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NUMBER OF ZEROS OF A CERTAIN CLASS OF POLYNOMIALS IN A SPECIFIC REGION

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Abstract. We obtain results giving bounds concerning the number of zeros of polynomials with restricted coefficients in a specific region. Our results generalize and improve several well-known results concerning the number of zeros of polynomials in certain regions.

1. Introduction

The study of zeros of complex polynomials is an old theme in analytic theory of polynomials, has spawned a vast amount of research over the past millennium includes its applications both within and outside of mathematics. In addition to having numerous applications, this study has been the inspiration for much theoretical research (including being the initial motivation for modern algebra). Algebraic and analytic methods for finding zeros of a polynomial, in general, can be quite complicated, so it is desirable to put some restrictions on polynomials. This motivated the study of identifying suitable regions in the complex plane containing the zeros of a polynomial when their coefficients are restricted with special conditions. A classical result on the location of zeros of a polynomial with restricted coefficients known as Eneström-Kakeya theorem states that for a polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$, if the coefficients satisfy $0 < a_0 \leqslant a_1 \leqslant a_2 \ldots \leqslant a_n$, then $P(z)$ has all its zeros in $|z| \leqslant 1$ (see section 8.3 of [6]). In literature (see [4], [7]) there exist several generalizations of Eneström-Kakeya theorem. There is always a need for better and better results in this subject because of its application in many areas including signal processing, communication theory, cryptography, control theory, combinatorics and mathematical biology. In this paper, by using standard techniques we shall obtain results of finding the number of zeros of the polynomials with restricted coefficients in a specific region. The following result concerning the number of zeros of a polynomial in a closed disk can be found in Titchmarsh's Classic "The Theory of Functions" (see [8] page 171).

THEOREM A. *Let $F(z)$ be analytic in $|z| \leqslant R$. Let $|F(z)| \leqslant M$ in $|z| \leqslant R$ and $F(0) \neq 0$. Then for $0 < \delta < 1$, the number of zeros of $F(z)$ in the disk $|z| \leqslant R\delta$ does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|F(0)|}.$$

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By putting a restriction on the coefficients of a polynomial similar to that of the Eneström-Kakeya Theorem, Mohammad [3] used a special case of Theorem A to prove the following result:

THEOREM B. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with real coefficients such that* $a_n \geqslant a_{n-1} \geqslant a_{n-2} \geqslant \ldots \geqslant a_1 \geqslant a_0 > 0$*, then the number of zeros of* $P(z)$ *in* $|z| \leqslant \frac{1}{2}$ *does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}.$$

Dewan [1] generalized Theorem B to the polynomials with complex coefficients and obtained the following results:

THEOREM C. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients such that for some real* β, $|\arg a_i - \beta| \leqslant \alpha \leqslant \pi/2$, $i = 0, 1, 2, \ldots, n$ *and*

$$|a_n| \geqslant |a_{n-1}| \geqslant \ldots \geqslant |a_2| \geqslant |a_1| \geqslant |a_0| > 0,$$

then the number of zeros of $P(z)$ *in* $|z| \leqslant \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{|a_n|(1 + \cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

THEOREM D. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients. If* $Re(a_i) = \alpha_i$, $Im(a_i) = \beta_i$, *for* $i = 0, 1, 2, \ldots, n$

$$\alpha_n \geqslant \alpha_{n-1} \ldots \geqslant \alpha_2 \geqslant \alpha_1 \geqslant \alpha_0 > 0,$$

then the number of zeros of $P(z)$ *in* $|z| \leqslant \frac{1}{2}$ *does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{\alpha_n + \sum_{i=0}^{n} |\beta_i|}{|a_0|}.$$

In this paper, we wish to weaken the hypothesis of the above results by considering a larger class of polynomials and obtain results with a relaxed hypothesis that improves the zero bounds in several ways. Besides, our results generalize several well-known results concerning the number of zeros of polynomials. We prove the following results.

2. Main results

THEOREM 1. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$, $a_0 \neq 0$ *be a polynomial of degree n with real coefficients such that for some* $k_j \geqslant 1$, $j = 0, 1, 2, 3, \ldots, r$ *where* $0 \leqslant r \leqslant n-1$,

$$k_0 a_n \geqslant k_1 a_{n-1} \geqslant k_2 a_{n-2} \geqslant \ldots \geqslant k_{r-1} a_{n-r+1} \geqslant k_r a_{n-r} \geqslant a_{n-r-1} \ldots \geqslant a_1 \geqslant a_0,$$

then for $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in* $|z| \leqslant \delta$ *does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0|},$$

where $M = k_0(|a_n| + a_n) + 2\sum_{j=1}^{r} (k_j - 1)|a_{n-j}| - a_0 + |a_0|$.

