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RELATIVE $(p,q,t)L$–TH ORDER ORIENTED SOME GROWTH PROPERTIES OF WRONSKIAN

TANMAY BISWAS AND CHINMAY BISWAS*

Abstract. In the paper we establish some new results depending on the comparative growth properties of composite transcendental entire and meromorphic functions using relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order and wronskian generated by one of the factors.

1. Introduction, definitions and notations

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [6, 9, 14, 15]. We also use the standard notations and definitions of the theory of entire functions which are available in [13] and therefore we do not explain those in details. Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to f is defined on $|z| = r$ as $M_f(r) = \max_{|z|=r} |f(z)|$. If f is non-constant then it has the following property:

PROPERTY (A). [2]: A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large values of r, $[M_f(r)]^2 \leqslant M_f(r^\sigma)$ holds.

For examples of functions with or without the Property (A), one may see [2].

When f is meromorphic, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of f, playing the same role as $M_f(r)$.

The integrated counting function $N_f(r,a)(\overline{N}_f(r,a))$ of a-points (distinct a-points) of f is defined as

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t}\, dt + n_f(0,a)\log r,$$
$$\left(\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(r,a)}{t}\, dt + \overline{n}_f(0,a)\log r\right),$$

where we denote by $n_f(t,a)\,(\overline{n}_f(t,a))$ the number of a -points (distinct a-points) of f in $|z| \leqslant t$ and an ∞ -point is a pole of f. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are

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* Corresponding author.

denoted by $N_f(r)$ and $\overline{N}_f(r)$ respectively. The function $N_f(r,a)$ is called the enumerative function. On the other hand, the function $m_f(r) \equiv m_f(r,\infty)$ known as the proximity function is defined as

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta,$$

where $\log^+ x = \max(\log x, 0)$ for all $x \geqslant 0$ and an ∞-point is a pole of f. Analogously, $m_{\frac{1}{f-a}}(r) \equiv m_f(r,a)$ is defined when a is not an ∞-point of f.

Thus the Nevanlinna's characteristic function $T_f(r)$ corresponding to f is defined as

$$T_f(r) = N_f(r) + m_f(r).$$

When f is entire, $T_f(r)$ coincides with $m_f(r)$ as $N_f(r) = 0$. However, for a meromorphic function f, the Wronskian determinant $W(f) = W(a_1, a_2, \ldots, a_k, f)$ is defined as

$$W(f) = \begin{vmatrix} a_1 & a_2 & \cdots & a_k & f \\ a_1' & a_2' & \cdots & a_k' & f' \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_1^{(k)} & a_2^{(k)} & \cdots & a_k^{(k)} & f^{(k)} \end{vmatrix},$$

where $a_1, a_2, \ldots, a_k$ are linearly independent meromorphic functions and small with respect to f (i.e., $T_{a_i}(r) = S(r,f)$ for $i = 1,2,3\ldots k$). From the Nevanlinna's second fundamental theorem, it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a;f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) \leqslant 2$ (*cf.* [6, p. 43]), where

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T_f(r)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T_f(r)}.$$

If in particular $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$, we say that f has the maximum deficiency sum. Moreover, if f is non-constant entire then $T_f(r)$ is strictly increasing and continuous function of r. Also its inverse $T_f^{-1} : (T_f(0),\infty) \to (0,\infty)$ exists and is such that $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$. Also the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of the Nevanlinna's characteristic functions of the meromorphic functions f and g. However, let us consider that $x \in [0,\infty)$ and $k \in \mathbb{N}$. We define

$$\exp^{[k]} x = \exp(\exp^{[k-1]} x), \quad \text{and} \quad \log^{[k]} x = \log(\log^{[k-1]} x).$$

We also denote $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. Further we assume that throughout the present paper l, p, q, m and n always denote positive integers and $t \in \mathbb{N} \cup \{-1,0\}$. Now considering this, we just recall that Shen *et al.* [12] defined the (m,n)-φ order and (m,n)-φ lower order of entire functions f.

