APPLICATIONS OF BRIOT–BOUQUET DIFFERENTIAL SUBORDINATION

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Abstract. Sharp estimates on $\beta$ in Briot-Bouquet differential subordination

$$p(z) + \beta z p'(z)/p(z) \prec h(z)$$

are obtained so that its solution $p$ is subordinate to some specific Carathéodory functions. As an application, the estimates on $\beta$ are obtained so that the integral operator

$$\beta - 1 \int_0^z f_1/\beta(t) - 1 dt$$

defines a sufficient condition for parabolic starlikeness.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions $f$ in the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ be the class of univalent functions in $\mathcal{A}$. For $f, g \in \mathcal{A}$, $f \prec g$ if there is an analytic function $w : D \rightarrow D$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ for all $z \in D$, where $\prec$ denotes the subordination. In particular, if $g$ is univalent in $D$, then the following equivalence condition holds:

$$f \prec g \iff f(0) = g(0) \text{ and } f(D) \subset g(D).$$

By making use of subordination, Ma and Minda[13] unified various subclasses of class of starlike functions in which the quantity $zf'(z)/f(z)$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function $\varphi$ with positive real part in the unit disk $D$ and normalized by $\varphi(0) = 1$ and $\varphi'(0) > 0$. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $zf'(z)/f(z) \prec \varphi(z)$ and is denoted by $\mathcal{S}^*(\varphi)$. In 1993, Ronning [19] introduced the class $\mathcal{S}_p^* := \mathcal{S}^*(\varphi_{PAR}(z))$ of the parabolic starlike functions, where

$$\varphi_{PAR}(z) = 1 + (2/\pi^2)(\log((1 - \sqrt{z})/(1 + \sqrt{z})))^2$$

is the analytic function defined on $D$. Analytically, it is noted that

$$f \in \mathcal{S}_p^* \iff \left| \frac{zf'(z)}{f(z)} - 1 \right| < \Re\left(\frac{zf'(z)}{f(z)}\right), \quad z \in D.$$
In [22], Sokół and Stankiewicz introduced and studied the class $\mathcal{S}_b^* := \mathcal{S}^*(\sqrt{1+z})$ which is the subclass of $\mathcal{S}^*$ consisting of functions $f \in \mathcal{S}$ such that for each $z \in \mathbb{D}$, $w = zf'(z)/f(z)$ lies in the region bounded by right half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. In 2015, Mendiratta et al. [16] established and studied the class $\mathcal{S}_e^* := \mathcal{S}^*(e^z)$ consisting of functions $f \in \mathcal{S}$ satisfying the condition $|\log(zf'(z)/f(z))| < 1$. In 2015, Raina and Sokól [18] studied a class $\mathcal{S}_q^* := \mathcal{S}^*(\varphi_q)$ where $\varphi_q(z) := z + \sqrt{1+z^2}$. Further, Kumar and Ravichandran [10], introduced and studied the geometric properties of the class $\mathcal{S}_R^* = \mathcal{S}^*(\varphi_0)$ associated with the rational function $\varphi_0(z) := 1 + (z/k)((k+z)/(k-z))$, $(k = \sqrt{2} + 1)$. Recently, Cho et al. [6] discussed various geometric estimates related to the class $\mathcal{S}_s^* = \mathcal{S}^*(\varphi_s)$ where $\varphi_s(z) := 1 + \sin z$.

The class $\mathcal{P}$ consists of analytic functions $p : \mathbb{D} \to \mathbb{C}$ with positive real part in $\mathbb{D}$ and normalized by the condition $p(0) = 1$. Let $\beta$, $\gamma$ be real or complex scalars and $h$ be a univalent function defined on unit disk $\mathbb{D}$. Miller and Mocanu [14] studied the properties of Briot-Bouquet differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} < h(z)$$

which has been used in the theory of univalent functions. For particular values of the scalars $\beta$, $\gamma$ and the function $h$, many researchers have discussed the first order differential subordination. In 1989, Nunokawa et al. [17] proved that if $1 + zp'(z) < 1 + z$, then $p(z) < 1 + z$. In [2, 3], Ali et al. obtained some sufficient conditions for normalized analytic functions to be in some subclasses of Janowski starlike functions and lemniscate starlike functions. Recently, authors [11] computed sharp bounds on parameters involved in certain first order differential subordinations related to the functions with positive real part. For details, see [1, 4, 7, 5, 8, 12, 21].