Taking $k_j = 1,\ j = 0,1,2,\dots,r$ in Theorem 1, we obtain the following result:

COROLLARY 1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j,\ a_0 \neq 0$ be a polynomial of degree n with real coefficients such that $a_n \geqslant a_{n-1} \geqslant a_{n-2} \geqslant \dots \geqslant a_1 \geqslant a_0$, then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \leqslant \delta$ does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{|a_n| + a_n - a_0 + |a_0|}{|a_0|}.$$

REMARK 1. On setting $a_0 > 0$ and $\delta = 1/2$, Corollary 1 reduces to Theorem B.

THEOREM 2. *Let $P(z) = \sum_{j=0}^{n} a_j z^j,\ a_0 \neq 0$ be a polynomial of degree n with complex coefficients such that for some real β, $|\arg a_i - \beta| \leqslant \alpha \leqslant \pi/2,\ i = 0,1,2,\dots,n$ and $k_j \geqslant 1,\ j = 0,1,2,\dots,r$ where $0 \leqslant r \leqslant n-1$,*

$$k_0|a_n| \geqslant k_1|a_{n-1}| \geqslant \dots \geqslant k_{r-1}|a_{n-r+1}| \geqslant k_r|a_{n-r}| \geqslant |a_{n-r-1}| \geqslant \dots \geqslant |a_2| \geqslant |a_1| \geqslant |a_0|, \tag{1}$$

then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \leqslant \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{K}{|a_0|},$$

where

$$K = k_0|a_n|\,(1 + \cos\alpha + \sin\alpha) + 2\sin\alpha \left\{ \sum_{j=1}^{r} k_j |a_{n-j}| + \sum_{j=r+1}^{n} |a_{n-j}| \right\} + 2\sum_{j=1}^{r} (k_j - 1)|a_{n-j}|.$$

REMARK 2. If we take $k_j = 1,\ j = 0,1,2,\dots,r$ and $\delta = 1/2$, Theorem 2 reduces to Theorem C.

For $\alpha = \beta = 0$, we obtain following result:

COROLLARY 2. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients such that for some $k_j \geqslant 1,\ j = 0,1,2,3,\dots,r$ where $0 \leqslant r \leqslant n-1$,*

$$k_0 a_n \geqslant k_1 a_{n-1} \geqslant k_2 a_{n-2} \geqslant \dots \geqslant k_{r-1} a_{n-r+1} \geqslant k_r a_{n-r} \geqslant a_{n-r-1} \dots \geqslant a_1 \geqslant a_0 > 0,$$

then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \leqslant \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{2k_0 a_n + 2\sum_{j=1}^{r} (k_j - 1) a_{n-j}}{a_0}.$$

THEOREM 3. *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$, $a_0\neq 0$ *be a polynomial of degree* n *with complex coefficients. If* $Re(a_i)=\alpha_i$, $Im(a_i)=\beta_i$, *for* $i=0,1,2,\ldots,n$ *and* $k_j\geqslant 1$, $j=0,1,2,\ldots,r$ *where* $0\leqslant r\leqslant n-1$,

$$k_0\alpha_n\geqslant k_1\alpha_{n-1}\geqslant\ldots\geqslant k_{r-1}\alpha_{n-r+1}\geqslant k_r\alpha_{n-r}\geqslant\alpha_{r-1}\geqslant\ldots\geqslant\alpha_2\geqslant\alpha_1\geqslant\alpha_0, \tag{2}$$

then for $0<\delta<1$, *the number of zeros of* $P(z)$ *in* $|z|\leqslant\delta$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{N}{|a_0|},$$

where $N=k_0(|\alpha_n|+\alpha_n)+2\sum_{i=1}^{r}(k_i-1)|\alpha_{n-i}|+2\sum_{i=0}^{n}|\beta_i|-\alpha_0+|\alpha_0|$.

On setting $\beta_i=0$, Theorem 3 reduces to Theorem 1.

