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ON WEIGHTED β–ABSOLUTE CONVERGENCE OF DOUBLE FOURIER SERIES

K. N. DARJI AND R. G. VYAS

Abstract. In this paper, we obtain a sufficient condition for the weighted β-absolute convergence ($0<\beta<2$) of the double Fourier series of a function f of (ϕ,ψ)-(Λ^1,Λ^2)-bounded variation.

1. Introduction

One of the most striking trends in analysis is the study of the Fourier coefficients properties of functions of various generalized bounded variations. Extending a classical result of Zygmund, Schramm and Waterman [6] obtained sufficient condition for the absolute convergence of single Fourier series of functions of the classes $\Lambda BV^{(p)}(\overline{\mathbb{T}})$ $(p \geqslant 1)$ and $\phi\Lambda BV(\overline{\mathbb{T}})$, where $\overline{\mathbb{T}}=[0,2\pi]$ is the torus. In 2007 [8], these results of Schramm and Waterman were extended to β-absolute convergence of single Fourier series. Gogoladze and Meskhia [3] obtained sufficient conditions for the weighted β-absolute convergence of single Fourier series for different function spaces. In 2008, Móricz and Veres [4] extended these results of Gogoladze and Meskhia from single to multiple Fourier series. In 2013 [9], we have obtained sufficient conditions for the β-absolute convergence of multiple Fourier series which includes a multidimensional analogue of one dimensional result proved by Schramm and Waterman [6, Theorem 1, p. 274]. In this paper, we obtain a sufficient condition for the weighted β-absolute convergence of the double Fourier series of a function f of (ϕ,ψ)-(Λ^1,Λ^2)-bounded variation. Our sufficient condition gives generalized two dimensional analogue of one dimensional result proved in [6, Theorem 2, p. 274] by Schramm and Waterman, [8, Theorem 2, with $n_k=k$, for all k, p. 770] and [12, Theorem 1, with $n_k=k$, for all k].

In the sequel, $\mathbb{L}$ is a class of non-decreasing sequences $\Lambda=\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\lim_{n\to\infty}\Lambda_n=\infty$, where $\Lambda_n=\sum_{k=1}^{n}\lambda_k^{-1}$, and C represents a constant vary time to time.

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2. Results for functions of one variable

For a 2π-periodic complex valued function $f \in L^1(\overline{\mathbb{T}})$, its Fourier series is defined as

$$f(x) \sim \sum_{m\in\mathbb{Z}} \hat{f}(m)\, e^{imx},$$

where the Fourier coefficients $\hat{f}(m)$ are defined by

$$\hat{f}(m) = \frac{1}{2\pi}\int_{\overline{\mathbb{T}}} f(x)\, e^{-imx}\, dx.$$

A Fourier series of f is said to be β-absolute convergent if

$$\sum_{m\in\mathbb{Z}} |\hat{f}(m)|^{\beta} < \infty.$$

For $\beta = 1$, one gets the absolute convergence of the Fourier series of f.

For a given $f \in L^p(\overline{\mathbb{T}})$ $(p \geqslant 1)$, the p-integral modulus of continuity of f is defined as

$$\omega^{(p)}(f;\delta) = \sup_{0<h\leqslant\delta} \|T_h f - f\|_p,$$

where $T_h f(x) = f(x+h)$ for all x and $\|\cdot\|_p$ denotes the L^p-norm over $\overline{\mathbb{T}}$.

For $p = \infty$, we omit writing p, one gets $\omega(f;\delta)$, the modulus of continuity of f.

Given a convex function ϕ, defined on $[0,\infty)$ and strictly increasing from 0 to ∞, and a sequence $\Lambda \in \mathbb{L}$, a complex valued function f defined on $\overline{\mathbb{T}}$ is said to be of ϕ-Λ-bounded variation (that is, $f \in \phi\Lambda BV(\overline{\mathbb{T}})$) if

$$V_{\Lambda_\phi}(f,\overline{\mathbb{T}}) = \sup_{\mathscr{I}} \left(\sum_k \frac{\phi(|f(I_k)|)}{\lambda_k} \right) < \infty,$$

where $\mathscr{I}$ is a finite collections of non-overlapping subintervals $\{I_k = [a_k,b_k]\}$ in $\overline{\mathbb{T}}$ and $f(I_k) = f(b_k) - f(a_k)$.

Note that, for $\phi(x) = x$ and $\Lambda = \{1\}$ one gets the class $BV(\overline{\mathbb{T}})$; for $\phi(x) = x$ one gets the class $\Lambda BV(\overline{\mathbb{T}})$; and for $\phi(x) = x^p$ $(p \geqslant 1)$ one gets the class $\Lambda BV^{(p)}(\overline{\mathbb{T}})$.

It is customary to consider ϕ an N-function which is defined as follows.

A convex function ϕ defined on $[0,\infty)$ such that $\phi(0) = 0$, $\frac{\phi(x)}{x} \to 0$ as $x \to 0_+$, and $\frac{\phi(x)}{x} \to \infty$ as $x \to \infty$, is called an N-function.

We note that an N-function is necessarily continuous and strictly increasing on $[0,\infty)$.

An N-function ϕ is said to be a Δ_2 function if there is a constant $d \geqslant 2$ such that $\phi(2x) \leqslant d\phi(x)$ for all $x \geqslant 0$.

Following the definition in [3], a sequence $\gamma = \{\gamma_m :\ m \in \mathbb{N}_+\}$ of nonnegative numbers is said to belongs to the class $\mathscr{A}_\alpha$ for some $\alpha \geqslant 1$ if

$$\left(\sum_{m\in\mathscr{D}_\mu} \gamma_m^{\alpha} \right)^{1/\alpha} \leqslant \kappa 2^{\mu(1-\alpha)/\alpha} \sum_{m\in\mathscr{D}_{\mu-1}} \gamma_m, \quad \mu \in \mathbb{N}_+, \tag{2.1}$$

where

$$\mathscr{D}_{-1} := \mathscr{D}_0 = \{1\}, \quad \mathscr{D}_\mu := \{2^{\mu-1}+1, 2^{\mu-1}+2, \dots, 2^\mu\}, \quad \mu \in \mathbb{N}_+, \tag{2.2}$$

and the constant κ does not depend on μ. Without loss of generality, we assume that $\kappa \geqslant 1$.

Note that,

$$\mathscr{A}_{\alpha_2} \subset \mathscr{A}_{\alpha_1}, \quad \text{where} \quad 1 \leqslant \alpha_1 < \alpha_2 < \infty. \tag{2.3}$$

If a sequences $\gamma = \{\gamma_m \geqslant 0\}$ is such that

$$\max\{\gamma_m : m \in \mathscr{D}_\mu\} \leqslant \kappa \min\{\gamma_m : m \in \mathscr{D}_{\mu-1}\}, \qquad \mu \in \mathbb{N}_+,$$

then $\gamma \in \mathscr{A}_\alpha$ for every $\alpha \geqslant 1$. This inequality was introduced by Ul'yanov [7].

For convenience in writing, put

$$\gamma_{-m} := \gamma_m, \qquad m \in \mathbb{N}_+.$$

We prove the following results.

