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INTEGRATING THE TAILS OF TWO MACLAURIN SERIES

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Abstract. The values of four improper integrals containing the squares of the tails of the Maclaurin series for the sine and cosine functions are computed using standard residue theory for contour integrals. Using a very different approach, we then provide solutions to some open questions concerning two related improper integrals.

1. Introduction

In a recent paper ([3]), Stewart used Fourier transforms to evaluate some integrals involving the squares of the tails of the Maclaurin series for sine and cosine. Our goal for the first part of this paper is to establish these same results using some basic concepts from residue theory for functions defined on the set $\mathbb{C}$ of complex numbers. In particular, we seek to verify the following results (from [3]) for each nonnegative integer n:

$$\int_0^\infty \frac{1}{x^{4n+6}}\left(\sin x-\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}x^{2k+1}\right)^2 dx=\frac{\pi/2}{(4n+5)((2n+2)!)^2}; \tag{1}$$

$$\int_0^\infty \frac{1}{x^{4n+4}}\left(\sin x-\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}x^{2k+1}\right)^2 dx=\frac{\pi/2}{(4n+3)((2n+1)!)^2}; \tag{2}$$

$$\int_0^\infty \frac{1}{x^{4n+4}}\left(\cos x-\sum_{k=0}^{n}\frac{(-1)^k}{(2k)!}x^{2k}\right)^2 dx=\frac{\pi/2}{(4n+3)((2n+1)!)^2}; \tag{3}$$

$$\int_0^\infty \frac{1}{x^{4n+2}}\left(\cos x-\sum_{k=0}^{n}\frac{(-1)^k}{(2k)!}x^{2k}\right)^2 dx=\frac{\pi/2}{(4n+1)((2n)!)^2}. \tag{4}$$

To do so, we first summarize the necessary results from residue theory and then use these results to evaluate the integrals. Some further details behind the theory are included after the computations for those readers desiring a brief review of this aspect of complex variables. After completing this portion of the paper, we answer the open questions presented in [3].

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2. Alternative evaluation for improper integrals (1) to (4)

It is assumed that the reader is familiar with analytic functions, poles, and residues. Let f be a function defined on $\mathbb{C}$ and assume that f is analytic except for a possible simple pole at $z=0$. We consider the positively oriented, closed integration path

$$\gamma_{r,\rho} = [-\rho,-r] \cup S_r \cup [r,\rho] \cup C_\rho,$$

where $\rho > r > 0$, and the paths S_r and C_ρ are semicircles with radii r and ρ, respectively, which lie in the upper half-plane. It follows that

$$0 = \int_{\gamma_{r,\rho}} f(z)\,dz = \int_{-\rho}^{-r} f(z)\,dz + \int_{S_r} f(z)\,dz + \int_r^{\rho} f(z)\,dz + \int_{C_\rho} f(z)\,dz.$$

It is known that

$$\lim_{r\to 0^+} \int_{S_r} f(z)\,dz = -i\pi\,\mathrm{Res}(f,0),$$

where $i=\sqrt{-1}$ is the imaginary unit. Assuming that the integral over C_ρ goes to 0 as ρ goes to infinity, we find that

$$\int_{-\infty}^{\infty} f(x)\,dx = i\pi\,\mathrm{Res}(f,0).$$

(We are using the Cauchy principal value of the improper integral for a function defined on the real line.) These facts are sufficient to evaluate our four integrals.