DEFINITION 1. [12] Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function and $m \geqslant n$. The (m,n)-φ order $\rho^{(m,n)}(f,\varphi)$ and (m,n)-φ lower order

$\lambda^{(m,n)}(f,\varphi)$ of entire function f are defined as:

$$\rho^{(m,n)}(f,\varphi) = \limsup_{r\to\infty} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)}; \quad \lambda^{(m,n)}(f,\varphi) = \liminf_{r\to\infty} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)}.$$

If f is a meromorphic function, then

$$\rho^{(m,n)}(f,\varphi) = \limsup_{r\to\infty} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]} \varphi(r)}; \quad \lambda^{(m,n)}(f,\varphi) = \liminf_{r\to\infty} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]} \varphi(r)}.$$

Further for any non-decreasing unbounded function $\varphi : [0,+\infty) \to (0,+\infty)$, if we assume $\lim\limits_{r\to+\infty} \frac{\log^{[n]} \varphi(ar)}{\log^{[n]} \varphi(r)} = 1$ for all $a > 0$, then for any entire function f, using the inequality $T_f(r) \leqslant \log M_f(r) \leqslant 3T_f(2r)$ $\{cf.$ [6]$\}$, one can easily verify that [12]

$$\rho^{(m,n)}(f,\varphi) = \limsup_{r\to\infty} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)} = \limsup_{r\to\infty} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]} \varphi(r)},$$
$$\lambda^{(m,n)}(f,\varphi) = \liminf_{r\to\infty} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)} = \liminf_{r\to\infty} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]} \varphi(r)},$$

when $m > 1$.

If we take $m = p$, $n = 1$ and $\varphi(r) = \log^{[q-1]} r$, then the above definitions reduce to the following definitions :

DEFINITION 2. The (p,q)-th order and (p,q)-th lower order of an entire function f are defined as:

$$\rho^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}; \quad \lambda^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}.$$

If f is a meromorphic function, then

$$\rho^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}; \quad \lambda^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}.$$

Definition 2 avoids the restriction $p \geqslant q$ of the original definition of (p,q)-th order (respectively (p,q)-th lower order) of entire functions introduced by Juneja et al. [7]. However the above definition is very useful for measuring the growth of entire and meromorphic functions. If $p = l$ and $q = 1$ then we write $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire or meromorphic function f. For details about generalized order one may see [11]. Also for $p = 2$ and $q = 1$, we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire or meromorphic function f.

In this connection we just recall the following definition of index-pair where we will give a minor modification to the original definition [7]:

DEFINITION 3. An entire function f is said to have index-pair (p,q) if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover, if $0 < \rho^{(p,q)}(f) < \infty$, then

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & n < p, \\ \rho^{(p,q-n)}(f) = 0 & n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & n \in \mathbb{N} \end{cases}.$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & n < p, \\ \lambda^{(p,q-n)}(f) = 0 & n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & n \in \mathbb{N} \end{cases}.$$

Analogously one can easily verify that Definition 3 of index-pair can also be applicable to a meromorphic function f. However, the function f is said to be of regular (p,q) growth when (p,q)-th order and (p,q)-th lower order of f are the same. Functions which are not of regular (p,q) growth are said to be of irregular (p,q) growth.

For entire functions, Somasundaram and Thamizharasi [10] introduced the notions of the growth indicators L-order and L-lower order where $L \equiv L(r)$ is a positive continuous slowly increasing function, which means that $\lim_{r\to\infty} L(ar)/L(r) = 1$ for all $a > 0$. The more generalized concept of L-order and L-lower order for entire function are L^*-order and L^*-lower order. Their definitions are as follows:

DEFINITION 4. [10] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log[re^{L(r)}]}; \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log[re^{L(r)}]}.$$

When f is meromorphic one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log[re^{L(r)}]}; \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log[re^{L(r)}]}.$$

If we take $m = p$, $n = 1$ and $\varphi(r) = \log^{[q-1]} r \cdot \exp^{[t+1]} L(r)$, then Definition 1 turn into the definitions of $(p,q,t)L$-th order and $(p,q,t)L$-th lower order of an entire function f which are:

$$\rho_f^L(p,q,t) = \limsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)};$$
$$\lambda_f^L(p,q,t) = \liminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

If f is a meromorphic function, then

$$\rho_f^L(p,q,t) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)};$$

$$\lambda_f^L(p,q,t) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

In order to compare the relative growth of two entire functions having same non zero finite $(p,q,t)L$-th order, one may introduce the definitions of $(p,q,t)L$-th type (respectively $(p,q,t)L$-th lower type) of entire functions having finite positive finite $(p,q,t)L$-th order.

