Journal of Classical Analysis
Volume 17, Number 2 (2021), 189–198
doi:10.7153/jca-2021-17-12

# SIMULTANEOUS APPROXIMATION PROPERTIES OF DE LA VALLÉE–POUSSIN MEANS IN WEIGHTED ORLICZ SPACES

SADULLA Z. JAFAROV

*Abstract.* We investigate the simultaneous approximation properties of the de la Vallée-Poussin means in weighted Orlicz spaces in terms of the modulus of smoothness. In terms of the modulus of smoothness the direct theorem of simultaneous approximation is proved. Also, in weighted Orlicz spaces the modulus of smoothness are estimated from below and above in terms of $n$-th partial Fourier sums and de la Vallée-Poussin means.

## 1. Introduction, some auxiliary results and main results

Let $M(u)$ be a continuous increasing convex function on $[0,\infty)$ such that $M(u)/u \to 0$ if $u \to 0$, and $M(u)/u \to \infty$ if $u \to \infty$. We denote by $N$ *the complementary of* $M$ *in Young's sense,* i.e. $N(u) = \max\{uv - M(v) : v \geqslant 0\}$ if $u \geqslant 0$. We will say that $M$ satisfies the $\Delta_2-$condition if $M(2u) \leqslant cM(u)$ for any $u \geqslant u_0 \geqslant 0$ with some constant $c$, independent of $u$.

Let $\mathbb{T}$ denote the interval $[-\pi,\pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$.

For a given Young function $M$, let $\widetilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which

$$\int_{\mathbb{T}} M(|f(x)|)\,dx < \infty.$$

Let $N$ be the complementary Young function of $M$. It is well-known [27, p. 69], [40, pp. 52–68] that the linear span of $\widetilde{L}_M(\mathbb{T})$ equipped with the *Orlicz norm*

$$\|f\|_{L_M(\mathbb{T})} := \sup\left\{ \int_{\mathbb{T}} |f(x)g(x)|\,dx : g \in \widetilde{L}_N(\mathbb{T}), \int_{\mathbb{T}} N(|g(x)|)\,dx \leqslant 1 \right\},$$

or with the *Luxemburg norm*

$$\|f\|^*_{L_M(\mathbb{T})} := \inf\left\{ k > 0 : \int_{\mathbb{T}} M\left(\frac{|f(x)|}{k}\right) dx \leqslant 1 \right\}$$

*Mathematics subject classification* (2010): 30E10, 41A10, 41A25, 46E30.
*Keywords and phrases*: Boyd indices, weighted Orlicz space, Muckenhoupt weight, modulus of smoothness, de la Vallée-Poussin means.


Paper JCA-17-12

becomes a Banach space. This space is denoted by $L_M(\mathbb{T})$ and is called an *Orlicz space* [27, p. 26]. The Orlicz spaces are known as the generalizations of the Lebesgue spaces $L_p(\mathbb{T}),\ 1<p<\infty$. If $M(x)=M(x,p):=x^p,\ 1<p<\infty$, then Orlicz spaces $L_M(\mathbb{T})$ coincides with the usual Lebesque spaces $L_p(\mathbb{T}),\ 1<p<\infty$. Note that the Orlicz spaces play an important role in many areas such as applied mathematics, mechanics, regularity theory, fluid dynamics and statistical physics. Therefore, the approximation of the functions by means of Fourier trigonometric series in Orlicz spaces is also important in these areas of research.

The Luxemburg norm is equivalent to the Orlicz norm and the equivalence

$$\|f\|^*_{L_M(\mathbb{T})} \leqslant \|f\|_{L_M(\mathbb{T})} \leqslant 2\,\|f\|^*_{L_M(\mathbb{T})}, \quad f\in L_M(\mathbb{T})$$

holds true [27, p. 80].

If we choose $M(u)=u^p/p,\ 1<p<\infty$ then the complementary function is $N(u)=u^q/q$ with $1/p+1/q=1$ and we have the relation

$$p^{-1/p}\,\|u\|_{L_p(\mathbb{T})} = \|u\|^*_{L_M(\mathbb{T})} \leqslant \|u\|_{L_M(\mathbb{T})} \leqslant q^{1/q}\,\|u\|_{L_p(\mathbb{T})},$$

where $\|u\|_{L_p(\mathbb{T})}=\left(\int\limits_{\mathbb{T}}|u(x)|^p\,dx\right)^{1/p}$ stands for the usual norm of the $L_p(\mathbb{T})$ space.