Fix a nonnegative integer n. For our first set of calculations, we define the following three functions for all real numbers x:

$$E(x) = \Big(e^{ix} - \sum_{k=0}^{2n+1} \frac{i^k}{k!} x^k\Big)^2;$$

$$S(x) = \Big(\sin x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}\Big)^2;$$

$$C(x) = \Big(\cos x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k}\Big)^2.$$

It is easy to verify that $E(x) = C(x) - S(x) + iS'(x)$. For complex numbers z, we note that $E(z)/z^{4n+4}$ is analytic in the complex plane and that

$$\mathrm{Res}\Big(\frac{E(z)}{z^{4n+5}}, 0\Big) = \frac{1}{((2n+2)!)^2}.$$

It follows that

$$\int_{-\infty}^{\infty} \frac{E(x)}{x^{4n+4}}\,dx = 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{E(x)}{x^{4n+5}}\,dx = \frac{i\pi}{((2n+2)!)^2}.$$

Equating real and imaginary parts for these integrals, we find that

$$\int_{-\infty}^{\infty} \frac{C(x)-S(x)}{x^{4n+4}}\,dx = 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{S'(x)}{x^{4n+5}}\,dx = \frac{\pi}{((2n+2)!)^2}.$$

Noting that the integrands for both integrals are even functions, it follows that

$$\int_0^{\infty} \frac{C(x)}{x^{4n+4}}\,dx = \int_0^{\infty} \frac{S(x)}{x^{4n+4}}\,dx \qquad \text{and} \qquad \int_0^{\infty} \frac{S'(x)}{x^{4n+5}}\,dx = \frac{\pi/2}{((2n+2)!)^2}.$$

The equality on the left explains why the integrals in equations (2) and (3) are equal. Using integration by parts on the integral on the right, we have

$$\frac{\pi/2}{((2n+2)!)^2} = \int_0^{\infty} \frac{S'(x)}{x^{4n+5}}\,dx = \frac{S(x)}{x^{4n+5}}\bigg|_0^{\infty} + (4n+5)\int_0^{\infty} \frac{S(x)}{x^{4n+6}}\,dx = (4n+5)\int_0^{\infty} \frac{S(x)}{x^{4n+6}}\,dx.$$

(Note that $S(x)$ behaves like x^{4n+6} near 0 and like x^{4n+2} as x goes to infinity.) The value of the integral in equation (1) then follows immediately.

Before moving on to our next set of calculations, we note that

$$\int_0^{\infty} \frac{1}{x^{4n+5}}\left(\sin x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}x^{2k+1}\right)\left(\cos x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}x^{2k}\right)dx = \frac{\pi/4}{((2n+2)!)^2}.$$

This equality is established by noting the form of $S'(x)$ in the previous integral equation. It is interesting to compare this result to the integral given in Theorem 7.

We now begin anew. Fix a nonnegative integer n. For this second set of calculations, we define the following three functions (note the slight changes in the functions E and S) for all real numbers x:

$$E(x) = \left(e^{ix} - \sum_{k=0}^{2n} \frac{i^k}{k!}x^k\right)^2;$$

$$S(x) = \left(\sin x - \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)!}x^{2k+1}\right)^2;$$

$$C(x) = \left(\cos x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}x^{2k}\right)^2.$$

It is easy to verify that $E(x) = C(x) - S(x) - iC'(x)$. For complex numbers z, we note that $E(z)/z^{4n+2}$ is analytic in the complex plane and that

$$\operatorname{Res}\left(\frac{E(z)}{z^{4n+3}}, 0\right) = \frac{-1}{((2n+1)!)^2}.$$

It follows that

$$\int_{-\infty}^{\infty} \frac{E(x)}{x^{4n+2}}\,dx = 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{E(x)}{x^{4n+3}}\,dx = \frac{-i\pi}{((2n+1)!)^2}.$$

Equating real and imaginary parts for these integrals, we find that

$$\int_{-\infty}^{\infty} \frac{C(x)-S(x)}{x^{4n+2}}\,dx = 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{C'(x)}{x^{4n+3}}\,dx = \frac{\pi}{((2n+1)!)^2}.$$

Noting that the integrands for both integrals are even functions, it follows that

$$\int_0^{\infty} \frac{C(x)}{x^{4n+2}}\,dx = \int_0^{\infty} \frac{S(x)}{x^{4n+2}}\,dx \qquad \text{and} \qquad \int_0^{\infty} \frac{C'(x)}{x^{4n+3}}\,dx = \frac{\pi/2}{((2n+1)!)^2}.$$