DEFINITION 5. [4] Let f be an entire function with non-zero finite $(p,q,t)L$-th order $\rho_f^L(p,q,t)$. The $(p,q,t)L$-th type denoted by $\sigma_f^L(p,q,t)$ and $(p,q,t)L$-th lower type denoted by $\overline{\sigma}_f^L(p,q,t)$ are respectively defined as follows:

$$\sigma_f^L(p,q,t) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_f^L(p,q,t)}},$$

and

$$\overline{\sigma}_f^L(p,q,t) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_f^L(p,q,t)}}.$$

Mainly the growth investigation of entire or meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire or meromorphic function with respect to a new entire function, the notions of relative growth indicators [2, 8] will come. Extending this notion, one may introduce the definitions of relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order of a meromorphic function f with respect to an entire function g in the following way:

DEFINITION 6. [4] Let f be a meromorphic function and g be an entire function. Then relative $(p,q,t)L$-th order denoted as $\rho_g^{(p,q,t)L}(f)$ and relative $(p,q,t)L$-th lower order denoted as $\lambda_g^{(p,q,t)L}(f)$ of f with respect to g are defined by

$$\rho_g^{(p,q,t)L}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r + \exp^{[t]} L(r)},$$

$$\lambda_g^{(p,q,t)L}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the wronskians. Actually in the paper we establish some new results depending on the comparative growth properties of composite transcendental entire and meromorphic function using relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order of meromorphic function with respect to another entire function and that of wronskian generated by one of the factors.

2. Lemmas

In this section we present two lemmas which will be needed in the sequel.

LEMMA 1. [1] *Let f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T_{f\circ g}(r) \leqslant \{1+o(1)\}\frac{T_g(r)}{\log M_g(r)}T_f(M_g(r)).$$

LEMMA 2. [5] *Let f be an entire function which satisfies the* Property (A)*, $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f(r) < T_f(\alpha r^{\delta}).$$

LEMMA 3. [3] *Assume that f is a transcendental meromorphic function with $\sum_{a\neq\infty}\sum\delta(a;f)+\delta(\infty;f)=2$ and let g be a transcendental entire function having the maximum deficiency sum with regular (m,p) growth where $m>1$. Then*

$$\lim_{r\to\infty}\frac{\log^{[p]} T^{-1}_{W(g)}(T_{W(f)}(r))}{\log^{[p]} T_g^{-1}(T_f(r))} = 1.$$

LEMMA 4. *Let f be transcendental meromorphic function which satisfies the constraint $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be a transcendental entire function having the maximum deficiency sum with regular (m,p) growth where $m>1$. Then the relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order of $W(f)$ with respect to $W(g)$ are same as those of f with respect to g i.e.,*

$$\rho_{W(g)}^{(p,q,t)L}(W(f)) = \rho_g^{(p,q,t)L}(f) \text{ and } \lambda_{W(g)}^{(p,q,t)L}(W(f)) = \lambda_g^{(p,q,t)L}(f).$$

Proof. In view of Lemma 3, we obtain that

$$\begin{aligned}
\rho_{W(g)}^{(p,q,t)L}(W(f)) &= \overline{\lim_{r\to\infty}}\frac{\log^{[p]} T^{-1}_{W(g)}(T_{W(f)}(r))}{\log^{[q]} r+\exp^{[t]} L(r)} \\
&= \overline{\lim_{r\to\infty}}\frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r+\exp^{[t]} L(r)}\cdot\lim_{r\to\infty}\frac{\log^{[p]} T^{-1}_{W(g)}(T_{W(f)}(r))}{\log^{[p]} T_g^{-1}(T_f(r))} \\
&= \rho_g^{(p,q,t)L}(f)\cdot 1 = \rho_g^{(p,q,t)L}(f).
\end{aligned}$$

In a similar manner, $\lambda_{W(g)}^{(p,q,t)L}(W(f)) = \lambda_g^{(p,q,t)L}(f)$.

This completes the proof of the assertion. □

3. Theorems

THEOREM 1. *Suppose that f is a transcendental meromorphic function for which $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, and let g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $\rho_g^L(m,n,t)<\lambda_h^{(p,q,t)L}(f)\leqslant\rho_h^{(p,q,t)L}(f)<\infty$ where $q\geqslant m$ and $a>1$. If h satisfies the* Property (A), *then*

$$\lim_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-m]}T_{W(h)}^{-1}(T_{W(f)}(r))}=0,$$

when for some $\alpha<\lambda_h^{(p,q,t)L}(f)$, there holds

$$\exp^{[t]}L(M_g(r))=o\{\exp^{[m-1]}[(\log^{[q-1]}r)\exp^{[t+1]}L(r)]^{\alpha}\}$$

as $r\to\infty$.