If $N$ is complementary to $M$ in Young's sense and $f\in L_M(\mathbb{T}),\ g\in L_N(\mathbb{T})$ then the so-called strong Hölder inequalities [27, p. 80]

$$\int\limits_{\mathbb{T}}|f(x)g(x)|\,dx \leqslant \|f\|_{L_M(\mathbb{T})}\,\|g\|^*_{L_N(\mathbb{T})},$$

$$\int\limits_{\mathbb{T}}|f(x)g(x)|\,dx \leqslant \|f\|^*_{L_M(\mathbb{T})}\,\|g\|_{L_N(\mathbb{T})}$$

are satisfied.

The Orlicz space $L_M(\mathbb{T})$ is *reflexive* if and only if the $N-$function $M$ and its complementary function $N$ both satisfy the $\Delta_2-$condition [40, p. 113].

Let $M^{-1}:[0,\infty)\to[0,\infty)$ be the inverse function of the $N-$function $M$. The *lower* and *upper indices* [8, p. 350]

$$\alpha_M := \lim_{t\to+\infty} -\frac{\log h(t)}{\log t},\ \beta_M := \lim_{t\to o^+} -\frac{\log h(t)}{\log t}$$

of the function

$$h:(0,\infty)\to(0,\infty],\quad h(t):=\lim_{y\to\infty}\sup\frac{M^{-1}(y)}{M^{-1}(ty)},\ t>0$$

first considered by Matuszewska and Orlicz [38], are called the *Boyd indices* of the Orlicz spaces $L_M(T)$.

It is known that the indices $\alpha_M$ and $\beta_M$ satisfy $0\leqslant\alpha_M\leqslant\beta_M\leqslant1$, $\alpha_N+\beta_M=1$, $\alpha_M+\beta_N=1$ and the space $L_M(\mathbb{T})$ is reflexive if and only if $0<\alpha_M\leqslant\beta_M<1$. The detailed information about the Boyd indices can be found in [7], [9]–[11], [35].

A measurable function $\omega : \mathbb{T} \to [0,\infty]$ is called a *weight function* if the set $\omega^{-1}(\{0,\infty\})$ has Lebesgue measure zero. With any given weight $\omega$ we associate the $\omega$*-weighted Orlicz space* $L_M(\mathbb{T},\omega)$ consisting of all measurable functions $f$ on $\mathbb{T}$ such that

$$\|f\|_{L_M(\mathbb{T},\omega)} := \|f\omega\|_{L_M(\mathbb{T})}.$$

Let $1 < p < \infty$, $1/p + 1/p' = 1$ and let $\omega$ be a weight function on $\mathbb{T}$. $\omega$ is said to satisfy *Muckenhoupt's* $A_p$*-condition* on $\mathbb{T}$ if

$$\sup_J \left( \frac{1}{|J|} \int_J \omega^p(t)\,dt \right)^{1/p} \left( \frac{1}{|J|} \int_J \omega^{-p'}(t)\,dt \right)^{1/p'} < \infty,$$

where $J$ is any subinterval of $\mathbb{T}$ and $|J|$ denotes its length.

Let us indicate by $A_p(\mathbb{T})$ the set of all weight functions satisfying Muckenhoupt's $A_p$-condition on $\mathbb{T}$.

According to [31], [32, Lemma 3.3], and [32, Section 2.3] and [33] if $L_M(\mathbb{T})$ is reflexive and $\omega$ weight function satisfying the condition $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$, then the space $L_M(\mathbb{T},\omega)$ is also reflexive.

Let $L_M(\mathbb{T},\omega)$ be a weighted Orlicz space, let $0 < \alpha_M \leqslant \beta_M < 1$ and let $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. For $f \in L_M(\mathbb{T},\omega)$ we set

$$(\nu_h f)(x) := \frac{1}{2h} \int_{-h}^{h} f(x+t)\,dt, \quad 0 < h < \pi, \quad x \in T.$$

By reference [15, Lemma 1], the shift operator $\nu_h$ is a bounded linear operator on $L_M(\mathbb{T},\omega)$:

$$\|\nu_h(f)\|_{L_M(\mathbb{T},\,\omega)} \leqslant c\,\|f\|_{L_M(\mathbb{T},\,\omega)}.$$

The function

$$\Omega^l_{M,\omega}(f,\delta) := \sup_{\substack{0<h_i\leqslant\delta \\ 1\leqslant i\leqslant l}} \left\| \prod_{i=1}^{l} \left(I - \nu_{h_i}\right) f \right\|_{L_M(\mathbb{T},\omega)}, \quad \delta > 0, \quad l = 1,2,\ldots$$

is called $l$*-th modulus of smoothness* of $f \in L_M(\mathbb{T},\omega)$, where I is the identity operator.