Taking $k_j=1,\ j=0,1,2,\ldots,r$ in Theorem 3, we obtain the following result:

COROLLARY 3. *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$, $a_0\neq 0$ *be a polynomial of degree* n *with complex coefficients. If* $Re(a_i)=\alpha_i$, $Im(a_i)=\beta_i$, *for* $i=0,1,2,\ldots,n$ *and*

$$\alpha_n\geqslant\alpha_{n-1}\ldots\geqslant\alpha_2\geqslant\alpha_1\geqslant\alpha_0,$$

then for $0<\delta<1$, *the number of zeros of* $P(z)$ *in* $|z|\leqslant\delta$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|\alpha_n|+\alpha_n+2\sum_{i=0}^{n}|\beta_i|-\alpha_0+|\alpha_0|}{|a_0|}.$$

REMARK 3. If we assume $\alpha_0>0$ and $\delta=1/2$, Corollary 3 reduces to Theorem D.

3. Lemmas

For the proof of these theorems, we require the following lemma which is due to Govil and Rahman [2].

LEMMA 1. *If for some real* β,

$$|\arg a_j-\beta|\leqslant\alpha\leqslant\pi/2, \qquad a_j\neq 0$$

and for some positive real numbers t_1 *and* t_2, $t_1|a_j|\geqslant t_2|a_{j-1}|$, *then*

$$|t_1a_j-t_2a_{j-1}|\leqslant(t_1|a_j|-t_2|a_{j-1}|)\cos\alpha+(t_1|a_j|+t_2|a_{j-1}|)\sin\alpha.$$

4. Proof of the theorems

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z)\\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots + (a_{n-r} - a_{n-r-1})z^{n-r} + \ldots + (a_1 - a_2)z + a_0\\
&= -a_n z^{n+1} + (k_0 a_n - k_1 a_{n-1} - (k_0-1)a_n + (k_1-1)a_{n-1})z^n\\
&\quad + (k_1 a_{n-1} - k_2 a_{n-2} - (k_1-1)a_{n-1} + (k_2-1)a_{n-2})z^{n-1}\\
&\quad + \ldots + (k_{r-1}a_{n-r+1} - k_r a_{n-r} - (k_{r-1}-1)a_{n-r+1} + (k_r-1)a_{n-r})z^{n-r-1}\\
&\quad + (k_r a_{n-r} - a_{n-r-1} - (k_r-1)a_{n-r})z^{n-r} + (a_{n-r-1} - a_{n-r-2})z^{n-r-1}\\
&\quad + \ldots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0.\\
|F(z)| &= |-a_n z^{n+1} - (k_0-1)a_n z^n + (k_0 a_n - k_1 a_{n-1})z^n + (k_1-1)a_{n-1}z^n\\
&\quad + (k_1 a_{n-1} - k_2 a_{n-2})z^{n-1} - (k_1-1)a_{n-1}z^{n-1} + (k_2-1)a_{n-2}z^{n-1} + \ldots +\\
&\quad + (k_{r-1}a_{n-r+1} - k_r a_{n-r})z^{n-r+1} - (k_{r-1}-1)a_{n-r+1}z^{n-r+1} + (k_r-1)a_{n-r}z^{n-r+1}\\
&\quad + (k_r a_{n-r} - a_{n-r-1})z^{n-r} - (k_r-1)a_{n-r}z^{n-r} + (a_{n-r-1} - a_{n-r-2})z^{n-r-1}\\
&\quad + \ldots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0|.
\end{aligned}$$

By using hypothesis, we have for $|z| \leqslant 1$,

$$\begin{aligned}
|F(z)| &\leqslant |a_n| + (k_0-1)|a_n| + k_0 a_n - k_1 a_{n-1} + (k_1-1)|a_{n-1}| + k_1 a_{n-1} - k_2 a_{n-2}\\
&\quad + (k_1-1)|a_{n-1}| + (k_2-1)|a_{n-2}| + \ldots + k_{r-1}a_{n-r+1} - k_r a_{n-r}\\
&\quad + (k_{r-1}-1)|a_{n-r+1}| + (k_r-1)|a_{n-r}| + k_r a_{n-r} - a_{n-r-1} + (k_r-1)|a_{n-r}|\\
&\quad + a_{n-r-1} - a_{n-r-2} + \ldots + a_2 - a_1 + a_1 - a_0 + |a_0|\\
&= |a_n| + (k_0-1)|a_n| + k_0 a_n + 2\sum_{j=1}^{r}(k_j-1)|a_{n-j}| - a_0 + |a_0|\\
&= k_0(|a_n| + a_n) + 2\sum_{j=1}^{r}(k_j-1)|a_{n-j}| - a_0 + |a_0|\\
&= M(say).
\end{aligned}$$