Proof. Let us suppose that $\gamma>2$ and $\delta\to 1+$ in Lemma 2. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 1, Lemma 2 and the inequality $T_g(r)\leqslant\log^+M_g(r)$ {cf. [6]} for all sufficiently large values of r that

$$T_h^{-1}(T_{f\circ g}(r))\leqslant T_h^{-1}[\{1+o(1)\}T_f(M_g(r))]$$
$$i.e.,\ T_h^{-1}(T_{f\circ g}(r))\leqslant\gamma[T_h^{-1}(T_f(M_g(r)))]^{\delta}$$
$$i.e.,\ \log^{[p]}T_h^{-1}(T_{f\circ g}(r))\leqslant\log^{[p]}T_h^{-1}(T_f(M_g(r)))+O(1)\tag{1}$$
$$i.e.,\ \log^{[p]}T_h^{-1}(T_{f\circ g}(r))\leqslant(\rho_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[q]}M_g(r)+\exp^{[t]}L(M_g(r))]+O(1)\tag{2}$$
$$i.e.,\ \log^{[p]}T_h^{-1}(T_{f\circ g}(r))\leqslant(\rho_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[m]}M_g(r)+\exp^{[t]}L(M_g(r))]+O(1)$$
$$i.e.,\ \log^{[p]}T_h^{-1}(T_{f\circ g}(r))\leqslant(\rho_h^{(p,q,t)L}(f)+\varepsilon)$$
$$\times[\exp^{[m-1]}[(\log^{[n-1]}r)\exp^{[t+1]}L(r)]^{(\rho_g^L(m,n,t)+\varepsilon)}+\exp^{[t]}L(M_g(r))]+O(1).\tag{3}$$

Also in view of Lemma 4, we obtain for all sufficiently large values of r that

$$\log^{[p-m]}T_{W(h)}^{-1}(T_{W(f)}(r))\geqslant\exp^{[m-1]}[(\log^{[q-1]}r)\exp^{[t+1]}L(r)]^{(\lambda_{W(h)}^{(p,q,t)L}(W(f))-\varepsilon)}.$$

$$i.e.,\ \log^{[p-m]}T_{W(h)}^{-1}(T_{W(f)}(r))\geqslant\exp^{[m-1]}[(\log^{[q-1]}r)\exp^{[t+1]}L(r)]^{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)}.\tag{4}$$

Now from (3) and (4) we get for all sufficiently large values of r that

$$\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-m]}T_{W(h)}^{-1}(T_{W(f)}(r))}$$
$$\leqslant\frac{(\rho_h^{(p,q,t)L}(f)+\varepsilon)[\exp^{[m-1]}[(\log^{[n-1]}r)\exp^{[t+1]}L(r)]^{(\rho_g^L(m,n,t)+\varepsilon)}}{\exp^{[m-1]}[(\log^{[q-1]}r)\exp^{[t+1]}L(r)]^{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)}}$$
$$+\frac{\exp^{[t]}L(M_g(r))]+O(1)}{\exp^{[m-1]}[(\log^{[q-1]}r)\exp^{[t+1]}L(r)]^{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)}}.\tag{5}$$

Since $\rho_g^L(m,n,t) < \lambda_h^{(p,q,t)L}(f)$, we can choose $\varepsilon(>0)$ in such a way that

$$\rho_g^L(m,n,t)+\varepsilon < \lambda_h^{(p,q,t)L}(f)-\varepsilon. \tag{6}$$

Now let for some $\alpha < \lambda_h^{(p,q,t)L}(f)$,
$\exp^{[t]} L(M_g(r)) = o\{\exp^{[m-1]}[(\log^{[q-1]} r)\exp^{[t+1]} L(r)]^{\alpha}\}$ as $r\to\infty$.

As $\alpha < \lambda_h^{(p,q,t)L}(f)$ we can choose $\varepsilon(>0)$ in such a way that

$$\alpha < \lambda_h^{(p,q,t)L}(f)-\varepsilon. \tag{7}$$

Since $\exp^{[t]} L(M_g(r)) = o\{\exp^{[m-1]}[(\log^{[q-1]} r)\exp^{[t+1]} L(r)]^{\alpha}\}$ as $r\to\infty$ we get on using (7) that

$$\frac{\exp^{[t]} L(M_g(r))}{\exp^{[m-1]}[(\log^{[q-1]} r)\exp^{[t+1]} L(r)]^{\alpha}} \to 0 \text{ as } r\to\infty$$

$$i.e., \ \frac{\exp^{[t]} L(M_g(r))}{\exp^{[m-1]}[(\log^{[q-1]} r)\exp^{[t+1]} L(r)]^{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)}} \to 0 \text{ as } r\to\infty. \tag{8}$$