It can easily be shown that $\Omega^l_{M,\omega}(f,\cdot)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0} \Omega^l_{M,\omega}(f,\delta) = 0, \quad \Omega^l_{M,\omega}(f+g,\delta) \leqslant \Omega^l_{M,\omega}(f,\delta) + \Omega^l_{M,\omega}(g,\delta)$$

for $f,\ g \in L_M(\mathbb{T},\omega)$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x,f) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{1}$$

be the Fourier series of the function $f \in L_1(\mathbb{T})$, where

$$A_k(x,f) := (a_k(f)\cos kx + b_k(f)\sin kx), \quad k \in \mathbb{N},$$

$a_k(f)$ and $b_k(f)$ are the Fourier coefficients of the function $f \in L_1(\mathbb{T})$.

The *n-th partial Fourier sums*, and *de la Vallée-Poussin means* [49] of series (1.1) are defined, respectively, as

$$S_n(f) := S_n(x,f) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k(x,f) = \sum_{k=-n}^{n} c_k e^{ikx}, \quad n = 1,2,3,\ldots,$$

$$V_n(f) := V_n(x,f) = \frac{1}{n}\sum_{\nu=n}^{2n-1} S_\nu(x,f).$$

Note that for the *de la Vallée-Poussin means* the integral representation

$$V_n(f) := V_n(x,f) = \int_{-\pi}^{\pi} f(x-t) K_n(t)\, dt,$$

holds with kernel

$$K_n(t) := \frac{1}{\pi} \frac{\sin\left(\frac{3nt}{2}\right)\sin\left(\frac{nt}{2}\right)}{2n\sin^2\left(\frac{t}{2}\right)}.$$

The best approximation to $f \in L_M(\mathbb{T},\omega)$ in the class $\prod_n$ of trigonometric polynomials of degree not exceeding $n$ is defined by

$$E_n(f)_{M,\omega} := \inf\left\{ \|f - T_n\|_{L_M(\mathbb{T},\,\omega)} : T_n \in \Pi_n \right\}.$$

Note that the existence of $T_n^* \in \Pi_n$ such that $E_n(f)_{M,\omega} = \|f - T_n^*\|_{L_M(\mathbb{T},\omega)}$ follows, for example, from Theorem 1.1 in [14, p. 59].

Let $W_M^r(\mathbb{T},\omega)$, $(r = 1,2,\ldots)$ be the class of functions such that $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L_M(\mathbb{T},\omega)$ becomes a Banach space under the consideration of the norm

$$\| f \|_{W_M^r(\mathbb{T},\omega)} := \| f \|_{L_M(\mathbb{T},\omega)} + \| f^{(r)} \|_{L_M(\mathbb{T},\omega)}.$$

We use $c, c_1, c_2, \ldots$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the questions of interest.

In the proof of the main results we need the following results.

THEOREM 1.1. [5] *Let $L_M(T,\omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \leqslant \beta_M < 1$, and let $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$ and let $T_n^*$ be the best approximation polynomial to $f$. Then for every $f \in W_M^r(T,\omega)$, $r = 0,1,2,\ldots$ and $n \in \mathbb{N}$ the inequality*

$$\| f^{(r)} - (T_n^*)^{(r)} \|_{L_M(\mathbb{T},\omega)} \leqslant c E_n(f^{(r)})_{M,\omega}$$

*holds with a constant $c > 0$ independent of $n$.*

THEOREM 1.2. [15] *Let* $L_M(T,\omega)$ *be a weighted Orlicz space with Boyd indices* $0<\alpha_M \leqslant \beta_M<1$, *and let* $\omega \in A_{1/\alpha_M}(T)\cap A_{1/\beta_M}(T)$. *Then for every* $f \in W^r_M(T,\omega)$ $(r=0,1,2,\ldots)$ *the inequality*

$$E_n(f)_{M,\omega} \leqslant \frac{c_1}{(n+1)^r} E_n(f^{(r)})_{M,\omega}$$

*holds with a constant* $c_1>0$ *independent of* $n$.

