Big\}\Big|\\
&\leqslant\frac{1}{stu}(\alpha+1)(\beta+1)(\gamma+1)h_ph_qh_r\delta
\end{aligned}$$

and since for $\frac{1}{m}\alpha h_p\leqslant 1,\ \frac{1}{m}\beta h_q\leqslant 1$ and $\frac{1}{m}\gamma h_r\leqslant 1$, we have

$$\frac{1}{stu}\Big|\Big\{0\leqslant j\leqslant s-1,0\leqslant k\leqslant t-1,0\leqslant l\leqslant u-1:$$
$$\widetilde{\phi}\left(d\left(x,A_{\sigma^j(m),\sigma^k(n),\sigma^l(o)}\right)-d(x,A)\right)\geqslant\varepsilon\Big\}\Big|\leqslant 8.\delta.$$

Thus , by Theorem 2, $W_3^{\widetilde{\phi}}S_\sigma^{\theta_3}\subset W_3^{\widetilde{\phi}}S_\sigma$. It is easy to see that $W_3^{\widetilde{\phi}}S_\sigma\subset W_3^{\widetilde{\phi}}S_\sigma^{\theta_3}$. This completes the proof. □

From Theorem 3, we have

THEOREM 4. *Let* $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ *be an Orlicz function and* $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ *be a lacunary triple sequence. Then* $\{A_{jkl}\}$ *is Wijsman lacunary invariant convergent to* A *iff* $\{A_{jkl}\}$ *is Wijsman strongly invariant* $\widetilde{\phi}$*-convergent to* A.

When $(\sigma(m), \sigma(n), \sigma(o)) = (m+1, n+1, o+1)$, from Definitions 5-10, we have the definitions of almost Wijsman, almost convergence, Wijsman strongly almost $\widetilde{\phi}$-convergence, Wijsman almost statistical $\widetilde{\phi}$-convergence, Wijsman lacunary almost convergence, Wijsman strongly lacunary almost $\widetilde{\phi}$-convergence, Wijsman lacunary almost statistical $\widetilde{\phi}$-convergence for the sets of triple sequences.

So, similar inclusions to Theorems 1-4 hold between Wijsman strongly lacunary almost $\widetilde{\phi}$-convergent triple set sequences, Wijsman lacunary almost statistical $\widetilde{\phi}$-convergent triple set sequences, Wijsman almost statistical $\widetilde{\phi}$-convergent triple set sequences, Wijsman lacunary almost convergent triple set sequences and Wijsman strongly almost $\widetilde{\phi}$-convergent triple set sequences.

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