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QUASI–INVARIANT CONVERGENCE FOR DOUBLE SEQUENCE

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Abstract. In this paper we introduce the concept of quasi-invariant convergence and quasi-invariant statistical convergence of double sequence in a normed space and we shall present a characterization of a bounded sequence to be quasi-invariant convergent.

1. Introduction

The concept of σ-convergent was first introduced by Raimi [18]. There is another notion of convergence called statistical convergence which was first introduced by H. Fast [11] and independently by Schoenberg [12] and since then several generalization of this notion was investigated by several authors ([3], [5], [6], [10], [19]).

Recently Savas and Nurry ([7]) introduced the concept of σ-statistical convergence for real or complex sequences. Many generalization results had been done by several authors ([1], [4], [15], [16], [17] etc.) for single and double sequence.

Motivated by their works we would like to extend the idea of σ-convergent and σ-statistical convergence of single sequence to double sequence of real or complex numbers in a norm linear spaces.

First we introduce the notion of σ-convergent and σ-statistical convergence from the article of E. Savas and F. Nuray ([7], [9]) for single sequence.

Let $\mathbb{N}$ be the set of all positive integers. Let us consider a injection map

$$\sigma : \mathbb{N} \longrightarrow \mathbb{N} \quad \text{by} \quad \sigma^i(m) \neq m,\ \forall i \in \mathbb{N}$$

and

$$\sigma^i(m) = \sigma(\sigma^{i-1}(m)),\ \forall m \in \mathbb{N},\ \ i = 1,2,3,\dots$$

Let Ω be a real normed space. A continuous linear functional ζ on the space l_∞ of all bounded sequences in Ω is said to be an invariant mean or σ-limit if

(i) $\zeta(x) \geqslant 0$ for the sequence $x = \{x_n\} \in \Omega$ with $x_n \geqslant 0\ \forall n$

(ii) $\zeta(1,1,1,1,\dots) = 1$

(iii) $\zeta(x) = \zeta(x_{\sigma(n)})$ for all bounded sequence x.

DEFINITION 1. A sequence $x = \{x_n\} \in \Omega$ is said to be σ-convergent to the number L if all its σ-limits coincide with L, i.e., $\zeta(x) = L\ \forall \zeta \in l_\infty$.

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DEFINITION 2. A bounded sequence $x = \{x_n\} \in \Omega$ is said to be invariant convergent to $L \in \Omega$ if

$$\lim_{p \longrightarrow \infty} \| \frac{1}{p} \sum_{i=0}^{p-1} x_{\sigma^i(k)} - L \| = 0.$$

We denote by V_σ the set of all bounded sequences whose invariant means are equal. The function σ is a one-to-one map from $\mathbb{N}$ to $\mathbb{N}$ and then $\sigma^i(n) \neq n$ for all i, $n \in \mathbb{N}$, where $\sigma^i(n) = \sigma(\sigma^{i-1}(n))$, $i = 1, 2, 3, \dots$.

When $\sigma(n) = n+1$, i.e, if σ is a translation, then the σ-limit is called a Banach-limit ([1]) and σ-convergent is reduced to almost convergent ([13], [14]). It is well-known that $c \subset V_\sigma \subset l_\infty$ where c is the space of all convergent sequences, l_∞ the space of all bounded sequences in a real normed space and V_σ is the set of all almost convergent sequences.

DEFINITION 3. A sequence $x = (x_i) \in \Omega$ is said to be statistically convergent to $s \in \Omega$ if for each $\varepsilon > 0$,

$$\lim_{p \longrightarrow \infty} \frac{1}{p} |\{i \leqslant p : \|x_i - s\| \geqslant \varepsilon\}| = 0,$$

where the vertical bars $|.|$ denotes the cardinality of the enclosed set.

Savas and Nuray ([7]) introduced the notion of σ-statistical convergence for real and complex single sequences as follows:

DEFINITION 4. A sequence (x_i) is said to be invariant or σ-statistically convergent to a real or complex number s if for each $\varepsilon > 0$

$$\lim_{m \longrightarrow \infty} \frac{1}{m} \mid \{0 \leqslant i \leqslant m : \mid x_{\sigma^i(k)} - s \mid \geqslant \varepsilon\} \mid = 0,$$

uniformly in k.

It is denoted by $S_\sigma - \lim x = L$ or $x_k \rightarrow L(S_\sigma)$.

This definition was generalized by Nuray ([9]) for sequences in real normed space Ω as follows:

DEFINITION 5. A sequence $x = (x_i) \in \Omega$ is said to be invariant or σ- statistically convergent $s \in \Omega$ if for each $\varepsilon > 0$

$$\lim_{m \longrightarrow \infty} \frac{1}{m} \mid \{i \leqslant m : \| x_{\sigma^i(k)} - s \| \geqslant \varepsilon\} \mid = 0$$

uniformly in k.