Now in view of (5), (6) and (8) we obtain that

$$\lim_{r\to\infty}\frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-m]} T_{W(h)}^{-1}(T_{W(f)}(r))} = 0.$$

Thus the theorem follows. □

THEOREM 2. *Let g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and k be a transcendental entire function having the maximum deficiency sum with regular (m,l) growth. Also let f be a meromorphic function and h is an entire function such that $\rho_h^{(p,q,t)L}(f)<\infty$, $\lambda_k^{(l,n,t)L}(g)>0$ and $\rho_g^L(m,n,t)<\infty$ where $m>q$. If h satisfy the Property (A), then*

$$\varlimsup_{r\to\infty}\frac{\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))+\exp^{[t]} L(M_g(r))} \leqslant \frac{\rho_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)},$$

when $\exp^{[t]} L(M_g(r)) = o\{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))\}$ as $r\to\infty$.

Proof. From (2) for all sufficiently large values of r, we have

$$\log^{[p]} T_h^{-1}(T_{f\circ g}(r)) \leqslant (\rho_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[q]} M_g(r)+\exp^{[t]} L(M_g(r))+O(1)]$$

$$i.e., \log^{[p]} T_h^{-1}(T_{f\circ g}(r)) \leqslant (\rho_h^{(p,q,t)L}(f)+\varepsilon)\cdot\log^{[q]} M_g(r)$$
$$+(\rho_h^{(p,q,t)L}(f)+\varepsilon)\cdot[\exp^{[t]} L(M_g(r))+O(1)]$$

$$i.e., \log^{[p]} T_h^{-1}(T_{f\circ g}(r)) \leqslant (\rho_h^{(p,q,t)L}(f)+\varepsilon)\cdot \log^{[q]} M_g(r)\Big[\frac{(\rho_h^{(p,q,t)L}(f)+\varepsilon)\cdot \log^{[q]} M_g(r)}{(\rho_h^{(p,q,t)L}(f)+\varepsilon)\cdot \log^{[q]} M_g(r)} + \frac{(\rho_h^{(p,q,t)L}(f)+\varepsilon)\cdot [\exp^{[t]} L(M_g(r))+O(1)}{(\rho_h^{(p,q,t)L}(f)+\varepsilon)\cdot \log^{[q]} M_g(r)}\Big]$$

$$i.e., \log^{[p]} T_h^{-1}(T_{f\circ g}(r)) \leqslant (\rho_h^{(p,q,t)L}(f)+\varepsilon)\cdot \log^{[q]} M_g(r)\Big[1+\frac{\exp^{[t]} L(M_g(r))+O(1)}{\log^{[q]} M_g(r)}\Big]$$

$$i.e., \log^{[p+1]} T_h^{-1}(T_{f\circ g}(r)) \leqslant \log(\rho_h^{(p,q,t)L}(f)+\varepsilon)+\log^{[q+1]} M_g(r) + \log\Big[1+\frac{\exp^{[t]} L(M_g(r))+O(1)}{\log^{[q]} M_g(r)}\Big]$$

Taking $\log\left(1+\frac{\exp^{[t]} L(M_g(r))+O(1)}{\log^{[q]} M_g(r)}\right) \sim \frac{\exp^{[t]} L(M_g(r))+O(1)}{\log^{[q]} M_g(r)}$, we get for all sufficiently large values of r,

$$\log^{[p+1]} T_h^{-1}(T_{f\circ g}(r)) \leqslant \log^{[q+1]} M_g(r)+\log(\rho_h^{(p,q,t)L}(f)+\varepsilon) + \frac{\exp^{[t]} L(M_g(r))+O(1)}{\log^{[q]} M_g(r)}$$

$$i.e.,\ \log^{[p+1]} T_h^{-1}(T_{f\circ g}(r)) \leqslant \log^{[q+1]} M_g(r)\Big[1+\frac{\exp^{[t]} L(M_g(r))+O(1)+\log^{[q]} M_g(r)\cdot\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\log^{[q]} M_g(r)\cdot \log^{[q+1]} M_g(r)}\Big].$$

$$i.e.,\ \log^{[p+2]} T_h^{-1}(T_{f\circ g}(r)) \leqslant \log^{[q+2]} M_g(r)\log\Big[1+\frac{\exp^{[t]} L(M_g(r))+O(1)+\log^{[q]} M_g(r)\cdot\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\log^{[q]} M_g(r)\cdot \log^{[q+1]} M_g(r)}\Big].$$