THEOREM 1.3. [15] *Let* $L_M(T,\omega)$ *be a weighted Orlicz space with Boyd indices* $0<\alpha_M \leqslant \beta_M<1$, *and let* $\omega \in A_{1/\alpha_M}(T)\cap A_{1/\beta_M}(T)$. *Then for every* $f \in L_M(T,\omega)$ *the inequality*

$$E_n(f)_{M,\omega} \leqslant c_2 \Omega^l_{M.\omega}\Big(f,\frac{1}{n+1}\Big)$$

*holds with a constant* $c_2>0$ *independent of* $n$.

Using Theorem 1.2 and 1.3 we have the following Corollary:

COROLLARY 1.1. *Let* $L_M(T,\omega)$ *be a weighted Orlicz space with Boyd indices* $0<\alpha_M \leqslant \beta_M<1$, *and let* $\omega \in A_{1/\alpha_M}(T)\cap A_{1/\beta_M}(T)$. *Then for every* $f \in W^r_M(T,\omega)$ $(r=0,1,2,\ldots)$ *the inequality*

$$E_n(f)_{M,\omega} \leqslant \frac{c_3}{(n+1)^r}\Omega^l_{M.\omega}\Big(f^{(r)},\frac{1}{n+1}\Big)$$

*holds with a constant* $c_3>0$ *independent of* $n$.

THEOREM 1.4. [15] *Let* $L_M(T,\omega)$ *be a weighted Orlicz space with Boyd indices* $0<\alpha_M \leqslant \beta_M<1$, *and let* $\omega \in A_{1/\alpha_M}(T)\cap A_{1/\beta_M}(T)$. *Then for each trigonometric polynomial* $T_n$ *of degree* $n$ *the inequality*

$$\| (T_n)^{(r)} \|_{L_M(\mathbb{T},\omega)} \leqslant c_4 n^r \| T_n \|_{L_M(\mathbb{T},\omega)}, \quad r=1,2,3,\ldots,$$

*holds with a constant* $c_4>0$ *independent of* $n$.

Using the method of proof of [43,Theorem 2.1] and Theorem 1.3 we can prove the following Theorem:

THEOREM 1.5. *Let* $L_M(T,\omega)$ *be a weighted Orlicz space with Boyd indices* $0<\alpha_M \leqslant \beta_M<1$, *and let* $\omega \in A_{1/\alpha_M}(T)\cap A_{1/\beta_M}(T)$. *Then for every* $f \in W^r_M(T,\omega)$ $(r=0,1,2,\ldots)$ *the inequality*

$$\| f-V_n(f) \|_{L_M(\mathbb{T},\omega)} \leqslant \frac{c_5}{(n+1)^r}\Omega^l_{M.\omega}\Big(f^{(r)},\frac{1}{n+1}\Big)$$

*holds with a constant* $c_5>0$ *independent of* $n$.

The problems of approximation theory in weighted and nonweighted Orlicz spaces have been investigated by several authors (see, for example, [1]–[4], [6], [13], [15]–[17], [20]–[26], [38], [39]).

In the present paper we investigate the simultaneous approximation properties of de la Vallée-Poussin means in weighted Orlicz spaces in terms of modulus of smoothness.

Also, we estimate the modulus of smoothness from below and above in terms $n$-th partial sums and de la Vallée-Poussin means in weighted Orlicz spaces. Similar problems in different spaces have been investigated by several researchers (see, for example, [18], [19], [21], [25], [29], [30], [36], [37], [41]–[51]).

Our main results are as follows.

THEOREM 1.6. *Let $L_M(T,\omega)$ be a weighted Orlicz space with Boyd indices $0<\alpha_M\leqslant\beta_M<1$, and let $\omega\in A_{1/\alpha_M}(T)\cap A_{1/\beta_M}(T)$. Then for every $f\in W_M^r(\mathbb{T},\omega)$ and $m=0,1,2,\ldots,r$ the estimate*

$$\| f^{(m)}-V_n^{(m)}(f)\|_{L_M(\mathbb{T},\omega)}\leqslant\frac{c_6}{n^{r-m}}\Omega_{M,\omega}^l\left(f^{(r)},\frac{1}{n}\right) \tag{2}$$

*holds with a constant $c_6>0$ independent of $n$.*

THEOREM 1.7. *Let $L_M(T,\omega)$ be a weighted Orlicz space with Boyd indices $0<\alpha_M\leqslant\beta_M<1$, and let $\omega\in A_{1/\alpha_M}(T)\cap A_{1/\beta_M}(T)$. Then for every $f\in L_M(\mathbb{T},\omega)$ and $l=1,2,\ldots$ the following inequalities hold:*

1.