THEOREM 2.1. *If* $\phi \in \Delta_2$, $f \in \phi\Lambda BV(\overline{\mathbb{T}})$, $1 \leqslant p < 2r$, $1 \leqslant r < \infty$, *and* $\gamma = \{\gamma_m\} \in \mathscr{A}_{2/(2-\beta)}$ *for some* $\beta \in (0,2)$, *then*

$$\sum_{|m|\geqslant 1} \gamma_m \, |\hat{f}(m)|^\beta \leqslant \kappa \, C \sum_{\mu=0}^{\infty} 2^{-\mu\beta/2} \, \Gamma_{\mu-1} \left(\phi^{-1} \left(\frac{(\omega^{((2-p)s+p)}(f; \frac{\pi}{2^\mu}))^{2r-p}}{\Lambda_{2^\mu}} \right) \right)^{\beta/2r},$$

where $\frac{1}{r} + \frac{1}{s} = 1$, κ *is from* (2.1) *corresponding to* $\alpha = 2/(2-\beta)$, C *is a constant,*

$$\Gamma_\mu := \sum_{m \in \mathscr{D}_\mu} \gamma_m \text{ for } \mu \in \mathbb{N}, \text{ and } \Gamma_{-1} := \Gamma_0 := \{\gamma_1\}.$$

Proof. Since $f \in \phi\Lambda BV(\overline{\mathbb{T}})$ is bounded, we have $f \in L^2(\overline{\mathbb{T}})$. For any $\mu \in \mathbb{N}$, consider

$$f_j\left(x; \frac{\pi}{2^\mu}\right) = f\left(x + \frac{j\pi}{2^\mu}\right) - f\left(x + \frac{(j-1)\pi}{2^\mu}\right).$$

Then, for each $m \in \mathbb{Z}$, we have

$$\widehat{f_j}(m) = 2i\hat{f}(m) \, e^{im(j-\frac{1}{2})\frac{\pi}{2^\mu}} \, \sin\left(\frac{m\pi}{2^{\mu+1}}\right).$$

By Parseval formula, we get

$$\sum_{m \in \mathbb{Z}} \left| \hat{f}(m) \, \sin\left(\frac{m\pi}{2^{\mu+1}}\right) \right|^2 = O(\|f_j\|_2^2).$$

Since

$$\frac{\pi}{4} < \frac{|m|\pi}{2^{\mu+1}} \leqslant \frac{\pi}{2}, \quad |m| \in \mathscr{D}_\mu, \tag{2.4}$$

we have

$$S_\mu := \sum_{|m| \in \mathscr{D}_\mu} |\hat{f}(m)|^2 = O(\|f_j\|_2^2), \tag{2.5}$$

for all $j = 1, \ldots, 2^\mu$.

Suppose $r > 1$. Since

$$2 = \left(2 - \frac{p}{r}\right) + \frac{p}{r} = \frac{2s - p(s-1)}{s} + \frac{p}{r} = \frac{(2-p)s + p}{s} + \frac{p}{r},$$

applying Hölder's inequality on the right side of the inequality (2.5), we have

$$\begin{aligned} S_\mu &= O\left(\left(\int_{\mathbb{T}} \left| f_j\left(x; \frac{\pi}{2^\mu}\right)\right|^{(2-p)s+p} dx\right)^{1/s} \|f_j\|_p^{p/r}\right) \\ &= O\left(\Omega_{\frac{\pi}{2^\mu}}^{1/r} \|f_j\|_p^{p/r}\right), \end{aligned}$$

where $\Omega_{\frac{\pi}{2^\mu}} = \left(\omega^{(2-p)s+p}\left(f; \frac{\pi}{2^\mu}\right)\right)^{2r-p}$. Thus,

$$S_\mu^r = O\left(\Omega_{\frac{\pi}{2^\mu}} \int_{\mathbb{T}} \left| f_j\left(x; \frac{\pi}{2^\mu}\right)\right|^p dx\right). \tag{2.6}$$

Since multiplying f by a positive constant changes $\omega^{(p)}\left(f; \frac{\pi}{2^\mu}\right)$ by the same constant, f is bounded, and $\phi \in \Delta_2$, we may assume that $\|f\|_\infty \leqslant \frac{1}{2}$. Thus $\|f_j\|_\infty \leqslant 1$ and hence from equation (2.6), we get

$$S_\mu^r \leqslant C\Omega_{\frac{\pi}{2^\mu}} \int_{\mathbb{T}} \left| f_j\left(x; \frac{\pi}{2^\mu}\right)\right| dx, \tag{2.7}$$

where constant C depends on f, r, s and p.

Since ϕ is convex on $[0, \infty)$ and $\phi(0) = 0$, for any $0 < \alpha < 1$ and $x > 0$ we have

$$\phi(\alpha x) = \phi(\alpha \cdot x + (1-\alpha) \cdot 0) \leqslant \alpha\phi(x) + (1-\alpha)\phi(0) = \alpha\phi(x). \tag{2.8}$$

Further, as $\phi(2x) \leqslant d\phi(x)$, for all $x \geqslant 0$, it follows that

$$\phi(ax) \leqslant d^{\log_2 a + 1}\phi(x) \tag{2.9}$$

for all $x \geqslant 0$ and for all $a \geqslant 1$. For, using induction on k we get

$$\phi(2^k x) \leqslant d^k \phi(x),$$

for all $x \geqslant 0$ and for all $k \in \mathbb{N}$. Next, if $a \geqslant 1$ is any real number, choosing $k \in \mathbb{N}$ such that $2^{k-1} \leqslant a < 2^k$, we get $0 < \frac{a}{2^k} < 1$. Therefore for all $x \geqslant 0$ we have

$$\phi(ax) = \phi\left(\frac{a}{2^k} \cdot 2^k x\right) \leqslant \frac{a}{2^k}\phi(2^k x) \leqslant \frac{a}{2^k} d^k \phi(x) < d^k \phi(x) \leqslant d^{\log_2 a + 1}\phi(x).$$

Since $C\Omega_{\frac{\pi}{2^\mu}} \geqslant 0$, if $C\Omega_{\frac{\pi}{2^\mu}} < 1$ then from equation (2.7) and equation (2.8) we get

$$\begin{aligned}\phi\left(\frac{S_\mu^r}{2\pi}\right) &\leqslant \phi\left(\frac{C\Omega_{\frac{\pi}{2^\mu}}}{2\pi}\int_{\mathbb{T}}\left|f_j\left(x;\frac{\pi}{2^\mu}\right)\right|dx\right)\\ &\leqslant C\Omega_{\frac{\pi}{2^\mu}}\phi\left(\frac{1}{2\pi}\int_{\mathbb{T}}\left|f_j\left(x;\frac{\pi}{2^\mu}\right)\right|dx\right). \qquad (2.10)\end{aligned}$$