Our intention is to generalize these concepts for double sequence in real normed space. First we give some definitions for pursuing our work. We present some results on quasi-invariant convergent and quasi-invariant statistical convergent for double sequence in real normed space Ω.

2. Definitions and notations

Let us now define σ-convergent and σ-statistical convergence for double sequence in a real normed space Ω as follows:

DEFINITION 6. A double sequence $x = (x_{ij})$ of real numbers is said to be σ-convergent or invariant convergent to a number s if,

$$\lim_{m,n\longrightarrow\infty} \frac{1}{mn} \sum_{i=0}^{m} \sum_{j=0}^{n} x_{\sigma^i(k)\sigma^j(l)} = s$$

uniformly in k, l $(= 1,2,3,\ldots)$.

In this case we write $\sigma_2 - \lim x = s$.

DEFINITION 7. A double sequence $x = (x_{ij})$ in real normed space Ω is said to be σ-statistical convergent to a number s if for each $\varepsilon > 0$,

$$\lim_{m,n\longrightarrow\infty} \frac{1}{mn} \mid \{(i \leqslant m, j \leqslant n : \|x_{\sigma^i(k)\sigma^j(l)} - s\| \geqslant \varepsilon\} \mid = 0$$

uniformly in k, l $(= 1,2,3,\ldots)$.

Let l^d_∞ be the set of all bounded double sequences in a real normed space Ω.

We define the function f on l^d_∞ by

$$f(x) \equiv f(x_{ij}) = \overline{\lim}_{p,q\longrightarrow\infty} \left\{ \sup_{m,n} \frac{1}{pq} \parallel \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} x_{\sigma^i(mp)\sigma^j(nq)} \parallel \right\}$$
$$\forall x = (x_{ij}) \in l^d_\infty.$$

The function f is clearly a real valued function and satisfies the following properties:

(i) $f(x) \geqslant 0$

(ii) $f(\alpha x) = |\alpha| f(x)$, $\forall\ \alpha \in \mathbb{R}$

(iii) $f(x+y) \leqslant f(x) + f(y)$, $x, y \in l^d_\infty$.

Therefore f is a symmetric convex functional on the space l^d_∞. According to the corollary of Hahn-Banach theorem there must exist non-trivial linear functional F on Ω such that

$$\mid F(x_{ij}) \mid \leqslant f(x_{ij}).$$

We now state a well known lemma ([9]) as follows:

LEMMA 1. *Let Ω be a real norm linear space and $f : \Omega \longrightarrow \mathbb{R}$ be a functional such that the following assertion holds*

(i) $f(x) \geqslant 0$

(ii) $f(\alpha x) = \mid \alpha \mid f(x)$, $\forall \alpha \in \mathbb{R}$

(iii) $f(x+y) \leqslant f(x) + f(y)$, $x, y\ \in l^d_\infty$

Then for each $x_0 \in \Omega$, there exist a linear functional F on Ω such that

$$\forall x \in \Omega, \ \mid F(x) \mid \leqslant f(x), \ F(x_0) = f(x_0).$$

Let Θ be the family of functionals satisfying the above conditions. Then for each $s \in \Omega$, we can write

$$\forall F \in \Theta, \ \ F(x_{ij} - s) = 0 \ \text{ iff } \ f(x_{ij} - s) = 0, \ \ (x_{ij}) \in l_\infty^d. \tag{1}$$

In view of the Lemma 1 we now state the following theorem which is well known in literature.

THEOREM 1. *In l_∞^d, there exist a non-trivial functional F such that for all $a, b, \in \mathbb{R}$, each $s \in \Omega$ and all $(x_{ij}), (y_{ij}) \in l_\infty^d$, the following assertions hold*
(i) $F(ax_{ij} + by_{ij}) = aF(x_{ij}) + bF(y_{ij})$
(ii) $F(x_{\sigma(i)\sigma(j)}) = F(x_{ij})$
(iii) $|F(x_{ij})| \leqslant f(x_{ij})$
(iv) $F(x_{ij} - s) = 0$ *if and only if* $f(x_{ij} - s) = 0$.

Now we give the following definitions:

DEFINITION 8. A sequence $(x_{ij}) \in l_\infty^d$ is said to be quasi invariant convergent to $s \in \Omega$ or quasi σ-summable to s if

$$\forall F \in \Theta, \ \ F(x_{ij} - s) = 0 \tag{2}$$

and in this case we write $(Q - \sigma) - \lim x_{ij} = s$.

3. Main results

In this paper we extend the results of F. Nuray ([9]) for double sequence in a real normed space Ω.

THEOREM 2. *A bounded sequence (x_{ij}) is quasi invariant convergent to $s \in \Omega$ iff*

$$\frac{1}{pq} \| \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} x_{\sigma^i(mp)\sigma^j(nq)} - s \| \longrightarrow 0 \tag{3}$$

as $p, q \longrightarrow \infty$, uniformly in m, n $(= 1, 2, 3, \ldots)$.