The equality on the left explains why the integrals in equations (1) and (4) are equal when a shift from n to $n+1$ is made. Using integration by parts on the integral on the right, we have

$$\frac{\pi/2}{((2n+1)!)^2} = \int_0^{\infty} \frac{C'(x)}{x^{4n+3}}\,dx = \frac{C(x)}{x^{4n+3}}\Big|_0^{\infty} + (4n+3)\int_0^{\infty} \frac{C(x)}{x^{4n+4}}\,dx = (4n+3)\int_0^{\infty} \frac{C(x)}{x^{4n+4}}\,dx.$$

(Note that $C(x)$ behaves like x^{4n+4} near 0 and like x^{4n} as x goes to infinity.) The value of the integral in equation (3) then follows immediately. Referring to the first set of calculations made earlier, we know that this integral value also gives equation (2). We have thus verified the values of all four integrals.

As with our first set of computations, we observe that

$$\int_0^{\infty} \frac{1}{x^{4n+3}}\left(\sin x - \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)!}x^{2k+1}\right)\left(\cos x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}x^{2k}\right)dx = \frac{-\pi/4}{((2n+1)!)^2}$$

by noting the form of $C'(x)$ in the previous integral equation. It is interesting to compare this result to the integral given in Theorem 6.

We now include a few of the necessary ideas from residue theory. There are many references for these facts, but we refer to the textbook [2] and include the key results that are needed for the computations that were used above. Further details and proofs of these results can be found in [2]. In Section 6.5 of [2], we have the following result (referring to the contours defined at the beginning of this section):

THEOREM 1. *If f has a simple pole at 0, then* $\displaystyle\lim_{r\to 0^+}\int_{S_r} f(z)\,dz = -i\pi \mathrm{Res}(f,0)$.

The other two results that are needed are recorded below (see Sections 6.4 and 6.3 of [2], respectively):

THEOREM 2. *If $m>0$ and P/Q is the ratio of two polynomials such that the degree of Q is greater than the degree of P, then* $\displaystyle\lim_{\rho\to\infty}\int_{C_\rho} \frac{e^{imz}P(z)}{Q(z)}\,dz = 0$.

THEOREM 3. *If P/Q is the ratio of two polynomials such that the degree of Q is at least two more than the degree of P, then* $\displaystyle\lim_{\rho\to\infty}\int_{C_\rho} \frac{P(z)}{Q(z)}\,dz = 0$.

The result of Theorem 2 is often called Jordan's Lemma.

The application of Theorem 1 in the integral computations needed above is clear. To see how Theorems 2 and 3 come into play, we consider just one of the scenarios, namely, the function

$$\frac{1}{z^{4n+5}}\left(e^{iz}-\sum_{k=0}^{2n+1}\frac{i^k}{k!}z^k\right)^2=\frac{e^{2iz}}{z^{4n+5}}-2\sum_{k=0}^{2n+1}\frac{i^k e^{iz}}{k!z^{4n+5-k}}+\frac{1}{z^{4n+5}}\left(\sum_{k=0}^{2n+1}\frac{i^k}{k!}z^k\right)^2;$$

the verifications for the other three complex integrands are similar. The terms involving the exponential clearly satisfy the hypotheses of Theorem 2. For the last term, the degree of the numerator is $4n+2$ and the degree of the denominator is $4n+5$ so the conditions for Theorem 3 are met. Together, these facts justify the computations made in the previous pages.