$$\begin{aligned} c_7\Omega_{M,\omega}^l\left(f,\frac{1}{n}\right) &\leqslant \left(n^{-2l}\left\|V_n^{(2l)}(f)\right\|_{L_M(\mathbb{T},\omega)}+\|f-V_n(f)\|_{L_M(\mathbb{T},\omega)}\right)\\ &\leqslant c_8\Omega_{M,\omega}^l\left(f,\frac{1}{n}\right), \end{aligned} \tag{3}$$

*where the constants $c_7$ and $c_8$ independent of $n$.*

2.

$$\begin{aligned} c_9\Omega_{M,\omega}^l\left(f,\frac{1}{n}\right) &\leqslant \left(n^{-2l}\left\|S_n^{(2l)}(f)\right\|_{L_M(\mathbb{T},\omega)}+\|f-S_n(f)\|_{L_M(\mathbb{T},\omega)}\right)\\ &\leqslant c_{10}\Omega_{M,\omega}^l\left(f,\frac{1}{n}\right), \end{aligned} \tag{4}$$

*where the constants $c_9$ and $c_{10}$ independent of $n$.*

## 2. Proofs of the main results

*Proof of Theorem 1.6.* $f\in W_M^r(\mathbb{T},\omega)$ and $T_n^*\in\prod_n$ $(n=0,1,2,\ldots)$ be the polynomial of best approximation to $f$. The following inequality holds:

$$\begin{aligned} &\| f^{(m)}-V_n^{(m)}(f)\|_{L_M(\mathbb{T},\omega)}\\ &\leqslant\| f^{(m)}-(T_n^*)^{(m)}\|_{L_M(\mathbb{T},\omega)}+\|(T_n^*)^{(m)}-V_n^{(m)}(f)\|_{L_M(\mathbb{T},\omega)}. \end{aligned} \tag{5}$$

Then according to Theorem 1.1 and 1.3 we get

$$\| f^{(m)} - (T_n^*)^{(m)} \|_{L_M(\mathbb{T},\omega)} \leqslant c_{11} E_n(f^{(m)})_{M,\omega.} \leqslant \frac{c_{12}}{n^{r-m}} E_n(f^{(r)})_{M,\omega.} \leqslant \frac{c_{13}}{n^{r-m}} \Omega^l_{M,\omega}\left(f^{(r)}, \frac{1}{n}\right). \tag{6}$$

On the other hand using Theorem 1.4 and 1.5 , Corollary 1.1 we obtain that

$$\begin{aligned} \| (T_n^*)^{(m)} - V_n^{(m)}(f) \|_{L_M(\mathbb{T},\omega)} &\leqslant c_{14} n^m \left\{ \|V_n(f) - f\|_{L_M(\mathbb{T},\omega)} + \|f - T_n^*\|_{L_M(\mathbb{T},\omega)} \right\} \\ &\leqslant c_{15} n^m \left\{ \frac{c_{16}}{n^r} \Omega^l_{M,\omega}\left(f^{(r)}, \frac{1}{n}\right) + E_n(f)_{M,\omega.} \right\} \\ &\leqslant \frac{c_{17}}{n^{r-m}} \Omega^l_{M,\omega}\left(f^{(r)}, \frac{1}{n}\right). \end{aligned} \tag{7}$$

Using (2.1), (2.2) and (2.3), we finally conclude that

$$\| f^{(m)} - T_n^{(m)} \|_{L_M(\mathbb{T},\omega)} \leqslant \frac{c_{18}}{n^{r-m}} \Omega^l_{M,\omega}\left(f^{(r)}, \frac{1}{n}\right).$$

Thus, the inequality (1.2) of Theorem 1.6 is proved. □

*Proof of Theorem 1.7.* Considering [15] the inequality

$$\Omega^l_{M,\omega}\left(V_n(f), \frac{1}{n}\right) \leqslant c_{19} n^{-2l} \left\| V_n^{(2l)}(f) \right\|_{L_M(\mathbb{T},\omega)} \tag{8}$$

holds. Taking into account the properties of modulus of smoothness $\Omega^l_{M,\omega}\left(f, \frac{1}{n}\right)$ and (2.4), we conclude that

$$\begin{aligned} \Omega^l_{M,\omega}\left(f, \frac{1}{n}\right) &\leqslant \left( \Omega^l_{M,\omega}\left(f - V_n(f), \frac{1}{n}\right) + \Omega^l_{M,\omega}\left(V_n(f), \frac{1}{n}\right) \right) \\ &\leqslant c_{20} \left( \|f - V_n(f)\|_{L_M(\mathbb{T},\omega)} + n^{-2l} \left\| V_n^{(2l)}(f) \right\|_{L_M(\mathbb{T},\omega)} \right). \end{aligned} \tag{9}$$