Further when $C\Omega_{\frac{\pi}{2^\mu}} \geqslant 1$, in view of equation (2.7) and equation (2.9) we get

$$\begin{aligned}\phi\left(\frac{S_\mu^r}{2\pi}\right) &\leqslant \phi\left(\frac{C\Omega_{\frac{\pi}{2^\mu}}}{2\pi}\int_{\mathbb{T}}\left|f_j\left(x;\frac{\pi}{2^\mu}\right)\right|dx\right)\\ &\leqslant d^{\log_2\left(C'\Omega_{\frac{\pi}{2^\mu}}\right)+1}\phi\left(\frac{1}{2\pi}\int_{\mathbb{T}}\left|f_j\left(x;\frac{\pi}{2^\mu}\right)\right|dx\right), \qquad (2.11)\end{aligned}$$

where $C' = \frac{C}{2\pi}$. Now, in view of the formula $x^\alpha = 2^{\alpha\log_2 x}$ for $x > 0$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}d^{\log_2\left(C'\Omega_{\frac{\pi}{2^\mu}}\right)+1} &= d\cdot d^{\log_2 C'}\cdot d^{\log_2\Omega_{\frac{\pi}{2^\mu}}} = d\cdot d^{\log_2 C'}\cdot 2^{\left(\log_2\Omega_{\frac{\pi}{2^\mu}}\right)(\log_2 d)}\\ &= d\cdot d^{\log_2 C'}\cdot\left(2^{\log_2\Omega_{\frac{\pi}{2^\mu}}}\right)^{\log_2 d} = C''\Omega_{\frac{\pi}{2^\mu}}^{\log_2 d}, \qquad (2.12)\end{aligned}$$

where $C'' = d\cdot d^{\log_2 C'}$. Now, using equation (2.12) in equation (2.11) and denoting the constant C'' by C itself, we get

$$\begin{aligned}\phi\left(\frac{S_\mu^r}{2\pi}\right) &\leqslant C\Omega_{\frac{\pi}{2^\mu}}^{\log_2 d}\phi\left(\frac{1}{2\pi}\int_{\mathbb{T}}\left|f_j\left(x;\frac{\pi}{2^\mu}\right)\right|dx\right)\\ &= C\Omega_{\frac{\pi}{2^\mu}}^{\log_2 d-1}\Omega_{\frac{\pi}{2^\mu}}\phi\left(\frac{1}{2\pi}\int_{\mathbb{T}}\left|f_j\left(x;\frac{\pi}{2^\mu}\right)\right|dx\right)\\ &\leqslant C\Omega_{\frac{\pi}{2^\mu}}\phi\left(\frac{1}{2\pi}\int_{\mathbb{T}}\left|f_j\left(x;\frac{\pi}{2^\mu}\right)\right|dx\right) \qquad (2.13)\end{aligned}$$

because of the fact that $\Omega_{\frac{\pi}{2^\mu}}^{\log_2 d-1} \leqslant 1$, as $\|f\|_\infty \leqslant \frac{1}{2}$ and $\log_2 d - 1 \geqslant 0$. In either case, from (2.10) and (2.13), in view of Jensen's inequality for integral, we have

$$\phi\left(\frac{S_\mu^r}{2\pi}\right) \leqslant C\Omega_{\frac{\pi}{2^\mu}}\int_{\mathbb{T}}\phi\left(\left|f_j\left(x;\frac{\pi}{2^\mu}\right)\right|\right)dx.$$

Multiplying both the sides of the above inequality by $\frac{1}{\lambda_j}$ and then summing over $j=1$ to 2^μ, we have

$$\phi\left(\frac{S_\mu^r}{2\pi}\right) = O\left(\frac{\Omega_{\frac{\pi}{2^\mu}}}{\Lambda_{2^\mu}}\int_{\mathbb{T}}\sum_{j=1}^{2^\mu}\frac{\phi\left(\left|f_j\left(x;\frac{\pi}{2^\mu}\right)\right|\right)}{\lambda_j}dx\right),$$

where $\Lambda_{2^\mu} = \sum_{j=1}^{2^\mu} \frac{1}{\lambda_j}$. Since $f \in \phi\Lambda BV(\overline{\mathbb{T}})$ implies

$$\sum_{j=1}^{2^\mu} \frac{\phi\left(\left|f_j\left(x;\frac{\pi}{2^\mu}\right)\right|\right)}{\lambda_j} = O(1)$$

and hence

$$S_\mu = O\left(\left(\phi^{-1}\left(\frac{\Omega_{\frac{\pi}{2^\mu}}}{\Lambda_{2^\mu}}\right)\right)^{1/r}\right).$$

Since $1 = \frac{\beta}{2} + \frac{2-\beta}{2}$, by Hölder's inequality, we have

$$\begin{aligned} R_\mu &:= \sum_{|m|\in\mathscr{D}_\mu} \gamma_m |\hat{f}(m)|^\beta \\ &\leqslant \left(\sum_{|m|\in\mathscr{D}_\mu} |\hat{f}(m)|^2\right)^{\beta/2} \left(\sum_{|m|\in\mathscr{D}_\mu} \gamma_m^{2/(2-\beta)}\right)^{(2-\beta)/2} \\ &\leqslant C\left(\phi^{-1}\left(\frac{\Omega_{\frac{\pi}{2^\mu}}}{\Lambda_{2^\mu}}\right)\right)^{\beta/2r} \left(\sum_{|m|\in\mathscr{D}_\mu} \gamma_m^{2/(2-\beta)}\right)^{(2-\beta)/2}. \end{aligned} \tag{2.14}$$

Thus, for $\mu \geqslant 1$, in view of (2.1), with $\alpha = 2/(2-\beta)$, and (2.14), we get

$$R_\mu \leqslant C\kappa 2^{-\mu\beta/2}\,\Gamma_{\mu-1}\left(\phi^{-1}\left(\frac{\Omega_{\frac{\pi}{2^\mu}}}{\Lambda_{2^\mu}}\right)\right)^{\beta/2r}.$$

If $\mu = 0$, then from equation (2.14) it follows that

$$R_0 = \gamma_1(|\hat{f}(-1)|^\beta + |\hat{f}(1)|^\beta) = O\left(\gamma_1\left(\phi^{-1}\left(\frac{\Omega_\pi}{\frac{1}{\lambda_1}}\right)\right)^{\beta/2r}\right).$$

Hence, for $r > 1$, the result follows from

$$\sum_{|m|\geqslant 1} \gamma_m |\hat{f}(m)|^\beta = \sum_{\mu=0}^{\infty} R_\mu.$$

For the case $r = 1$, $s = \infty$, simply note that

$$\left|f_j\left(x;\frac{\pi}{2^\mu}\right)\right|^2 \leqslant \left(\omega\left(f;\frac{\pi}{2^\mu}\right)\right)^{2-p}\left|f_j\left(x;\frac{\pi}{2^\mu}\right)\right|^p,\ x \in \overline{\mathbb{T}},$$

and proceed as above from equation (2.6) onwards. □

COROLLARY 2.2. *Under the hypothesis of Theorem 2.1, we have*

$$\sum_{|m|\geqslant 1} \gamma_m\,|\hat{f}(m)|^\beta \leqslant \kappa\,C\sum_{m=1}^{\infty} m^{-\beta/2}\,\gamma_m\,\left(\phi^{-1}\left(\frac{(\omega^{((2-p)s+p)}(f;\frac{\pi}{m}))^{2r-p}}{\Lambda_m}\right)\right)^{\beta/2r}.$$

In the case when $\{\gamma_m\} = \{1\}$, it follows from Corollary 2.2 that

$$\sum_{|m|\geqslant 1} |\hat{f}(m)|^\beta \leqslant \kappa\, C \sum_{m=1}^{\infty} m^{-\beta/2} \left(\phi^{-1}\left(\frac{(\omega^{((2-p)s+p)}(f;\frac{\pi}{m}))^{2r-p}}{\Lambda_m}\right)\right)^{\beta/2r}.$$

This was proved in [8, Theorem 2, with $n_k = k$, for all k, p. 770].