Motivated by these works, we obtain sharp estimates on $\beta$ such that the first order differential subordination

$$p(z) + \beta \frac{zp'(z)}{p(z)} < 1 + z$$

implies $p(z) < \mathcal{D}(z)$, where $\mathcal{D}(z) = e^z, \sqrt{1+z}, \varphi_0(z), \varphi_s(z)$ or $\varphi_q(z)$. For a starlike function satisfying $zf'(z)/f(z) < 1 + z$, we have shown that, for $\beta \geq 0.291936$, the integral $\beta^{-1} \int_0^z f^{1/\beta}(t)^{-1} dt$ belongs to the class $\mathcal{S}_e^*$. Similar bounds were obtained for the integral to belong to the classes $\mathcal{S}_L^*$, $\mathcal{S}_R^*$ and $\mathcal{S}_s^*$. Moreover, we estimate a bound on $\beta$ so that $p < \varphi_{PAR}$ whenever the subordination relation $1 + \beta zp'(z) < e^z$ holds and this subordination relation provides a sufficient condition for parabolic starlikeness. Our results are proved by applying the theory of differential subordination developed by Miller and Mocanu and using some properties of confluent hypergeometric functions.
2. Subordination and hypergeometric functions

The confluent (or Kummer) hypergeometric function \( \Phi(a,c;z) \) is given by the convergent power series

\[
\Phi(a,c;z) = \, _1F_1(a,c;z) := 1 + \frac{a}{c} \frac{z}{1!} + \frac{a(a+1)}{c(c+1)} \frac{z^2}{2!} + \cdots, \tag{1}
\]

where \( a \) and \( c \) are complex numbers with \( c \neq 0, -1, -2, \ldots \). The function \( \Phi \) is analytic in \( \mathbb{C} \) and satisfies the Kummer’s differential equation

\[
zw''(z) + (c-z)w'(z) - aw(z) = 0.
\]

Let \( (d)_k \) denotes the Pochhammer symbol given by \( (d)_k = \Gamma(d+k)/\Gamma(d) = d(d+1) \cdots (d+k-1) \) and \( (d)_0 = 1 \), then (1) can be written in the form

\[
\Phi(a,c;z) = \sum_{k=0}^{\infty} \frac{a^k}{(c)_k \, k!} = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c-a)} \sum_{k=0}^{\infty} \Gamma(a+k) \frac{z^k}{k!}.
\]

Also \( c\Phi'(a,c;z) = a\phi(a+1,c+1;z) \) and the following integral representation of \( \Phi \) [15, p. 5] given by

\[
\Phi(a,c;z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}e^{zt}dt = \int_0^1 e^{zt}d\mu(t),
\]

is well known, where \( d\mu(t) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c-a)} t^{a-1}(1-t)^{c-a-1}dt \) is a probability measure on \([0,1]\) and \( \text{Re}\, c > \text{Re}\, a > 0 \).

Let \( c \) be a complex number such that \( \text{Re}\, c > 0 \), let \( n \) be a positive integer, and let

\[
C_n = C_n(c) = \frac{n}{\text{Re}\, c} \left| c \sqrt{1 + 2\text{Re}\, c/n + 1\text{Im}\, c} \right|.
\]

If \( R(z) \) is the univalent function defined in \( \mathbb{D} \) by \( R(z) = 2C_nz/(1-z^2) \), then the open door function is defined by

\[
R_{c,n}(z) = R \left( \frac{z+b}{1+bz} \right) = 2C_n \frac{(z+b)(1+bz)}{(1+bz)^2 - (z+b)^2},
\]

where \( b = R^{-1}(c) \). We note that if \( c > 0 \), then the open door function reduces to

\[
R_{c,n}(z) = c \frac{1+z}{1-z} + \frac{2nz}{1-z^2}. \tag{2}
\]