We estimate the modulus of smoothness $\Omega^l_{M,\omega}(f,\cdot)$ from below. By [15] the following inequalities hold:

$$E_n(f)_{M,\omega} \leqslant c_{21} \Omega^l_{M,\omega}\left(f, \frac{1}{n+1}\right), \tag{10}$$

$$n^{-2l} \left\| V_n^{(2l)}(f) \right\|_{L_M(\mathbb{T},\omega)} \leqslant c_{22} \Omega^l_{M,\omega}\left(f, \frac{1}{n+1}\right). \tag{11}$$

Let $V_v(f,x)$ be the de la Vallée-Poussin sums of the series (1.1) and let $T_n^* \in \Pi_n$ be the polynomial of best approximation to $f$ in $L_M(\mathbb{T},\omega)$, that is $\|f - T_n^*\|_{L_M(\mathbb{T},\omega)} = E_n(f)_{M,\omega}$. Then we can write the following inequality:

$$\begin{aligned} \|f - V_n(f)\|_{L_M(\mathbb{T},\omega)} &\leqslant \|f - T_n^*\|_{L_M(\mathbb{T},\omega)} + \|T_n^* - V_n(f)\|_{L_M(\mathbb{T},\omega)} \\ &\leqslant c_{23} E_n(f)_{M,\omega} + \|V_n(T_n^* - f, \cdot)\|_{L_M(\mathbb{T},\omega)} \\ &\leqslant c_{24} E_n(f)_{M,\omega}. \end{aligned} \tag{12}$$

Consideration of (2.6), (2.7) and (2.8) gives us

$$\begin{aligned}
&n^{-2l}\left\|V_n^{(2l)}(f)\right\|_{L_M(\mathbb{T},\omega)}+\|f-V_n(f)\|_{L_M(\mathbb{T},\omega)}\\
&\leqslant c_{25}\left(\Omega_{M,\omega}^{l}\left(V_n(f),\frac{1}{n+1}\right)+E_n(f)_{M,\omega}\right)\\
&\leqslant c_{26}\left(\Omega_{M,\omega}^{l}\left(f,\frac{1}{n+1}\right)+\Omega_{M,\omega}^{l}\left(f-V_n(f),\frac{1}{n+1}\right)+E_n(f)_{M,\omega}\right)\\
&\leqslant c_{27}\Omega_{M,\omega}^{l}\left(f,\frac{1}{n+1}\right).
\end{aligned} \tag{13}$$

From (2.5) and (2.9) we obtain estimation (1.3) of Theorem 1.7.

According to [15] there exists a constant $c_{28}>0$ such that

$$\|f-S_n(f)\|_{L_M(\mathbb{T},\omega)}\leqslant c_{28}E_n(f)_{M,\omega}. \tag{14}$$

The proof of the estimation (1.4) is obtained in analogy to proof of the estimation (1.3) using the inequality (2.10).

So, Theorem 1.7 is proved. □

*Acknowledgement.* The author would like to thank the referee for his/her valuable comments and suggestions.

REFERENCES

[1] R. AKGÜN, *Improved converse theorems and fractional moduli of smoothness in Orlicz spaces*, Bull. Malays. Math. Sci. Soc. (2) **36**, (2013), no. 1, 49–62.

[2] R. AKGÜN, *Approximating polynomials for functions of weighted Smirnov-Orlicz spaces*, J. Funct. Spaces Appl. **2012** (2012) Article ID 982360, 41 pp.

[3] R. AKGÜN, D. M. ISRAFILOV, *Polynomial approximation in weighted Smirnov-Orlicz space*, Proc. A. Razmadze Math. Inst. **139** (2005), 89–92.

[4] R. AKGÜN, D. M. ISRAFILOV, *Approximation and moduli of fractional orders in Smirnov-Orlicz classes*, Glas. Mat. Ser. III **43** (63) (2008), no. 1, 121–136.

[5] R. AKGÜN, D. M. ISRAFILOV, *Simultaneous and converse approximation theorems in weighted Orlicz spaces*, Bull. Belg. Math. Soc., **17** (2010), no. 1, 13–28.

[6] S. P. BAĬBORODOV, *Approximation of functions by de la Vallée-Poussin sums*, Matem. Zametki, **24** (1980), no. 1, 33–47, (in Russian).