Similarly, Corollary 2.2 reduces to the result concerning the absolute convergence of Fourier series of Schramm and Waterman [6, Theorem 2, p. 274] in the case when $\{\gamma_m\} = \{1\}$ and $\beta = 1$; and also reduces to the result proved in [12, Theorem 1, with $n_k = k$, for all k] in the case when $\{\gamma_m\} = \{1\}$, $\phi(x) = x$, $r = 1$ and $p = 1$.

3. Results for functions of two variables

For a complex valued function $f \in L^1(\overline{\mathbb{T}}^2)$, where f is 2π-periodic in each variable, its double Fourier series is defined as

$$f(x,y) \sim \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \hat{f}(m,n)\, e^{i(mx+ny)},$$

where the Fourier coefficients $\hat{f}(m,n)$ are defined by

$$\hat{f}(m,n) = \frac{1}{4\pi^2} \int\int_{\overline{\mathbb{T}}^2} f(x,y)\, e^{-i(mx+ny)}\, dx\, dy.$$

A double Fourier series of f is said to be β-absolute convergent if

$$\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\hat{f}(m,n)|^\beta < \infty,$$

where

$$\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\hat{f}(m,n)|^\beta = \sum_{|m|\geqslant 1} \sum_{|n|\geqslant 1} |\hat{f}(m,n)|^\beta + \sum_{m\in\mathbb{Z}} |\hat{f}(m,0)|^\beta + \sum_{n\in\mathbb{Z}} |\hat{f}(0,n)|^\beta - |\hat{f}(0,0)|^\beta. \tag{3.1}$$

In the special case, when $m = 0$ or $n = 0$, we write

$$\hat{f}(m,0) = \hat{f}_1(m), \quad \text{where} \quad f_1(x) := \frac{1}{2\pi} \int_{\overline{\mathbb{T}}} f(x,y)\, dy, \quad x \in \mathbb{T}; \tag{3.2}$$

and

$$\hat{f}(0,n) = \hat{f}_2(n), \quad \text{where} \quad f_2(y) := \frac{1}{2\pi} \int_{\overline{\mathbb{T}}} f(x,y)\, dx, \quad y \in \mathbb{T}. \tag{3.3}$$

We may write

$$\sum_{m\in\mathbb{Z}} |\hat{f}_1(m)|^\beta = \sum_{m\in\mathbb{Z}} |\hat{f}(m,0)|^\beta \quad \text{and} \quad \sum_{n\in\mathbb{Z}} |\hat{f}_2(n)|^\beta = \sum_{n\in\mathbb{Z}} |\hat{f}(0,n)|^\beta.$$

Combining this with (3.1) gives

$$\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\hat{f}(m,n)|^\beta = \sum_{|m|\geqslant 1} \sum_{|n|\geqslant 1} |\hat{f}(m,n)|^\beta + \sum_{m\in\mathbb{Z}} |\hat{f}_1(m)|^\beta + \sum_{n\in\mathbb{Z}} |\hat{f}_2(n)|^\beta - |\hat{f}(0,0)|^\beta.$$

Thus, the double Fourier series of f is β-absolute convergent if

$$\sum_{|m|\geqslant 1}\sum_{|n|\geqslant 1}|\hat{f}(m,n)|^{\beta}<\infty,\quad \sum_{m\in\mathbb{Z}}|\hat{f}_1(m)|^{\beta}<\infty \text{ and } \sum_{n\in\mathbb{Z}}|\hat{f}_2(n)|^{\beta}<\infty.$$

For $\beta=1$, one gets the absolute convergence of the double Fourier series of f.

For a given $f\in L^p(\overline{\mathbb{T}}^2)$ $(p\geqslant 1)$, the p-integral modulus of continuity of f is defined as

$$\omega^{(p)}(f;\delta_1,\delta_2)=\sup_{\substack{h\in(0,\delta_1]\\ k\in(0,\delta_2]}}\|T_{h,k}f-T_{0,k}f-T_{h,0}f+f\|_p,$$

where $(T_{h,k}f)(x,y)=f(x+h,y+k)$ for all x and $\|\cdot\|_p$ denotes the L^p-norm over $\overline{\mathbb{T}}^2$.

For $p=\infty$, we omit writing p, one gets $\omega(f;\delta_1,\delta_2)$, the modulus of continuity of f.

For $I=[a,b]$ and $J=[c,d]$, define

$$f(I\times J)=f(b,d)-f(a,d)-f(b,c)+f(a,c).$$

A complex valued measurable function f defined on $\overline{\mathbb{T}}^2$ is said to be of (ϕ,ψ)-(Λ^1,Λ^2)-bounded variation (that is, $f\in(\phi,\psi)(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2)$) if

$$V_{(\Lambda^1,\Lambda^2)_{(\phi,\psi)}}(f,\overline{\mathbb{T}}^2)=\sup_{\mathscr{I},\mathscr{J}}\left(\sum_j\frac{1}{\lambda_{2,j}}\psi\left(\sum_k\frac{\phi(|f(I_k\times J_j)|)}{\lambda_{1,k}}\right)\right)<\infty,$$

where $\Lambda^1=\{\lambda_{1,n}\}$, $\Lambda^2=\{\lambda_{2,n}\}\in\mathbb{L}$; functions ϕ and ψ are convex and strictly increasing on $[0,\infty)$ with $\phi(0)=\psi(0)=0$; and $\mathscr{I}$ and $\mathscr{J}$ are finite collections of non-overlapping subintervals $\{I_k\}$ and $\{J_j\}$ in $\overline{\mathbb{T}}$ respectively.

Consider a function $f:\overline{\mathbb{T}}^2\to\mathbb{R}$ defined by $f(x,y)=g(x)+h(y)$, where g and h are any two arbitrary not necessarily bounded functions from $\overline{\mathbb{T}}$ into $\mathbb{R}$. Then $V_{(\Lambda^1,\Lambda^2)_{(\phi,\psi)}}(f,\overline{\mathbb{T}}^2)=0$. Thus, a function f with $V_{(\Lambda^1,\Lambda^2)_{(\phi,\psi)}}(f,\overline{\mathbb{T}}^2)<\infty$ need not be bounded.

If $f\in(\phi,\psi)(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2)$ is such that the marginal functions $f(0,.)\in\phi\Lambda^2BV(\overline{\mathbb{T}})$ and $f(.,0)\in\phi\Lambda^1BV(\overline{\mathbb{T}})$ then f is said to be of (ϕ,ψ)-$(\Lambda^1,\Lambda^2)^*$-bounded variation (that is, $f\in(\phi,\psi)(\Lambda^1,\Lambda^2)^*BV(\overline{\mathbb{T}}^2)$).