We see that \( R_{c,n} \) is univalent in \( \mathbb{D} \), \( R_{c,n}(0) = c \) and \( R_{c,n}(\mathbb{D}) = R(\mathbb{D}) \) is the complex plane with slits along the half-lines \( \text{Re}\, w = 0 \) and \( |\text{Im}\, w| \geq C_n \).
THEOREM 1. Let \( p \) be analytic function in \( \mathbb{D} \) with \( p(0) = 1 \). Let
\[
 p(z) + \beta \frac{zp'(z)}{p(z)} < 1 + z, \quad \beta > 0.
\]
Then the following are true:

(a) If \( \beta \geq \beta_{c} \simeq 0.291936 \), then \( p(z) \prec e^{z} \).

(b) If \( \beta \geq \beta_{l} \simeq 1.76736 \), then \( p(z) \prec \sqrt{1 + z} \).

(c) If \( \beta \geq \beta_{s} \simeq 0.322915 \), then \( p(z) \prec \varphi_{s}(z) \).

(d) If \( \beta \geq \beta_{R} \simeq 4.3622 \), then \( p(z) \prec \varphi_{0}(z) \).

(e) If \( \beta \geq \beta_{L} \simeq 0.391235 \), then \( p(z) \prec \varphi_{q}(z) \).

The bounds on \( \beta \) are sharp.

In order to prove this result, we need the following lemma due to Miller and Mocanu:

**Lemma 1.** [15, Theorem 3.2j., p. 97] Let \( \beta, \gamma \in \mathbb{C} \) with \( \beta \neq 0 \), and let \( n \) be a positive integer. Let \( R_{\beta a + \gamma, n} \) be as given in (2) and let \( h \) be analytic in \( \mathbb{D} \), with \( h(0) = a \), \( \text{Re}(\beta a + \gamma) > 0 \) and \( \beta h(z) + \gamma \prec R_{\beta a + \gamma, n}(z) \). If \( q \) is the analytic solution of Briot-Bouquet differential equation
\[
 q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z)
\]
and if \( h \) is convex or \( Q(z) = zq'(z)/(\beta q(z) + \gamma) \) is starlike, then \( q \) and \( h \) are univalent. Furthermore, if \( p \in \mu[a, n] \) satisfies
\[
 p(z) + \frac{nzp'(z)}{\beta p(z) + \gamma} < h(z)
\]
then \( p \prec q \), and \( q \) is the best \((a, n)\)-dominant.

**Proof of Theorem 1.** Consider the following Briot-Bouquet differential equation
\[
 p(z) + \frac{\beta zp'(z)}{p(z)} = 1 + z.
\]
Using [15, Theorem 3.3d], its analytic solution is given by
\[
 q_{\beta}(z) = (\Phi(1, 1 + 1/\beta, -z/\beta))^{-1}.
\]
Clearly, the function \( h(z) = 1 + z \) is convex. As \( \text{Re}\ h(z) > 0 \),
\[
 \frac{1}{\beta} h(z) \prec \frac{1}{\beta} \left( \frac{1 + z}{1 - z} \right) + \frac{2z}{1 - z^2} = R_{1/\beta, 1}(z).
\]
From Lemma 1, taking \( n = 1, \gamma = 0, \) and \( \beta \to 1/\beta, \) we get \( p(z) \prec q_\beta(z). \) We now obtain lower bounds on \( \beta \) for the subordination \( q_\beta(z) \prec \mathcal{P}(z) \) to hold for different choices of \( \mathcal{P}(z). \) The transitivity will then imply \( p(z) \prec \mathcal{P}(z) \) for obtained values of \( \beta. \)

We note that \( q_\beta((-1, 1)) \subset \mathbb{R} \) and \( \mathcal{P}((-1, 1)) \subset \mathbb{R} \) for all \( \beta > 0 \) and all choices of \( \mathcal{P}. \) Also \( q_\beta(-1) \leq q_\beta(1). \) Hence the subordination \( q_\beta(z) \prec \mathcal{P}(z) \) implies

\[
\mathcal{P}(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \mathcal{P}(1).
\]