Since $F(z)$ is analytic in $|z| \leqslant 1$ with $F(0) = a_0 \neq 0$ and $|F(z)| \leqslant M$, then by Theorem A, the number of zeros of F(z) in $|z| \leqslant \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|F(0)|}.$$

Now as the number of zeros of P(z) in $|z| \leqslant \delta$ is equal to the number of zeros of F(z) in $|z| \leqslant \delta$, therefore it follows that the number of zeros of P(z) in $|z| \leqslant \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0|}.$$

This proves theorem 1. □

Proof of Theorem 2. Consider

$$\begin{aligned}G(z) &= (1-z)P(z)\\ &= -a_nz^{n+1} + (a_n - a_{n-1})z^n + \ldots + (a_{n-r} - a_{n-r-1})z^{n-r} + \ldots + (a_1 - a_2)z + a_0\\ &= -a_nz^{n+1} + (k_0a_n - k_1a_{n-1} - (k_0-1)a_n + (k_1-1)a_{n-1})z^n\\ &\quad + (k_1a_{n-1} - k_2a_{n-2} - (k_1-1)a_{n-1} + (k_2-1)a_{n-2})z^{n-1}\\ &\quad + \ldots + (k_{r-1}a_{n-r+1} - k_ra_{n-r} - (k_{r-1}-1)a_{n-r+1} + (k_r-1)a_{n-r})z^{n-r+1}\\ &\quad + (k_ra_{n-r} - a_{n-r-1} - (k_r-1)a_{n-r})z^{n-r} + (a_{n-r-1} - a_{n-r-2})z^{n-r-1}\\ &\quad + \ldots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0.\end{aligned}$$

$$\begin{aligned}|G(z)| = |&-a_nz^{n+1} - (k_0-1)a_nz^n + (k_0a_n - k_1a_{n-1})z^n + (k_1-1)a_{n-1}z^n\\ &+ (k_1a_{n-1} - k_2a_{n-2})z^{n-1} - (k_1-1)a_{n-1}z^{n-1} + (k_2-1)a_{n-1}z^{n-1} + \ldots +\\ &+ (k_{r-1}a_{n-r+1} - k_ra_{n-r})z^{n-r+1} - (k_{r-1}-1)a_{n-r+1}z^{n-r+1} + (k_r-1)a_{n-r}z^{n-r+1}\\ &+ (k_ra_{n-r} - a_{n-r-1})z^{n-r} - (k_r-1)a_{n-r}z^{n-r} + (a_{n-r-1} - a_{n_r-2})z^{n-r-1}\\ &+ \ldots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0|.\end{aligned}$$

For $|z| \leqslant 1$, we have

$$\begin{aligned}|G(z)| \leqslant\ &|a_n| + (k_0-1)|a_n| + |k_0a_n - k_1a_{n-1}| + (k_1-1)|a_{n-1}| + |k_1a_{n-1} - k_2a_{n-2}|\\ &+ (k_1-1)|a_{n-1}| + (k_2-1)|a_{n-2}| + \ldots + |k_{r-1}a_{n-r+1} - k_ra_{n-r}|\\ &+ (k_{r-1}-1)|a_{n-r+1}| + (k_r-1)|a_{n-r}| + |k_ra_{n-r} - a_{n-r-1}| + (k_r-1)|a_{n-r}|\\ &+ |a_{n-r-1} - a_{n_r-2}| + \ldots + |a_2 - a_1| + |a_1 - a_0| + |a_0|.\end{aligned}$$

In view (1), applying lemma 1, we have for $|z| \leqslant 1$,

$$\begin{aligned}|G(z)| &\leqslant k_0|a_n|(1+\cos\alpha+\sin\alpha) + 2\sin\alpha\left\{\sum_{j=1}^{r} k_j|a_{n-j}| + \sum_{j=r+1}^{n}|a_{n-j}|\right\}\\ &\quad + 2\sum_{j=1}^{r}(k_j-1)|a_{n-j}| - |a_0|(\cos\alpha+\sin\alpha-1)\\ &\leqslant k_0|a_n|(1+\cos\alpha+\sin\alpha) + 2\sin\alpha\left\{\sum_{j=1}^{r} k_j|a_{n-j}| + \sum_{j=r+1}^{n}|a_{n-j}|\right\}\\ &\quad + 2\sum_{j=1}^{r}(k_j-1)|a_{n-j}|\\ &= K(say).\end{aligned}$$

Since $G(z)$ is analytic in $|z| \leqslant 1$ with $G(0) = a_0 \neq 0$ and $|G(z)| \leqslant K$, then by Theorem A, the number of zeros of G(z) in $|z| \leqslant \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{K}{|G(0)|}.$$