Note that, for $\phi(x)=\psi(x)=x$ and $\Lambda^1=\Lambda^2=\{1\}$ classes $(\phi,\psi)(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2)$ and $(\phi,\psi)(\Lambda^1,\Lambda^2)^*BV(\overline{\mathbb{T}}^2)$ reduce to classes $BV_V(\overline{\mathbb{T}}^2)$ (the class of functions of bounded variation in the sense of Vitali (refer [5, p. 279] for the definition of $BV_V(\overline{\mathbb{T}}^2)$)) and $BV_H(\overline{\mathbb{T}}^2)$ (the class of functions of bounded variation in the sense of Hardy (refer [5, p. 280] for the definition of $BV_H(\overline{\mathbb{T}}^2)$)) respectively; for $\psi(x)=x$ classes $(\phi,\psi)(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2)$ and $(\phi,\psi)(\Lambda^1,\Lambda^2)^*BV(\overline{\mathbb{T}}^2)$ reduce to classes $\phi(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2)$ [10, Definition 1. p. 1153] and $\phi(\Lambda^1,\Lambda^2)^*BV(\overline{\mathbb{T}}^2)$ respectively; for $\psi(x)=x$ and $\phi(x)=x^p$ $(p\geqslant 1)$ classes $(\phi,\psi)(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2)$ and $(\phi,\psi)(\Lambda^1,\Lambda^2)^*BV(\overline{\mathbb{T}}^2)$ reduce to classes $(\Lambda^1,\Lambda^2)BV^{(p)}(\overline{\mathbb{T}}^2)$ [11, Definition 1.2, p. 28] and $(\Lambda^1,\Lambda^2)^*BV^{(p)}(\overline{\mathbb{T}}^2)$ respectively; for $\phi(x)=x^p$ $(p\geqslant 1)$ and $\psi(x)=x^{q/p}$ $(q\geqslant p)$ classes $(\phi,\psi)(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2)$ and $(\phi,\psi)(\Lambda^1,\Lambda^2)^*BV(\overline{\mathbb{T}}^2)$ reduce to classes $(\Lambda^1,\Lambda^2)BV^{(p,q)}(\overline{\mathbb{T}}^2)$ [2, Definition 2.1, p. 362] and $(\Lambda^1,\Lambda^2)^*BV^{(p,q)}(\overline{\mathbb{T}}^2)$ respectively; for $\phi(x)=\psi(x)=x$ classes $(\phi,\psi)(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2)$ and $(\phi,\psi)(\Lambda^1,\Lambda^2)^*BV(\overline{\mathbb{T}}^2)$ reduce to classes $(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2)$ [1, Definition 2, p. 8] and $(\Lambda^1,\Lambda^2)^*BV(\overline{\mathbb{T}}^2)$ respectively.

Following the definition in [4], a double sequence $\gamma=\{\gamma_{mn}\}:(m,n)\in\mathbb{N}_+^2\}$ of nonnegative numbers belongs to the class $\mathscr{A}_\alpha$ for some $\alpha\geqslant 1$ if

$$\left(\sum_{m\in\mathscr{D}_\mu}\sum_{n\in\mathscr{D}_\nu}\gamma_{mn}^{\alpha}\right)^{1/\alpha}\leqslant\kappa 2^{(\mu+\nu)(1-\alpha)/\alpha}\sum_{m\in\mathscr{D}_{\mu-1}}\sum_{n\in\mathscr{D}_{\nu-1}}\gamma_{mn} \tag{3.4}$$

for all $\mu,\nu\geqslant 0$, where $\mathscr{D}_\mu$ is as defined in (2.2) for $\mu\geqslant 0$.

For instance, if $\mu\geqslant 1$ and $\nu=0$, then inequality (3.4) is of the form

$$\left(\sum_{m\in\mathscr{D}_\mu}\gamma_{m1}^{\alpha}\right)^{1/\alpha}\leqslant\kappa 2^{\mu(1-\alpha)/\alpha}\sum_{m\in\mathscr{D}_{\mu-1}}\gamma_{m1}.$$

It is easy to check that the inclusion (2.3) remain valid; and if a double sequence $\gamma=\{\gamma_{mn}\geqslant 0\}$ is such that

$$\max\{\gamma_{mn}:\ m\in\mathscr{D}_\mu,\ n\in\mathscr{D}_\nu\}\leqslant\kappa\min\{\gamma_{mn}:\ m\in\mathscr{D}_{\mu-1},\ n\in\mathscr{D}_{\nu-1}\},\quad(\mu,\nu)\in\mathbb{N}^2,$$

where κ is a constant, then $\gamma\in\mathscr{A}_\alpha$ for every $\alpha\geqslant 1$.

For convenience in writing, put

$$\gamma_{-m,n}=\gamma_{m,-n}=\gamma_{-m,-n}=\gamma_{m,n},\qquad(m,n)\in\mathbb{N}_+^2.$$

We prove the following results.

THEOREM 3.1. *If $\phi,\psi\in\Delta_2$, $f\in(\phi,\psi)(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2)\cap L^\infty(\overline{\mathbb{T}}^2)$, $1\leqslant p<2r$, $1\leqslant r<\infty$, and $\gamma=\{\gamma_{mn}\}\in\mathscr{A}_{2/(2-\beta)}$ for some $\beta\in(0,2)$, then*

$$\sum_{|m|\geqslant 1}\sum_{|n|\geqslant 1}\gamma_{mn}\,|\hat f(m,n)|^\beta\leqslant\kappa C\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}2^{-(\mu+\nu)\beta/2}\,\Gamma_{\mu-1,\nu-1} \tag{3.5}$$
$$\times\left(\phi^{-1}\left(\frac{(\omega^{((2-p)s+p)}(f;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}))^{2r-p}}{\Lambda^1_{2^\mu}}\psi^{-1}\left(\frac{1}{\Lambda^2_{2^\nu}}\right)\right)\right)^{\beta/2r},$$

where $\frac{1}{r}+\frac{1}{s}=1$, κ is from (3.4) corresponding to $\alpha=2/(2-\beta)$,

$$\Gamma_{\mu\nu}:=\sum_{m\in\mathscr{D}_\mu}\sum_{n\in\mathscr{D}_\nu}\gamma_{mn}\quad\textit{for }\mu,\nu\geqslant -1, \tag{3.6}$$

$$\Gamma_{-1,\nu}:=\Gamma_{0\nu},\ \Gamma_{\mu,-1}:=\Gamma_{\mu 0}\ \textit{for }\mu,\nu\geqslant 0,\ \textit{and}\ \Gamma_{-1,-1}:=\Gamma_{00}=\{\gamma_{11}\}. \tag{3.7}$$

Proof. For any $\mu, \nu \in \mathbb{N}$, consider

$$f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right) = f\left(\left[x+\frac{(j-1)\pi}{2^\mu}, x+\frac{j\pi}{2^\mu}\right]\times\left[y+\frac{(k-1)\pi}{2^\nu}, y+\frac{k\pi}{2^\nu}\right]\right).$$