The lower bounds on \( \beta \) are obtained by solving this inequality. Though the converse need not hold, we show that for obtained lower bounds on \( \beta, q_\beta(z) \prec \mathcal{P}(z) \) for all our choices of function \( \mathcal{P}. \) We note that

\[
(\Phi(1, 1 + 1/\beta, 1/\beta))^{-1} = \left( \sum_{n=0}^{\infty} \frac{1}{(1 + \beta) \ldots (1 + n\beta)} \right)^{-1}
\]

and

\[
(\Phi(1, 1 + 1/\beta, -1/\beta))^{-1} = \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{(1 + \beta) \ldots (1 + n\beta)} \right)^{-1}.
\]

(a) For \( \mathcal{P}(z) = e^z, \) the inequalities \( e^{-1} \leq q_\beta(-1) \) and \( q_\beta(1) \leq e \) reduce to

\[
f(\beta) = (\Phi(1, 1 + 1/\beta, 1/\beta))^{-1} - \frac{1}{e} \geq 0
\]

and

\[
g(\beta) = e - (\Phi(1, 1 + 1/\beta, -1/\beta))^{-1} \geq 0.
\]

Here \( \lim_{\beta \to 0} f(\beta) = -1/e \) and \( \lim_{\beta \to \infty} f(\beta) = e - 1 > 0. \) Also \( f'(\beta) > 0 \) for all \( \beta \in (0, \infty). \) Hence \( f \) is strictly increasing in \((0, \infty).\) Let \( \beta_e \) denotes the unique zero of \( f \) in \((0, \infty).\) Then \( f(\beta) \geq 0 \) for every \( \beta \geq \beta_e \simeq 0.291936. \) Also \( g(\beta) > 0 \) for all \( \beta \in (0, \infty). \)

We now show that \( q_\beta(z) \prec e^z \) for \( \beta \geq \beta_e \simeq 0.291936. \) First we note that \( q_\beta(\mathbb{D}) \) is convex, symmetrical about real axis and \( q_\beta(\mathbb{D}) \subseteq q_\beta(\mathbb{D}) \) for \( \beta \geq \beta_e. \) Hence it suffices to prove that \( q_\beta(z) \prec e^z. \) But this is equivalent to showing that

\[
|\log(q_\beta(z))| \leq 1, \quad z \in \overline{\mathbb{D}}
\]

where \( \log \) denotes the principal branch of logarithm function. In fact, proving this for boundary points is enough. For \( z = e^{it}, \) let \( F(t) = |\log(q_\beta(e^{it}))|^2, -\pi \leq t \leq \pi. \) We note that \( F(t) = F(-t), \) so it is enough to consider the interval \( 0 \leq t \leq \pi. \)

\[
F(t) = (\ln|q_\beta(e^{it})|)^2 + (\arg(q_\beta(e^{it}))^2
= (\ln|\Phi(1, 1 + 1/\beta_e, -e^{it}/\beta_e)|)^2 + (\arg(\Phi(1, 1 + 1/\beta_e, -e^{it}/\beta_e))^2.
\]
We see that $F(t)$ is an increasing function of $t$ and $\max_{0 \leq t \leq \pi} F(t) = F(\pi) = 1$. See Figure 1.

This proves that $q_{\beta_e}(z) \prec e^z$ and hence $q_{\beta}(z) \prec e^z$ for $\beta \geq \beta_e$. Figure 2 illustrates our proof.

(b) For $\mathcal{P}(z) = \sqrt{1+z}$, the inequalities $0 \leq q_{\beta}(-1)$ and $q_{\beta}(1) \leq \sqrt{2}$ reduce to

$$f(\beta) = (\Phi(1, 1 + 1/\beta, 1/\beta))^{-1} \geq 0$$

and

$$g(\beta) = \sqrt{2} - (\Phi(1, 1 + 1/\beta, -1/\beta))^{-1} \geq 0.$$  

Clearly, $f(\beta) \geq 0$ for $\beta > 0$. Also, we see that $\lim_{\beta \searrow 0} g(\beta) < 0$ and $\lim_{\beta \nearrow \infty} g(\beta) = \sqrt{2} - 1 > 0$. Also $g'(\beta) > 0$ for all $\beta \in (0, \infty)$. Let $\beta_l$ denotes the unique zero of $g$ in $(0, \infty)$. Then we show that $q_{\beta}(z) \prec \mathcal{P}(z)$ for $\beta \geq \beta_l \simeq 1.76736$.