Again using $\log(1+x)\sim x$ for $x=\frac{\exp^{[t]} L(M_g(r))+O(1)+\log^{[q]} M_g(r)\cdot\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{q+1}\log^{[k]} M_g(r)}$, we get from above for all sufficiently large positive numbers of r,

$$\log^{[p+2]} T_h^{-1}(T_{f\circ g}(r)) \leqslant \log^{[q+2]} M_g(r) + \frac{\exp^{[t]} L(M_g(r))+O(1)+\log^{[q]} M_g(r)\cdot\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{q+1}\log^{[k]} M_g(r)}.$$

Continuing this process, we get

$$\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r)) \leqslant \log^{[q+m-q]} M_g(r) + \frac{\exp^{[t]} L(M_g(r)) + O(1) + \log^{[q]} M_g(r) \cdot \log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{q+m-q-1} \log^{[k]} M_g(r)}.$$

$$i.e., \log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r)) \leqslant \log^{[m]} M_g(r) + \frac{\exp^{[t]} L(M_g(r)) + O(1) + \log^{[q]} M_g(r) \cdot \log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{m-1} \log^{[k]} M_g(r)}.$$

$$i.e.,\ \log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r)) \leqslant (\rho_g^L(m,n,t)+\varepsilon)[\log^{[n]} r + \exp^{[t]} L(r)] + \frac{\exp^{[t]} L(M_g(r)) + O(1) + \log^{[q]} M_g(r) \cdot \log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{m-1} \log^{[k]} M_g(r)}. \tag{9}$$

Again in view of Lemma 4, we have for all sufficiently large values of r that

$$\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) \geqslant (\lambda_{W(k)}^{(l,n,t)L}(W(g))-\varepsilon)[\log^{[n]} r + \exp^{[t]} L(r)]$$

$$i.e.,\ \log^{[n]} r + \exp^{[t]} L(r) \leqslant \frac{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))}{(\lambda_k^{(l,n,t)L}(g)-\varepsilon)}. \tag{10}$$

Hence from (9) and (10), it follows for all sufficiently large values of r that

$$\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r)) \leqslant \left(\frac{\rho_g^L(m,n,t)+\varepsilon}{\lambda_k^{(l,n,t)L}(g)-\varepsilon}\right) \cdot \log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) + \frac{\exp^{[t]} L(M_g(r)) + O(1) + \log^{[q]} M_g(r) \cdot \log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{m-1} \log^{[k]} M_g(r)}$$

$$i.e,\ \frac{\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) + \exp^{[t]} L(M_g(r))} \leqslant \left(\frac{\rho_g^L(m,n,t)+\varepsilon}{\lambda_k^{(l,n,t)L}(g)-\varepsilon}\right) \cdot \frac{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) + \exp^{[t]} L(M_g(r))} + \frac{\exp^{[t]} L(M_g(r)) + O(1) + \log^{[q]} M_g(r) \cdot \log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{[\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) + \exp^{[t]} L(M_g(r))] \cdot \prod_{k=q}^{m-1} \log^{[k]} M_g(r)}$$

$$i.e,\ \frac{\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) + \exp^{[t]} L(M_g(r))}$$

$$\leqslant \frac{\frac{\rho_g^L(m,n,t)+\varepsilon}{\lambda_k^{(l,n,t)L}(g)-\varepsilon}}{1+\frac{\exp^{[t]} L(M_g(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))}} + \frac{1+\frac{O(1)+\log^{[q]} M_g(r)\cdot\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\exp^{[t]} L(M_g(r))}}{[1+\frac{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))}{\exp^{[t]} L(M_g(r))}]\cdot\prod\limits_{k=q}^{m-1}\log^{[k]} M_g(r)}.$$

$$i.e,\ \frac{\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) + \exp^{[t]} L(M_g(r))} \leqslant \frac{\frac{\rho_g^L(m,n,t)+\varepsilon}{\lambda_k^{(l,n,t)L}(g)-\varepsilon}}{1+\frac{\exp^{[t]} L(M_g(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))}}$$

$$+\frac{\frac{1}{\prod\limits_{k=q}^{m-1}\log^{[k]} M_g(r)} + \frac{O(1)}{\exp^{[t]} L(M_g(r))\cdot\prod\limits_{k=q}^{m-1}\log^{[k]} M_g(r)} + \frac{\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\exp^{[t]} L(M_g(r))\cdot\prod\limits_{k=q+1}^{m-1}\log^{[k]} M_g(r)}}{[1+\frac{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))}{\exp^{[t]} L(M_g(r))}]}. \quad (11)$$