Then, for each $m,n \in \mathbb{Z}$, we have

$$\widehat{f_{jk}}(m,n) = -4\hat{f}(m,n)\, e^{im(j-\frac{1}{2})\frac{\pi}{2^\mu}}\, e^{in(k-\frac{1}{2})\frac{\pi}{2^\nu}}\, \sin\left(\frac{m\pi}{2^{\mu+1}}\right)\sin\left(\frac{n\pi}{2^{\nu+1}}\right).$$

Since $f \in L^2(\overline{\mathbb{T}}^2)$, from Parseval formula, we get

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\left|\hat{f}(m,n)\, \sin\left(\frac{m\pi}{2^{\mu+1}}\right)\, \sin\left(\frac{n\pi}{2^{\nu+1}}\right)\right|^2 = O(\|f_{jk}\|_2^2).$$

In view of inequality (2.4) and an analogous inequality for $|n| \in \mathscr{D}_\nu$, we have

$$S_{\mu\nu} := \sum_{|m|\in\mathscr{D}_\mu}\sum_{|n|\in\mathscr{D}_\nu}|\hat{f}(m,n)|^2 = O(\|f_{jk}\|_2^2), \tag{3.8}$$

for all $j = 1,\ldots,2^\mu$ and for all $k = 1,\ldots,2^\nu$.

Suppose $r > 1$. Since

$$2 = \frac{(2-p)s+p}{s} + \frac{p}{r},$$

applying Hölder's inequality on the right side of the inequality (3.8), we have

$$\begin{aligned} S_{\mu\nu} &= O\left(\left(\int\int_{\overline{\mathbb{T}}^2}\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right|^{(2-p)s+p} dx\, dy\right)^{1/s} \|f_{jk}\|_p^{p/r}\right) \\ &= O\left(\Omega^{1/r}_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}\ \|f_{jk}\|_p^{p/r}\right), \end{aligned}$$

where $\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}} = \left(\omega^{(2-p)s+p}\left(f;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right)^{2r-p}$. Thus,

$$S^r_{\mu\nu} = O\left(\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}\int\int_{\overline{\mathbb{T}}^2}\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right|^p dx\, dy\right). \tag{3.9}$$

Since multiplying f by a positive constant changes $\omega^{(p)}\left(f;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)$ by the same constant, f is bounded, and $\phi \in \Delta_2$, we may assume that $\|f\|_\infty \leqslant \frac{1}{4}$. Thus $\|f_{jk}\|_\infty \leqslant 1$ and hence from equation (3.9), we get

$$S^r_{\mu\nu} \leqslant C\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}\int\int_{\overline{\mathbb{T}}^2}\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right| dx\, dy, \tag{3.10}$$

where constant C depends on f, r, s and p.

Since $C\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}} \geqslant 0$, if $C\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}} < 1$ then from equation (3.10) and equation (2.8) we get

$$\phi\left(\frac{S^r_{\mu\nu}}{4\pi^2}\right) \leqslant \phi\left(\frac{C\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}}{4\pi^2}\int\int_{\mathbb{T}^2}\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right| dx\,dy\right)$$
$$\leqslant C\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}\phi\left(\frac{1}{4\pi^2}\int\int_{\mathbb{T}^2}\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right| dx\,dy\right). \tag{3.11}$$

Further when $C\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}} \geqslant 1$, in view of equation (3.10) and equation (2.9) we get

$$\phi\left(\frac{S^r_{\mu\nu}}{4\pi^2}\right) \leqslant \phi\left(\frac{C\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}}{4\pi^2}\int\int_{\mathbb{T}^2}\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right| dx\,dy\right)$$
$$\leqslant d^{\log_2\left(C'\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}\right)+1}\phi\left(\frac{1}{4\pi^2}\int\int_{\mathbb{T}^2}\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right| dx\,dy\right), \tag{3.12}$$

where $C' = \frac{C}{4\pi^2}$. Since $d^{\log_2\left(C'\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}\right)+1} = C''\Omega^{\log_2 d}_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}$, where $C'' = d\cdot d^{\log_2 C'}$, and denoting the constant C'' by C itself, in view of equation (3.12), we get

$$\phi\left(\frac{S^r_{\mu\nu}}{4\pi^2}\right) \leqslant C\Omega^{\log_2 d}_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}\phi\left(\frac{1}{4\pi^2}\int\int_{\mathbb{T}^2}\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right| dx\,dy\right)$$
$$= C\Omega^{\log_2 d-1}_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}\phi\left(\frac{1}{4\pi^2}\int\int_{\mathbb{T}^2}\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right| dx\,dy\right)$$
$$\leqslant C\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}\phi\left(\frac{1}{4\pi^2}\int\int_{\mathbb{T}^2}\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right| dx\,dy\right) \tag{3.13}$$

because of the fact that $\Omega^{\log_2 d-1}_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}} \leqslant 1$, as $\|f\|_\infty \leqslant \frac{1}{4}$ and $\log_2 d - 1 \geqslant 0$. In either case, from (3.11) and (3.13), in view of Jensen's inequality for integral, we have

$$\phi\left(\frac{S^r_{\mu\nu}}{4\pi^2}\right) \leqslant C\frac{\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}}{4\pi^2}\int\int_{\mathbb{T}^2}\phi\left(\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right|\right)dx\,dy.$$

Multiplying both the sides of the above inequality by $\frac{1}{\lambda_{1,j}}$ and then summing over $j=1$ to 2^μ, we have

$$\frac{\Lambda^1_{2^\mu}}{\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}}\phi\left(\frac{S^r_{\mu\nu}}{4\pi^2}\right) = O\left(\frac{1}{4\pi^2}\int\int_{\mathbb{T}^2}\sum_{j=1}^{2^\mu}\frac{\phi\left(\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right|\right)}{\lambda_{1,j}}\,dx\,dy\right),$$

where $\Lambda^1_{2^\mu} = \sum_{j=1}^{2^\mu}\frac{1}{\lambda_{1,j}}$.

Again, using Jensen's inequality for integrals, we have

$$\psi\left(\frac{\Lambda^1_{2^\mu}}{\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}}\phi\left(\frac{S^r_{\mu\nu}}{4\pi^2}\right)\right) = O\left(\frac{1}{4\pi^2}\int\int_{\mathbb{T}^2}\psi\left(\sum_{j=1}^{2^\mu}\frac{\phi\left(\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right|\right)}{\lambda_{1,j}}\right)dx\,dy\right).$$