[7] C. BENNETT AND YU. I. SHARPLEY, *Interpolation of operators*, Academic Press, 1988.

[8] A. BÖTTCHER AND YU. I. KARLOVICH, *Carleson curves, Muckenhoupt weights and Teoplitz operators*, Birkhäuser Verlag, 1997.

[9] D. W. BOYD, *Spaces between a pair of reflexive Lebesgue spaces*, Proc. Amer. Math. Soc. **18** (1967), 215–219.

[10] D. W. BOYD, *Indices of function spaces and their relationship to interpolation*, Canad. J. Math. **21** (1969), 1245–1254.

[11] D. W. BOYD, *Indices for the Orlicz spaces*, Pacific J. Math. **38** (1971), 315–325.

[12] V. T. GAVRILJUK, *Linear summability methods for Fourier series and best approximation*, Ukrain. Math. Zh. **15** (1963), no. 4, 412–418, (in Russian).

[13] A. GUVEN, D. M. ISRAFILOV, *Approximation by means of Fourier trigonometric series in weighted Orlicz spaces*, Adv. Stud. Contemp. Math. (Kyungshang), **19** (2009), no. 2, 283–295.

[14] R. A DEVORE AND G. G. LORENTZ, *Constructive Approximation*, Springer Verlag, 1993.

[15] D. M. ISRAFILOV, ALI GUVEN, *Approximation by trigonometric polynomials in weighted Orlicz spaces*, Studia Math. **174** (2006), no. 2, 147–168.

[16] D. M. ISRAFILOV, B. OKTAY, R. AKGÜN, *Approximation in Smirnov-Orlicz classes*, Glas. Mat. Ser. III **40** 60 (2005), 87–102.

[17] D. M. ISRAFILOV, R. AKGÜN, *Approximation in weighted Smirnov-Orlicz classes*, J. Math. Kyoto Univ. **46** (2006), no. 4, 755–770.

[18] D. M. ISRAFILOV, A. TESTICI, *Simultaneous approximation in Lebesgue spaces with variable exponent*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **44** (2018), no. 1, 3–18.

[19] D. M. ISRAFILOV, A. TESTICI, *Some inverse and simultaneous approximation theorems in weighted variable exponent Lebesgue spaces*, Anal. Math. **44** (2018), no. 4, 475–492.

[20] S. Z. JAFAROV, *Approximation by rational functions in Smirnov-Orlicz classes*, J. Math. Anal. Appl. **379** (2011), 870–877.

[21] S. Z. JAFAROV, *Approximation of functions by de la Vallée-Poussin sums in weighted Orlizc spaces*, Arab. J. Math. **5** (2016), 125–137.

[22] S. Z. JAFAROV, *On approximation in weighted Smirnov-Orlicz classes*, Complex Var. Elliptic Equ. **57** (2012), no. 5, 567–577.

[23] S. Z. JAFAROV, *The inverse theorem of approximation of the function in Smirnov-Orlicz classes*, Math. Ineq. Appl. **12** (2012), no. 4, 835–844.

[24] S. Z. JAFAROV, *Approximation of conjugate functions by trigonometric polynomials in weighted Orlicz spaces*, J. Math. Ineq. **7** (2013), no. 2, 271–281.

[25] S. Z. JAFAROV, *Approximation by Fejér sums of Fourier trigonometric series in weighted Orlicz spaces*, Hacet. J. Math. Stat. **42** (2013), no. 3, 259–268.

[26] S. Z. JAFAROV, *On moduli of smoothness of functions in Orlicx spaces*, Tbilisi Math. J. **12** (2019), no. 3, 121–129.

[27] M. A. KRASNOSEL'SKIĬ, YA. B. RUTICKIĬ, *Convex Functions and Orlicz Spaces*, Translated from the first Russian edition by Leo F. Boron, P. Noordhoff Ltd., Groningen, 1961.

[28] V. M. KOKILASHVILI, *On analytic functions of Smirnov-Orlicz classes*, Studia Math. **31** (1968), 43–59.

[29] V. M. KOKILASHVILI, S. G. SAMKO, *Operators of harmonic analysis in weighted spaces with non-standart growth*, J. Math. Anal. Appl. **352** (2009), 15–34.

[30] V. M. KOKILASHVILI, S. G. SAMKO, *A refined inverse inequality of approximation in weighted variable exponent Lebesgue spaces*, Proc. A. Razmadze Math. Inst. **151** (2009), 132–138.