Now as the number of zeros of P(z) in $|z| \leqslant \delta$ is equal to the number of zeros of G(z) in $|z| \leqslant \delta$, therefore it follows that the number of zeros of P(z) in $|z| \leqslant \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{K}{|a_0|}.$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. Consider

$$\begin{aligned}
T(z) &= (1-z)P(z) \\
&= a_n z^{n+1} + \sum_{i=1}^{n} (a_i - a_{i-1}) z^i + a_0.
\end{aligned}$$

We have for $|z| \leqslant 1$,

$$\begin{aligned}
|T(z)| &\leqslant |a_n| + \sum_{i=1}^{n} |a_i - a_{i-1}| + |a_0| \\
&\leqslant |\alpha_n| + |\beta_n| + \sum_{i=1}^{n} \{|\alpha_i - \alpha_{i-1}| + |\beta_i - \beta_{i-1}|\} + |\alpha_0| + |\beta_0| \\
&\leqslant |\alpha_n| + |\beta_n| + \sum_{i=1}^{n} |\alpha_i - \alpha_{i-1}| + \sum_{i=1}^{n} |\beta_i - \beta_{i-1}| + |\alpha_0| + |\beta_0| \\
&\leqslant |\alpha_n| + |\beta_n| + \sum_{i=0}^{n-1} |\alpha_{n-i} - \alpha_{n-i-1}| + \sum_{i=1}^{n} (|\beta_i| + |\beta_{i-1}|) + |\alpha_0| + |\beta_0| \\
&= |\alpha_n| + \sum_{i=0}^{r} |k_i \alpha_{n-i} - k_{i+1} \alpha_{n-i-1} - (k_i - 1)\alpha_{n-i} + (k_{i+1} - 1)\alpha_{n-i-1}| \\
&\quad + \sum_{i=r+1}^{n-1} |\alpha_{n-i} - \alpha_{n-i-1}| + 2\sum_{i=0}^{n} |\beta_i| + |\alpha_0|, \quad k_{r+1} = 1 \\
&\leqslant |\alpha_n| + \sum_{i=0}^{r} |k_i \alpha_{n-i} - k_{i+1} \alpha_{n-i-1}| + \sum_{i=0}^{r} |(k_i - 1)\alpha_{n-i}| + \sum_{i=0}^{r} |(k_{i+1} - 1)\alpha_{n-i-1}| \\
&\quad + \sum_{i=r+1}^{n-1} |\alpha_{n-i} - \alpha_{n-i-1}| + 2\sum_{i=0}^{n} |\beta_i| + |\alpha_0|, \quad k_{r+1} = 1.
\end{aligned}$$

By using the hypothesis, we get

$$
\begin{aligned}
|T(z)| &= |\alpha_n| + \sum_{i=0}^{r}(k_i\alpha_{n-i} - k_{i+1}\alpha_{n-i-1}) + (k_0-1)|\alpha_n| + 2\sum_{i=1}^{r}(k_i-1)|\alpha_{n-i}| \\
&\quad + \sum_{i=r+1}^{n-1}(\alpha_{n-i} - \alpha_{n-i-1}) + 2\sum_{i=0}^{n}|\beta_i| + |\alpha_0|, \quad k_{r+1} = 1 \\
&= |\alpha_n| + k_0\alpha_n + (k_0-1)|\alpha_n| + 2\sum_{i=1}^{r}(k_i-1)|\alpha_{n-i}| - \alpha_0 \\
&\quad + 2\sum_{i=0}^{n}|\beta_i| + |\alpha_0| \\
&= k_0(|\alpha_n| + \alpha_n) + 2\sum_{i=1}^{r}(k_i-1)|\alpha_{n-i}| + 2\sum_{i=0}^{n}|\beta_i| - \alpha_0 + |\alpha_0| \\
&= N(say).
\end{aligned}
$$

Since $T(z)$ is analytic in $|z| \leqslant 1$ with $T(0) = a_0 \neq 0$ and $|T(z)| \leqslant N$, then by Theorem A, the number of zeros of T(z) in $|z| \leqslant \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{N}{|T(0)|}.$$

Now as the number of zeros of P(z) in $|z| \leqslant \delta$ is equal to the number of zeros of T(z) in $|z| \leqslant \delta$, therefore it follows that the number of zeros of P(z) in $|z| \leqslant \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{N}{|a_0|}.$$

This completes the proof of Theorem 3. □

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