Proof. First suppose that (x_{ij}) is quasi-invariant convergent to s. Then $(Q - \sigma) - \lim x_{ij} = s$. Therefore by (1) and (2) we can write

$$f(x_{ij} - s) = 0$$

i.e., we have

$$\overline{\lim}_{p,q \longrightarrow \infty} \left\{ \sup_{m,n} \frac{1}{pq} \| \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} x_{\sigma^i(mp)\sigma^j(nq)} - s \| \right\} = 0.$$

Hence for any $\varepsilon > 0$, there exist integers $p_0 > 0, q_0 > 0$ such that $\forall p > p_0$, $q > q_0$ and $m, n = 1, 2, 3, \ldots$ we find that

$$\frac{1}{pq} \| \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} x_{\sigma^i(mp)\sigma^j(nq)} - s \| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\frac{1}{pq}\|\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{\sigma^i(mp)\sigma^j(nq)} - s\| \longrightarrow 0$$

as $p \longrightarrow \infty$, $q \longrightarrow \infty$, uniformly in m,n.

Therefore the condition (3) is necessary.

Conversely suppose that the condition (3) hold, i.e.,

$$\sup_{m,n} \frac{1}{pq}\|\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{\sigma^i(mp)\sigma^j(nq)} - s\| \longrightarrow 0 \text{ as } p,q \longrightarrow \infty$$

or,

$$f(x_{ij} - s) = \overline{\lim}_{p,q\longrightarrow\infty}\left\{\sup_{m,n}\frac{1}{pq}\|\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{\sigma^i(mp)\sigma^j(nq)} - s\|\right\} = 0.$$

Hence by (1), we have,

$$\forall F \in \Theta, \ \ F(x_{ij} - s) = 0.$$

So by (2),

$$(Q-\sigma) - \lim x_{ij} = s.$$

Therefore the condition (3) is sufficient also. □

THEOREM 3. *If a bounded double sequence $x = (x_{ij})$ is invariant convergent to $s \in \Omega$, then the sequence is quasi invariant convergent to s.*

Proof. Let the bounded sequence $x = (x_{ij})$ be invariant convergent to s.

Therefore for any $\varepsilon > 0$ there exist integers $p_0 > 0$, $q_0 > 0$ such that

$$\frac{1}{pq}\|\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{\sigma^i(mp)\sigma^j(nq)} - s\| < \varepsilon \text{ for } p > p_0, \ q > q_0, \ m,n = 1,2,3,\ldots.$$

Since $\varepsilon > 0$ is arbitrary, we can write

$$\frac{1}{pq}\|\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{\sigma^i(mp)\sigma^j(nq)} - s\| \longrightarrow 0 \text{ as } p,q \longrightarrow \infty$$

and uniformly in m,n.

Hence by (3), the sequence (x_{ij}) is quasi invariant convergent to s. □

4. Quasi invariant statistical convergence for double sequence

Ganguly and Dafadar [3] investigated the quasi statistical convergence of double sequence of real numbers and described some important results related to double sequences. Also F. Nuray [9] explored some interesting results on quasi invariant statistical convergence of single sequences in normed spaces. Inspiring from the above articles we first give some definitions on quasi invariant statistical convergence for double sequences as follows:

DEFINITION 9. A double sequence $x=(x_{ij})$ is said to be quasi invariant statistically convergent to $s \in \Omega$ if for each $\varepsilon > 0$,

$$\lim_{p,q\longrightarrow\infty} \frac{1}{pq} \mid \{i \leqslant p, j \leqslant q : \|x_{\sigma^i(mp)\sigma^j(nq)} - s\| \geqslant \varepsilon\} \mid = 0$$

uniformly in m,n.

DEFINITION 10. A double sequence $x=(x_{ij})$ is said to be quasi almost statistically convergent to $s \in S$ if for each $\varepsilon > 0$,

$$\lim_{p,q\longrightarrow\infty} \frac{1}{pq} \mid \{i \leqslant p, j \leqslant q : \|x_{mp+i,nq+j} - s\| \geqslant \varepsilon\} \mid = 0$$

uniformly in m,n.

Now we present a theorem

THEOREM 4. *If a double sequence* $x=(x_{ij}) \in l_\infty^d$ *be quasi almost statistically convergent to* $s \in \Omega$*, then it is quasi invariant statistically convergent to* s.

Proof. Suppose that $x=(x_{ij}) \in l_\infty^d$ is quasi almost statistically convergent to $s \in \Omega$.