Again as done in part (a), it suffices to show that $q_{\beta_l}(z) \prec \sqrt{1+z}$. This is equivalent to showing that

$$|(q_{\beta_l}(z))^2 - 1| \leq 1, z \in \overline{D}.$$  

We prove it for boundary points. For $z = e^{it}$, $q_{\beta_l}(z) = (\Phi(1, 1 + 1/\beta_l, -e^{it}/\beta_l))^{-1}$, hence inequality (3) is equivalent to showing that

$$2 \Re \Phi(1, 1 + 1/\beta_l, -e^{it}/\beta_l)^2 - 1) \geq 0.$$
We define
\[ G(t) = \Re \Phi(1, 1 + 1/\beta, -e^{it}/\beta)^2, \quad t \in [0, \pi]. \]

We see that \( G(t) \) is an increasing function with \( \min_{0 \leq t \leq \pi} G(t) = G(0) = 1/2. \) See Figure 3.

(c) For \( \mathcal{P}(z) = \varphi_\beta(z) \), the inequalities \( 1 - \sin 1 \leq q_\beta(-1) \) and \( q_\beta(1) \leq 1 + \sin 1 \) reduce to
\[
    f(\beta) = (\Phi(1, 1 + 1/\beta, 1/\beta))^{-1} - 1 + \sin 1 \geq 0
\]
and
\[
    g(\beta) = 1 + \sin 1 - (\Phi(1, 1 + 1/\beta, -1/\beta))^{-1} \geq 0.
\]

Here we see that both \( f(\beta) \) and \( g(\beta) \) have unique root in \((0, \infty)\). In fact, \( f(\beta) \geq 0 \) for \( \beta \geq \beta_1 \approx 0.04435 \) and \( g(\beta) \geq 0 \) for \( \beta \geq \beta_2 \approx 0.322915 \). Let \( \beta_s = \max\{\beta_1, \beta_2\} = \beta_2 \). We show that \( q_\beta(z) \prec 1 + \sin z \) for \( \beta \geq \beta_s \). As done in previous parts, it suffices to show that \( q_{\beta_s}(z) \prec 1 + \sin z \). This is equivalent to showing that \( |\arcsin(q_{\beta_s}(z) - 1)| \leq 1, \ z \in \mathbb{D} \). We define
\[
    H(t) = |\arcsin(q_{\beta_s}(e^{it}) - 1)|, \quad t \in [0, \pi].
\]

Then \( H(t) \) is decreasing function of \( t \) with maximum value 1 attained at \( t = 0 \). See Figure 4.
(d) For $P(z) = \varphi_0(z)$, consider the inequalities $\varphi_0(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \varphi_0(1)$. Let

$$f(\beta) = (\Phi(1, 1 + 1/\beta, 1/\beta))^{-1} - 2(\sqrt{2} - 1) \geq 0$$

and

$$g(\beta) = 2 - (\Phi(1, 1 + 1/\beta, -1/\beta))^{-1} \geq 0.$$  

Since $\lim_{\beta \searrow 0} f(\beta) = -2(\sqrt{2} - 1) < 0$, $\lim_{\beta \nearrow \infty} f(\beta) = 3 - 2\sqrt{2} > 0$ and $f'(\beta) > 0$ for all $\beta \in (0, \infty)$, $f(\beta)$ has a unique zero in $(0, \infty)$ say $\beta_R \simeq 4.3622$. Also we see that $g(\beta) \geq 0$ for $\beta > 0$. We show that $q_\beta(z) < \varphi_0(z)$ for $\beta \geq \beta_R$.

Again it suffices to show that $q_{\beta_R}(z) < \varphi_0(z)$. This is equivalent to showing that

$$\frac{k}{2} \left| -q_{\beta_R}(z) + |q_{\beta_R}(z) - 2| \right| \leq 1, \quad z \in \overline{D},$$

where $k = \sqrt{2} + 1$. It is enough to show for boundary points. We define

$$M(t) = | -q_{\beta_R}(e^{it}) + |q_{\beta_R}(e^{it}) - 2| |, \quad t \in [0, \pi].$$

Then we see that $\max_{0 \leq t \leq 1} M(t) = M(0) = 0.405592$ which is less than $2/k = 0.828427$. See Figure 5.