[31] A. YU. KARLOVICH, *Algebras of singular integral operators with piecewise continuous coefficients on reflexive Orlicz spaces*, Math. Nachr. **179** (1996), 187–222.

[32] A. YU. KARLOVICH, *Singular integral operators with PC coefficients in reflexive rearrangement invariant spaces*, Integral Equations Operator Theory **32** (1998), 436–481.

[33] A. YU. KARLOVICH, *Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces*, J. Integral Equations Appl. **15** (2003), 263–320.

[34] W. MATUSZEWSKA, W. ORLICZ, *On certain properties of $\varphi$-functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **8** (1960), no. 7, 439–443.

[35] L. MALIGRANDA, *Indices and interpolation*, Dissertationes Math. (Rozprawy Mat.) **234** (1985), 49pp.

[36] LE. YU. OVSII, A. S. SERDYUK, *Approximation of continuous periodic functions by de la Vallée Poussin sums*, Zb. Pr. Inst. Mat. NAN Ukr. **8** (1) (2011), 151–161.

[37] J. PRESTIN, *On the approximation by de la Vallée Poussin sums and interpolatory polynomials in Lipschitz norms*, Anal. Math. **14** (1987) 251–259.

[38] V. G. PONOMARENKO, *Approximation of periodic functions in a Orlicz space*, Sibirsk. Math. J. **7** (1966), 1337–1346, (in Russian).

[39] A.-R. K. RAMAZANOV, *On approximation by polynomials and rational functions in Orlicz spaces*, Anal. Math. **10** (1984), 117–132.

[40] M. M. RAO AND Z. D. REN, *Theory of Orlicz spaces*, Marcel Dekker, New York, 1991.

[41] S. B. STECHKIN, *The approximation of periodic functions by Fejér sums*, Tr. Math. Inst. Steklova **62** (1961), 48–60, (in Russian).

[42] S. B. STECHKIN, *On the approximation of periodic functions by de la Vallée Poussin sums*, Anal. Math. **4** (1978), 61–74.

[43] I. I. SHARAPUDINOV, *On direct and inverse theorems of approximation theory in vector Lebesgue and Sobolev spaces*, Azerb. J. Math. 4 (2014), no. 1, 55–72.

[44] A. S. SERDYUK, LE. YU. OVSII, A. P. MUSIENKO, *Approximation of classes of analytic functions by de la Vallée Poussin sums in uniform metric*, Rend. Mat. Appl. **32** (2012), 1–15.

[45] B. V. SIMONOV AND S. YU. TIKHONOV, *On embeddings of function classes defined by constructive characteristics*, Approximation and probability (Bedlewo, Poland 2004), Banach Center Publ. vol. 72, Polish Acad. Sci., Warsaw 2006, pp. 285–307.

[46] B. V. SIMONOV, S. YU. TIKHONOV, *Embedding theorems in the constructive theory of approximations*, Mat. Sb. (2008), 199 (9), 107–148, (in Russian); translation in Sb. Math. **199** (2008), no. 9–10, 1367–1407.

[47] M. F. TIMAN, *Best approximation of a function and linear methods of summing Fourier series*, Izv. Akad. Nauk SSSR Ser: Math. **29** (1965), 587–604, (in Russian).

[48] M. F. TIMAN, *Some linear summation processes for Fourier series and best apptroximation*, Dokl. Akad. Nauk SSSR **145** (1962), 741–743.

[49] CH. LA. VALLÉE POUSSIN, *Sur la meilleure approximation des fonctions d'une variable reelle par des expressions d'ordre donné*, C. R. Acad. Sci., Paris **166** (1918), no. 4, 799–802.

[50] Y. E. YILDIRIR, D. M. ISRAFILOV, *Simultaneous and converse approximation theorems in weighted Lebesgue spaces*, Math. Inequal. Appl. **14** (2011), no. 2, 359–371.

[51] A. A. ZAKHAROV, *Bound on deviations of continuous periodic functions from their de la Vallée-Poussin sums*, Matem. Zametki, **3** (1968), no. 1, 77–84, (in Russian).

(Received December 28, 2020)

*Sadulla Z. Jafarov*
*Department of Mathematics and Science Education*
*Faculty of Education, Muş Alparslan University*
*49250, Muş, Turkey*
*and*
*Institute of Mathematics and Mechanics*
*National Academy of Sciences of Azerbaijan*
*9 B. Vahabzadeh str., AZ1141, Baku, Azerbaijan*
*e-mail:* s.jafarov@alparslan.edu.tr