Multiplying both the sides of the above inequality by $\frac{1}{\lambda_{2,k}}$ and then summing over $k=1$ to 2^ν, we have

$$\begin{aligned}&\Lambda^2_{2^\nu}\psi\left(\frac{\Lambda^1_{2^\mu}}{\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}}\phi\left(\frac{S^r_{\mu\nu}}{4\pi^2}\right)\right)\\ &=O\left(\frac{1}{4\pi^2}\int\int_{\mathbb{T}^2}\sum_{k=1}^{2^\nu}\frac{1}{\lambda_{2,k}}\psi\left(\sum_{j=1}^{2^\mu}\frac{\phi\left(\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right|\right)}{\lambda_{1,j}}\right)\,dx\,dy\right),\end{aligned}$$

where $\Lambda^2_{2^\nu}=\sum_{k=1}^{2^\nu}\frac{1}{\lambda_{2,k}}$. Since $f\in(\phi,\psi)(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2)$ implies

$$\sum_{k=1}^{2^\nu}\frac{1}{\lambda_{2,k}}\psi\left(\sum_{j=1}^{2^\mu}\frac{\phi\left(\left|f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right|\right)}{\lambda_{1,j}}\right)=O(1)$$

and hence

$$S_{\mu\nu}=O\left(\left(\phi^{-1}\left(\frac{\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}}{\Lambda^1_{2^\mu}}\psi^{-1}\left(\frac{1}{\Lambda^2_{2^\nu}}\right)\right)\right)^{1/r}\right).$$

Since $1=\frac{\beta}{2}+\frac{2-\beta}{2}$, by Hölder's inequality, we have

$$\begin{aligned}R_{\mu\nu}&:=\sum_{|m|\in\mathscr{D}_\mu}\sum_{|n|\in\mathscr{D}_\nu}\gamma_{mn}|\hat{f}(m,n)|^\beta\\ &\leqslant\left(\sum_{|m|\in\mathscr{D}_\mu}\sum_{|n|\in\mathscr{D}_\nu}|\hat{f}(m,n)|^2\right)^{\beta/2}\left(\sum_{|m|\in\mathscr{D}_\mu}\sum_{|n|\in\mathscr{D}_\nu}\gamma_{mn}^{2/(2-\beta)}\right)^{(2-\beta)/2}\\ &\leqslant C\left(\phi^{-1}\left(\frac{\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}}{\Lambda^1_{2^\mu}}\psi^{-1}\left(\frac{1}{\Lambda^2_{2^\nu}}\right)\right)\right)^{\beta/2r}\left(\sum_{|m|\in\mathscr{D}_\mu}\sum_{|n|\in\mathscr{D}_\nu}\gamma_{mn}^{2/(2-\beta)}\right)^{(2-\beta)/2}.\end{aligned}\tag{3.14}$$

Thus, for $\max\{\mu,\nu\}\geqslant 1$, in view of (3.4), with $\alpha=2/(2-\beta)$, and (3.14), we get

$$R_{\mu\nu}\leqslant C\kappa 2^{-(\mu+\nu)\beta/2}\,\Gamma_{\mu-1,\nu-1}\left(\phi^{-1}\left(\frac{\Omega_{\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}}}{\Lambda^1_{2^\mu}}\psi^{-1}\left(\frac{1}{\Lambda^2_{2^\nu}}\right)\right)\right)^{\beta/2r}.$$

If $\mu=\nu=0$, then from equation (3.14) it follows that

$$\begin{aligned}R_{00}&=\gamma_{11}(|\hat{f}(1,1)|^\beta+|\hat{f}(-1,1)|^\beta+|\hat{f}(1,-1)|^\beta+|\hat{f}(-1,-1)|^\beta)\\ &=O\left(\gamma_{11}\left(\phi^{-1}\left(\frac{\Omega_{\pi,\pi}}{\frac{1}{\lambda_{1,1}}}\psi^{-1}\left(\frac{1}{\frac{1}{\lambda_{2,1}}}\right)\right)\right)^{\beta/2r}\right).\end{aligned}$$

Hence, for $r>1$, the result follows from

$$\sum_{|m|\geqslant 1}\sum_{|n|\geqslant 1}\gamma_{mn}|\hat{f}(m,n)|^{\beta}=\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}R_{\mu\nu}.$$

For the case $r=1$, $s=\infty$, simply note that

$$\left|f_{jk}\left(x,y;\frac{\pi}{2^{\mu}},\frac{\pi}{2^{\nu}}\right)\right|^{2}\leqslant\left(\omega\left(f;\frac{\pi}{2^{\mu}},\frac{\pi}{2^{\nu}}\right)\right)^{2-p}\left|f_{jk}\left(x,y;\frac{\pi}{2^{\mu}},\frac{\pi}{2^{\nu}}\right)\right|^{p},\ (x,y)\in\overline{\mathbb{T}}^{2},$$

and proceed as above from equation (3.9) onwards. $\square$

LEMMA 3.2. *If $f\in(\phi,\psi)(\Lambda^{1},\Lambda^{2})^{*}BV(\overline{\mathbb{T}}^{2})$, then f is bounded on $\overline{\mathbb{T}}^{2}$.*

Proof. For any $f\in(\phi,\psi)(\Lambda^{1},\Lambda^{2})^{*}BV(\overline{\mathbb{T}}^{2})$,

$$\begin{aligned}|f(x,y)|&\leqslant|f([0,x]\times[0,y])|+|f(0,y)-f(0,0)|+|f(x,0)-f(0,0)|+|f(0,0)|\\&\leqslant\phi^{-1}(\lambda_{1,1}\psi^{-1}(\lambda_{2,1}V_{(\Lambda^{1},\Lambda^{2})_{(\phi,\psi)}}(f,\overline{\mathbb{T}}^{2})))+\phi^{-1}(\lambda_{2,1}V_{\Lambda^{2}_{\phi}}(f(0,.),\overline{\mathbb{T}}))\\&\quad+\phi^{-1}(\lambda_{1,1}V_{\Lambda^{1}_{\phi}}(f(.,0),\overline{\mathbb{T}}))+|f(0,0)|\end{aligned}$$

implies f is bounded on $\overline{\mathbb{T}}^{2}$. $\square$

COROLLARY 3.3. *If $\phi,\psi\in\Delta_{2}$ and a measurable $f\in(\phi,\psi)(\Lambda^{1},\Lambda^{2})^{*}BV(\overline{\mathbb{T}}^{2})$, then (3.5) holds true, where p,r,s,γ, β, κ, α and Γ are as in Theorem 3.1.*

Proof. It follows from Lemma 3.2 and Theorem 3.1. $\square$

COROLLARY 3.4. *Under the hypothesis of Theorem 3.1, we have*

$$\sum_{|m|\geqslant 1}\sum_{|n|\geqslant 1}\gamma_{mn}\,|\hat{f}(m,n)|^{\beta}$$

$$\leqslant\kappa C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-\beta/2}\,\gamma_{mn}\,\left(\phi^{-1}\left(\frac{(\omega^{((2-p)s+p)}(f;\frac{\pi}{m},\frac{\pi}{n}))^{2r-p}}{\Lambda^{1}_{m}}\psi^{-1}\left(\frac{1}{\Lambda^{2}_{n}}\right)\right)\right)^{\beta/2r}. \tag{3.15}$$