![Figure 5: Graph of $M(t)$.](image)

(e) For $P(z) = \varphi_q(z)$, the inequalities $-1 + \sqrt{2} \leq q_\beta(-1)$ and $q_\beta(1) \leq 1 + \sqrt{2}$ reduce to

$$f(\beta) = (\Phi(1, 1 + 1/\beta, 1/\beta))^{-1} + 1 - \sqrt{2} \geq 0$$

and

$$g(\beta) = 1 + \sqrt{2} - (\Phi(1, 1 + 1/\beta, -1/\beta))^{-1} \geq 0.$$  

Here we see that $g(\beta) \geq 0$ for $\beta > 0$ and $f(\beta)$ have unique root in $(0, \infty)$. In fact, $f(\beta) \geq 0$ for $\beta \geq \beta_L \simeq 0.391235$. 

We show that $q_\beta(z) \prec \phi_q(z)$ for $\beta \geq \beta_L$. It suffices to show that

$$\left| \frac{(q_{\beta_L}(z))^2 - 1}{2q_{\beta_L}(z)} \right| \leq 1, \quad z \in \mathbb{D}.$$ 

We define

$$N(t) = \left| 1 - \frac{\Phi(1, 1 + 1/\beta_L, -e^{it}/\beta_L)}{2\Phi(1, 1 + 1/\beta_L, -e^{it}/\beta_L)} \right|, \quad t \in [0, \pi].$$

Then it can be seen that $N(t)$ is an increasing function of $t$ with $\max_{0 \leq t \leq \pi} N(t) = 1$. See Figure 6.

![Figure 6: Graph of $N(t)$.](image)

The Briot-Bouquet differential equations and the integral operator $F$ defined by

$$F(z) = \frac{1}{\beta} \int_0^z f^{1/\beta}(t)t^{-1}dt. \quad (4)$$

are closely related. If we set $p(z) = zF'(z)/F(z)$, then

$$p(z) + \frac{\beta zp'(z)}{p(z)} = \frac{zf'(z)}{f(z)}.$$

Using this relation we have the following corollary:

**COROLLARY 1.** Let $F$ be as defined in (4). If $zf'(z)/f(z) \prec 1 + z$, then the following hold:

(a) If $\beta \geq \beta_c \approx 0.291936$, then $zF'(z)/F(z) \prec e^z$.

(b) If $\beta \geq \beta_L \approx 1.76736$, then $zF'(z)/F(z) \prec \sqrt{1+z}$.

(c) If $\beta \geq \beta_S \approx 0.322915$, then $zF'(z)/F(z) \prec \phi_S(z)$.

(d) If $\beta \geq \beta_R \approx 4.3622$, then $zF'(z)/F(z) \prec \phi_0(z)$. 
The bounds on $\beta$ are sharp.

In [20], authors obtained a condition on $\beta$ so that $p \prec \varphi_{\text{PAR}}$ whenever $1 + \beta z p'(z)$ is subordinate to $\sqrt{1 + z}$. To determine the estimate on $\beta$ so that $p \prec \varphi_{\text{PAR}}$ whenever $1 + \beta z p'(z) \prec e^z$, following lemmas are needed.

**Lemma 2.** [9, Lemma 4, p. 3] Let $z$ be a complex number. Then,

\[ |\log(1 + z)| \geq 1 \quad \text{if and only if} \quad |z| \geq e - 1. \]

**Lemma 3.** [15, Theorem 3.4h, p. 132] Let $q$ be analytic in $\mathbb{D}$ and let $\psi$ and $\nu$ be analytic in a domain $U$ containing $q(\mathbb{D})$ with $\psi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) := zq'(z)\psi(q(z))$ and $h(z) := \nu(q(z)) + Q(z)$. Suppose that

(i) either $h$ is convex, or $Q$ is starlike univalent in $\mathbb{D}$ and

(ii) $\Re(zh'(z)/Q(z)) > 0$ for $z \in \mathbb{D}$.