Since $\exp^{[t]} L(M_g(r)) = o\{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r))\}$ as $r\to\infty$ and $\varepsilon(>0)$ is arbitrary we obtain from (11) that

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(r)) + \exp^{[t]} L(M_g(r))} \leqslant \frac{\rho_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)}.$$

Thus the theorem is established. □

Now we state the following theorem without its proof as it can be carried out in the line of Theorem 2:

THEOREM 3. *Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2,$ g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $0<\lambda_h^{(p,q,t)L}(f)\leqslant\rho_h^{(p,q,t)L}(f)<\infty$ and $\rho_g^L(m,n,t)<\infty$ where $m>n=q$. If h satisfy the Property (A), then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p+m-q]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(r)) + \exp^{[t]} L(M_g(r))} \leqslant \frac{\rho_g^L(m,n,t)}{\lambda_h^{(p,q,t)L}(f)},$$

when $\exp^{[t]} L(M_g(r)) = o\{\log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(r))\}$ as $r\to\infty$.

THEOREM 4. *Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2,$ g be an entire function and h be a transcendental entire function having* *the maximum deficiency sum with regular* (a,p) *growth such that* $\rho_h^{(p,q,t)L}(f) < \infty$ *and* $\lambda_h^{(p,q,t)L}(f \circ g) = \infty$ *where* $a > 1$. *Then*

$$\lim_{r\to\infty} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(r))} = \infty.$$

Proof. If possible, let there exists a constant β such that for a sequence of values of r tending to infinity we have

$$\log^{[p]} T_h^{-1}(T_{f\circ g}(r)) \leqslant \beta \cdot \log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(r)). \tag{12}$$

Again from the definition of $\rho_{W(h)}^{(p,q,t)L}(W(f))$, it follows in view of Lemma 4 for all sufficiently large values of r that

$$\log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(r)) \leqslant (\rho_{W(h)}^{(p,q,t)L}(W(f)) + \varepsilon)[\log^{[q]} r + \exp^{[t]} L(r)]$$

$$i.e., \ \log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(r)) \leqslant (\rho_h^{(p,q,t)L}(f) + \varepsilon)[\log^{[q]} r + \exp^{[t]} L(r)]. \tag{13}$$

Now combining (12) and (13) we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} &\log^{[p]} T_h^{-1}(T_{f\circ g}(r)) \leqslant \beta \cdot (\rho_h^{(p,q,t)L}(f) + \varepsilon)[\log^{[q]} r + \exp^{[t]} L(r)] \\ &i.e., \ \lambda_h^{(p,q,t)L}(f \circ g) \leqslant \beta \cdot (\rho_h^{(p,q,t)L}(f) + \varepsilon), \end{aligned}$$

which contradicts the condition $\lambda_h^{(p,q,t)L}(f \circ g) = \infty$. So for all sufficiently large values of r we get that

$$\log^{[p]} T_h^{-1}(T_f(r)) \geqslant \beta \cdot \log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(r)),$$

from which the theorem follows. □

In the line of Theorem 4, one can easily prove the following theorem and therefore its proof is omitted.

THEOREM 5. *Let* g *be a transcendental entire function with* $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$ *and* k *be a transcendental entire function having the maximum deficiency sum with regular* (l,m) *growth where* $l > 1$. *Also let* f *be a meromorphic function and* h *is an entire function such that* $\rho_k^{(m,q,t)L}(g) < \infty$ *and* $\lambda_h^{(p,q,t)L}(f \circ g) = \infty$. *Then*

$$\lim_{r\to\infty} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[m]} T_{W(k)}^{-1}(T_{W(g)}(r))} = \infty.$$

THEOREM 6. *Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $0<\lambda_h^{(p,q,t)L}(f)\leqslant\rho_h^{(p,q,t)L}(f)<\infty$ and $\sigma_g^L(m,n,t)<\infty$ where $m-1\leqslant q$, $a>1$. If h satisfy the Property (A), then*

$$\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]}T_{W(h)}^{-1}(T_{W(f)}(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}))}\leqslant\frac{\rho_h^{(p,q,t)L}(f)\cdot\sigma_g^L(m,n,t)}{\lambda_h^{(p,q,t)L}(f)},$$

when for some positive $\alpha<\rho_g^L(m,n,t)$,
$\exp^{[t]}L(M_g(r))=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ *as* $r\to\infty$.