Proof. In case when $\mu,\nu\geqslant 1$, from (3.4) and (3.6), we get

$$2^{-(\mu+\nu)\beta/2}\,\Gamma_{\mu-1,\nu-1}\left(\phi^{-1}\left(\frac{(\omega^{((2-p)s+p)}(f;\frac{\pi}{2^{\mu}},\frac{\pi}{2^{\nu}}))^{2r-p}}{\Lambda^{1}_{2^{\mu}}}\psi^{-1}\left(\frac{1}{\Lambda^{2}_{2^{\nu}}}\right)\right)\right)^{\beta/2r}$$

$$\leqslant\sum_{m\in\mathscr{D}_{\mu-1}}\sum_{n\in\mathscr{D}_{\nu-1}}(mn)^{-\beta/2}\gamma_{mn}\left(\phi^{-1}\left(\frac{(\omega^{((2-p)s+p)}(f;\frac{\pi}{m},\frac{\pi}{n}))^{2r-p}}{\Lambda^{1}_{m}}\psi^{-1}\left(\frac{1}{\Lambda^{2}_{n}}\right)\right)\right)^{\beta/2r}.$$

In case $\mu \geqslant 1$ and $\nu = 0$, from (2.2) and (3.7) it follows that

$$2^{-\mu\beta/2}\,\Gamma_{\mu-1,-1}\left(\phi^{-1}\left(\frac{(\omega^{((2-p)s+p)}(f;\frac{\pi}{2^\mu},\pi))^{2r-p}}{\Lambda^1_{2^\mu}}\psi^{-1}\left(\frac{1}{\frac{1}{\lambda_{2,1}}}\right)\right)\right)^{\beta/2r}$$
$$\leqslant \sum_{m\in\mathscr{D}_{\mu-1}} m^{-\beta/2}\gamma_{m1}\left(\phi^{-1}\left(\frac{(\omega^{((2-p)s+p)}(f;\frac{\pi}{m},\pi))^{2r-p}}{\Lambda^1_m}\psi^{-1}\left(\frac{1}{\frac{1}{\lambda_{2,1}}}\right)\right)\right)^{\beta/2r}.$$

In case $\mu = 0$ and $\nu \geqslant 1$, an analogous inequality holds; while in case $\mu = \nu = 0$, we have

$$\Gamma_{-1,-1}\left(\phi^{-1}\left(\frac{(\omega^{((2-p)s+p)}(f;\pi,\pi))^{2r-p}}{\frac{1}{\lambda_{1,1}}}\psi^{-1}\left(\frac{1}{\frac{1}{\lambda_{2,1}}}\right)\right)\right)^{\beta/2r}$$
$$= \gamma_{11}\left(\phi^{-1}\left(\frac{(\omega^{((2-p)s+p)}(f;\pi,\pi))^{2r-p}}{\frac{1}{\lambda_{1,1}}}\psi^{-1}\left(\frac{1}{\frac{1}{\lambda_{2,1}}}\right)\right)\right)^{\beta/2r}.$$

Hence, the corollary follows from Theorem 3.1. □

COROLLARY 3.5. *Under the hypothesis of Corollary 3.3, the inequality (3.15) holds true.*

Proof. It follows from Lemma 3.2 and Corollary 3.4. □

Taking $\psi(x) = x$ in the Corollary 3.4 and Corollary 3.5, we have the following corollaries.

COROLLARY 3.6. *If $\phi \in \Delta_2$ and $f \in \phi(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2)\cap L^\infty(\overline{\mathbb{T}}^2)$, then*

$$\sum_{|m|\geqslant 1}\sum_{|n|\geqslant 1}\gamma_{mn}\,|\hat{f}(m,n)|^\beta$$
$$\leqslant \kappa C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-\beta/2}\,\gamma_{mn}\left(\phi^{-1}\left(\frac{(\omega^{((2-p)s+p)}(f;\frac{\pi}{m},\frac{\pi}{n}))^{2r-p}}{\Lambda^1_m\Lambda^2_n}\right)\right)^{\beta/2r}, \qquad (3.16)$$

where p,r,s,γ, β, κ and α are as in Theorem 3.1.

COROLLARY 3.7. *If $\phi \in \Delta_2$ and a measurable $f \in \phi(\Lambda^1,\Lambda^2)^*BV(\overline{\mathbb{T}}^2)$, then (3.16) holds true, where p,r,s,γ, β, κ and α are as in Theorem 3.1.*

In the case when $\{\gamma_{mn}\} = \{1\}$, $\psi(x) = \phi(x) = x$, $r = 1$ and $p = 1$, Corollary 3.4 and Corollary 3.5 were proved in [9, Theorem 3.3 and Corollary 3.4 (i)].

Combining Corollary 2.2 and Corollary 3.4, or Corollary 2.2 and Corollary 3.6, we can easily find sufficient conditions imposed on f, f_1 and f_2 for the convergence of the double series

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\gamma_{mn}|\hat{f}(m,n)|^\beta.$$

For $\{\gamma_{mn}\}=\{\gamma_m\}=\{\gamma_n\}=\{1\}$, combining Corollary 2.2 and Corollary 3.4, we obtain the following corollary.

COROLLARY 3.8. *If $\phi,\psi\in\Delta_2$, a measurable $f\in(\phi,\psi)(\Lambda^1,\Lambda^2)^*BV(\overline{\mathbb{T}}^2)$, $1\leqslant p<2r$, $1\leqslant r<\infty$, $\beta\in(0,2)$,*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-\beta/2}\left(\phi^{-1}\left(\frac{(\omega^{((2-p)s+p)}(f;\frac{\pi}{m},\frac{\pi}{n}))^{2r-p}}{\Lambda_m^1}\psi^{-1}\left(\frac{1}{\Lambda_n^2}\right)\right)\right)^{\beta/2r}<\infty,$$

$$\sum_{m=1}^{\infty}m^{-\beta/2}\left(\phi^{-1}\left(\frac{(\omega^{((2-p)s+p)}(f_1;\frac{\pi}{m}))^{2r-p}}{\Lambda_m^1}\right)\right)^{\beta/2r}<\infty,$$

and

$$\sum_{n=1}^{\infty}n^{-\beta/2}\left(\phi^{-1}\left(\frac{(\omega^{((2-p)s+p)}(f_2;\frac{\pi}{n}))^{2r-p}}{\Lambda_n^2}\right)\right)^{\beta/2r}<\infty,$$

where f_1 and f_2 are as defined in (3.2) and (3.3), respectively, and $\frac{1}{r}+\frac{1}{s}=1$, then the double Fourier series of f is β-absolute convergent.

Corollary 3.8 gives two dimensional analogue of one dimensional result proved in [8, Theorem 2, with $n_k=k$, for all k, p. 770] in the case when $\psi(x)=x$. Similarly, Corollary 3.8 gives two dimensional analogue of one dimensional result of Schramm and Waterman [6, Theorem 2, p. 274] in the case when $\psi(x)=x$ and $\beta=1$; and also gives two dimensional analogue of one dimensional result proved in [12, Theorem 1, with $n_k=k$, for all k] in the case when $\phi(x)=\psi(x)=x$, $r=1$ and $p=1$.

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K. N. Darji
Department of Science and Humanities,
Tatva Institute of Technological Studies
Modasa, Arvalli, Gujarat, India
e-mail: darjikiranmsu@gmail.com

R. G. Vyas
Department of Mathematics, Faculty of Science
The Maharaja Sayajirao University of Baroda
Vadodara, Gujarat, India
e-mail: drrgvyas@yahoo.com