If $p$ is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq U$ and $\nu(p(z)) + zp'(z)\psi(p(z)) \prec \nu(q(z)) + zq'(z)\psi(q(z))$, then $p(z) \prec q(z)$, and $q$ is best dominant.

**Theorem 2.** Let the function $p \in \mathcal{P}$ satisfying following subordination relation

\[ 1 + \beta z p'(z) \prec e^z \quad \text{for} \quad \beta \geq \pi(e - 1). \]

Then

\[ p(z) \prec \varphi_{\text{PAR}}(z). \]

**Proof.** Let the function $q : \mathbb{D} \to \mathbb{C}$ be defined by $q(z) = \varphi_{\text{PAR}}(z)$ that is univalent. Let $\nu(w) = 1$ and $\psi(w) = \beta$ be two functions. Since the function $q$ is convex univalent, then the function

\[ Q(z) = zq'(z)\psi(q(z)) = \beta zq'(z) = \beta \frac{4\sqrt{z}}{\pi^2(1 - z)} \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \]

is starlike in $\mathbb{D}$. The function $h$ is defined on $\mathbb{D}$ as

\[ h(z) := \nu(q(z)) + Q(z) = 1 + \beta \frac{4\sqrt{z}}{\pi^2(1 - z)} \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \]

satisfies the inequality $\Re(zh'(z)/Q(z)) > 0$ in $\mathbb{D}$. Note that the function $h$ is univalent. Using Lemma 3, we note that $1 + \beta z p'(z) \prec 1 + \beta z q'(z)$ implies $p \prec q$. We set $\rho(\mathbb{D}) = \{w \in \mathbb{C} : |\log w| \leq 1\}$. Therefore the required subordination $\rho \prec h$ holds if $\partial h(\mathbb{D}) \subseteq \mathbb{C} \setminus \rho(\mathbb{D})$ or $|\log h(z)| > 1$ or

\[ \left| \log \left( 1 + \beta \frac{4\sqrt{z}}{\pi^2(1 - z)} \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right| > 1 \]
holds. In view of Lemma 2, the last inequality is equivalent to
\[
\left| \frac{4\sqrt{z}}{\pi^2(1-z)} \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right|^2 > (e-1)^2.
\] (5)

For \( z = e^{i\theta} \) and \( \theta \in [-\pi, \pi] \), consider
\[
\left| \frac{4e^{i\theta/2}}{\pi^2(1-e^{i\theta})} \log \frac{1+e^{i\theta/2}}{1-e^{i\theta/2}} \right|^2 = \left| \frac{2i}{\pi^2} \csc(\theta/2) \log(i\cot(\theta/4)) \right|^2
\]
\[
= \frac{\beta^2}{\pi^2} \csc^2(\theta/2) + \frac{4\beta^2}{\pi^4} \csc^2(\theta/2) (\log(\cot(\theta/4)))^2
\]
\[
= \beta^2 g(\theta),
\] (6)

where
\[
g(\theta) = \frac{1}{\pi^2} \csc^2(\theta/2) \left( 1 + \frac{4}{\pi^2} (\log(\cot(\theta/4)))^2 \right).
\]

Using second derivative test, we note that the function \( g \) has minimum value at \( \theta = \pi \).

Therefore, inequality (6) becomes
\[
\left| \frac{4e^{i\theta/2}}{\pi^2(1-e^{i\theta})} \log \frac{1+e^{i\theta/2}}{1-e^{i\theta/2}} \right|^2 = \beta^2 g(\theta) \geq \beta^2 g(\pi) = \frac{\beta^2}{\pi^2}
\] (7)

In view of inequalities (5) and (7), we conclude that the desired subordination for \( \beta \geq \pi(e-1) \). □

As an application of Theorem 2, we have following sufficient condition for the parabolic starlike function:

**Corollary 2.** Let \( \beta \geq \pi(e-1) \). If the function \( f \in \mathcal{S} \) satisfies following subordination relation
\[
1 + \beta \frac{zf'(z)}{f(z)} \left( \frac{(zf'(z))'}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \preceq e^z
\]
then \( f \in \mathcal{S}_p^* \).

**References**


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