Then by definition, for any $\varepsilon > 0$ there exist integers $p_0 > 0$, $q_0 > 0$ such that

$$\lim_{p,q\longrightarrow\infty} \frac{1}{pq} \mid \{i \leqslant p, j \leqslant q : \|x_{mp+i,nq+j} - s\| \geqslant \varepsilon\} \mid < \varepsilon$$

for $p > p_0$, $q > q_0$, $m,n = 1,2,3,\ldots$

$$= \lim_{p,q\longrightarrow\infty} \frac{1}{pq} \mid \{i \leqslant p, j \leqslant q : \|x_{k+i,l+j} - s\| \geqslant \varepsilon\} \mid < \varepsilon$$

for $p > p_0$, $q > q_0$, $m,n = 1,2,3,\ldots$, where $k = mp$, $l = nq$.

$$= \lim_{p,q\longrightarrow\infty} \frac{1}{pq} \mid \{i \leqslant p, j \leqslant q : \|x_{\sigma^i(k)\sigma^j(l)} - s\| \geqslant \varepsilon\} \mid < \varepsilon$$

$$= \lim_{p,q\longrightarrow\infty} \frac{1}{pq} \mid \{i \leqslant p, j \leqslant q : \|x_{\sigma^i(mp)\sigma^j(nq)} - s\| \geqslant \varepsilon\} \mid < \varepsilon$$

uniformly in m,n.

As $\varepsilon > 0$ is arbitrary, we have

$$\lim_{p,q\longrightarrow\infty} \frac{1}{pq} \mid \{i \leqslant p, j \leqslant q : \|x_{\sigma^i(mp)\sigma^j(nq)} - s\| \geqslant \varepsilon\} \mid = 0$$

uniformly in m,n.

Hence the sequence $x=(x_{ij})$ is quasi invariant statistically convergent to s. □

REFERENCES

[1] BISWA RANJAN DATTA AND ABSOS ALI SHAIKH, *Generalized almost statistical convergence*, Real Anal. Exchange, **45**, 2 (2020), 439–452.

[2] C. CAKEN, B. ALTAY, M. MURSALEEN, *The σ-convergent and σ-core of double sequences*, Applied Mathematics Letter, **19**, (2006), 1122–1128.

[3] D. K. GANGULY AND ALAUDDIN DAFADAR, *On quasi statistical convergence of double sequence*, Gen. Math. Notes, **32**, 2 (February 2006), 42–53.

[4] D. K. GANGULY, B. BISWAS AND ALAUDDIN DAFADAR, *Some properties of a function connecting to exponent of convergence for double sequences*, Journal of Classical Analysis, **10**, 2 (November 2017), 109–117.

[5] D. K. GANGULY AND A. DAFADAR, *On strong matrix summable double sequence space with respect to modulus and statistical convergence*, J. Cal. Math. Soc., **12**, 1 (2016), 9–20.

[6] E. SAVAS, *Some Sequence Space and Statistical Convergence*, Int. J. Math. and Math. Sci., **29**, 5 (2002), 303–306.

[7] E. SAVAS, F. NURRY, *On σ-statistical convergence and lacunary σ-statistical convergence*, Math. Slovaca, **43**, 3 (1993), 309–315.

[8] E. SAVAS, *Some sequence space involving invariant means*, Indian J. Pure Appl. Math., **31**, (1989), 1–8.

[9] E. SAVAS, *Quasi-invariant convergent in normed space*, Annals of the University of Cralova, Mathematics and computer science series, **41**, 1 (2014), 1–5.

[10] F. NURRY, E.SAVAS, *Invariant statistical convergence ana A-invariant statistical convergence*, Indian J. Pure Appl. Math., **10**, (1994), 267–294.

[11] H. FAST, *Sur la convergence*, Colloq. Math., **2**, (1951), 241–244.

[12] I. J. SCHOENBERG, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66**, (1959), 361–375.

[13] M. BASARIR (ELAZIG), *On the strong almost convergence of double sequence*, Periodica Mathematica Hungarica, **30**, 2 (1995), 99–105.

[14] M. MURSALEEN AND O. H. H. EDELY, *Almost convergence and a core theorem for double sequences*, J. Math. Anal. Appl., **293**, (2004), 532–540.

[15] M. MURSALEEN AND OSAMA H. H. EDELY, *On the invariant mean and statistical convergence*, Appl. Math. Letters, **22**, (2009), 1700–1704.

[16] M. MURSALEEN et.al., *Generalized statistical convergence and statistical core of double sequences*, Acta Mathematica Sinica, **26**, (November 2010), 2131–2144.

[17] N. SUBRAMANIAN AND AYHAN ESI, *On triple sequence of Bernstine operator of weighted rough I_λ-convergence*, Journal of Classical Analysis, **13**, 1 (November 2018), 45–62.

[18] R. A. RAIMI, *Invariant means and invariant matrix method of summability*, Duke Math. J., **30**, (1963), 81–94.

[19] T. SALAT, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30**, (1980), 139–150.

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