Proof. Since $0<\rho_h^{(p,q,t)L}(f)<\infty$ and $T_h^{-1}(r)$ is an increasing function of r, it follows from (1) for all sufficiently large values of r that

$$\log^{[p]}T_h^{-1}(T_{f\circ g}(r))\leqslant(\rho_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[q]}M_g(r)+\exp^{[t]}L(M_g(r))]+O(1)$$

$$i.e.,\ \log^{[p]}T_h^{-1}(T_{f\circ g}(r))\leqslant(\rho_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[m-1]}M_g(r)+\exp^{[t]}L(M_g(r))]+O(1)$$

$$i.e.,\ \log^{[p]}T_h^{-1}(T_{f\circ g}(r))\leqslant(\rho_h^{(p,q,t)L}(f)+\varepsilon)\cdot$$
$$[(\sigma_g^L(m,n,t)+\varepsilon)[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}+\exp^{[t]}L(M_g(r))]+O(1)\,. \quad (14)$$

Also, we obtain in view of Lemma 4 for all sufficiently large values of r that

$$\begin{aligned}&\log^{[p]}T_{W(h)}^{-1}T_{W(f)}(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)})\\ \geqslant\ &(\lambda_{W(h)}^{(p,q,t)L}(W(f))-\varepsilon)[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}\\ &+(\lambda_{W(h)}^{(p,q,t)L}(W(f))-\varepsilon)\exp^{[t]}[L(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)})]\end{aligned}$$

$$\begin{aligned}i.e.,\quad&\log^{[p]}T_{W(h)}^{-1}(T_{W(f)}(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}))\\ \geqslant\ &(\lambda_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}\\ &+(\lambda_h^{(p,q,t)L}(f)-\varepsilon)\exp^{[t]}[L(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)})]\end{aligned}$$

$$\begin{aligned}&\log^{[p]}T_{W(h)}^{-1}(T_{W(f)}(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}))\\ \geqslant\ &(\lambda_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}.\end{aligned}$$

Now from (14) and above it follows for all sufficiently large values of r that

$$\frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}))}$$
$$\leqslant \frac{(\rho_h^{(p,q,t)L}(f)+\varepsilon)[(\sigma_g^L(m,n,t)+\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}}{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}}$$
$$+\frac{\exp^{[t]} L(M_g(r))]}{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}}$$
$$+\frac{O(1)}{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}} \tag{15}$$

As $\alpha < \rho_g^L(m,n,t)$ and $\exp^{[t]} L(M_g(r)) = o([\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\alpha})$ as $r\to\infty$, we obtain that

$$\lim_{r\to\infty}\frac{\exp^{[t]} L(M_g(r))}{[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}} = 0\,. \tag{16}$$

Since $\varepsilon(>0)$ is arbitrary, it follows from (15) and (16) that

$$\overline{\lim_{r\to\infty}}\frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]} T_{W(h)}^{-1}(T_{W(f)}(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}))}$$
$$\leqslant \frac{\rho_h^{(p,q,t)L}(f)\cdot\sigma_g^L(m,n,t)}{\lambda_h^{(p,q,t)L}(f)}. \quad \square$$

In the line of Theorem 6, one can easily prove the following theorem and therefore its proof is omitted.

THEOREM 7. *Let g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and k be a transcendental entire function having the maximum deficiency sum with regular (m,l) growth where $m>1$. Also let f be a meromorphic function and h is an entire function such that $\lambda_k^{(l,n,t)L}(g)>0$, $\rho_h^{(p,q,t)L}(f)<\infty$ and $\sigma_g^L(m,n,t)<\infty$ where $m-1\leqslant q$. If h satisfy the Property (A), then*

$$\overline{\lim_{r\to\infty}}\frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[l]} T_{W(k)}^{-1}(T_{W(g)}(\exp^{[n]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}))}$$
$$\leqslant \frac{\rho_h^{(p,q,t)L}(f)\cdot\sigma_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)},$$

when, for some positive $\alpha<\rho_g^L(m,n,t)$, $\exp^{[t]} L(M_g(r)) = o([\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\alpha})$ as $r\to\infty$.

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Tanmay Biswas
Rajbari, Rabindrapally
R. N. Tagore Road, P.O. Krishnagar, Dist. Nadia
PIN-741101, West Bengal, India
e-mail: tanmaybiswas_math@rediffmail.com

Chinmay Biswas
Department of Mathematics
Nabadwip Vidyasagar College
Nabadwip, Dist. Nadia, PIN-741302, West Bengal, India
e-mail: chinmay.shib@gmail.com