3. Solutions to some open problems

In the paper [3], Stewart made the conjectures that

$$\int_0^\infty \frac{1}{x^{4n+3}}\left(\cos x-\sum_{k=0}^{n}\frac{(-1)^k}{(2k)!}x^{2k}\right)^2 dx=\frac{2^{4n+1}}{(4n+2)!}\ln 2-b_n;$$

$$\int_0^\infty \frac{1}{x^{4n+5}}\left(\sin x-\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}x^{2k+1}\right)^2 dx=\frac{2^{4n+3}}{(4n+4)!}\ln 2-\beta_n;$$

for each nonnegative integer n, where b_n is a nonnegative rational number and β_n is a positive rational number. Since the integrands in each of these cases are odd functions, integrating over the entire real line is not of any help. In what follows, we will verify the expressions for the coefficients for the $\ln 2$ terms and find (with proof) expressions for the rational parts of the integrals. In our work, we make use of the well-known cosine integral function (see [1])

$$\operatorname{Ci}(x)=-\int_x^\infty \frac{\cos t}{t}\,dt=\gamma+\ln x-\int_0^x\frac{1-\cos t}{t}\,dt \quad \text{for which} \quad \operatorname{Ci}'(x)=\frac{\cos x}{x},$$

where γ is Euler's constant. We note that

$$\lim_{a\to 0^+}\left(\operatorname{Ci}(2a)-\operatorname{Ci}(a)\right)=\lim_{a\to 0^+}\left(\left(\gamma+\ln(2a)-\int_0^{2a}\frac{1-\cos t}{t}\,dt\right)\right.$$
$$\left.-\left(\gamma+\ln a-\int_0^{a}\frac{1-\cos t}{t}\,dt\right)\right)=\ln 2.$$

In addition, it is easy to verify that $\lim_{x\to\infty}\operatorname{Ci}(x)=0$.

We first prove two lemmas in which the following functions appear. For each positive integer n, let

$$U_n(x)=\sum_{k=1}^{n}\frac{(-1)^{k+1}(2k-1)!}{x^{2k}} \qquad \text{and} \qquad V_n(x)=\sum_{k=1}^{n}\frac{(-1)^k(2k-2)!}{x^{2k-1}}.$$

We note in passing that $V_n'(x) = U_n(x)$, but we will not be using this result. The first few explicit expressions for these functions are

$$U_1(x) = \frac{1}{x^2}; \qquad V_1(x) = -\frac{1}{x};$$
$$U_2(x) = \frac{1}{x^2} - \frac{6}{x^4}; \qquad V_2(x) = -\frac{1}{x} + \frac{2}{x^3};$$
$$U_3(x) = \frac{1}{x^2} - \frac{6}{x^4} + \frac{120}{x^6}; \qquad V_3(x) = -\frac{1}{x} + \frac{2}{x^3} - \frac{24}{x^5}.$$

For ease of writing, we omit the constant of integration for the indefinite integrals that appear in our work.

LEMMA 4. *For each positive integer n, we have*

$$\int \frac{\cos x}{x^{2n+1}}\,dx = \frac{(-1)^n}{(2n)!}\Big(U_n(x)\cos x + V_n(x)\sin x + \mathrm{Ci}(x)\Big);$$
$$\int \frac{\sin x}{x^{2n+2}}\,dx = \frac{(-1)^n}{(2n+1)!}\Big(U_n(x)\cos x + V_{n+1}(x)\sin x + \mathrm{Ci}(x)\Big).$$

Proof. For the integral involving $\cos x$, we proceed by induction. For $n = 1$, we use integration by parts twice to obtain

$$\begin{aligned}\int \frac{\cos x}{x^3}\,dx &= -\frac{\cos x}{2x^2} + \int \frac{\sin x}{-2x^2}\,dx \\ &= -\frac{\cos x}{2x^2} + \frac{\sin x}{2x} - \int \frac{\cos x}{2x}\,dx \\ &= \frac{(-1)^1}{2!}\Big(U_1(x)\cos x + V_1(x)\sin x + \mathrm{Ci}(x)\Big),\end{aligned}$$

the correct form for $n = 1$. Similarly, integration by parts yields

$$\begin{aligned}\int \frac{\cos x}{x^{2n+3}}\,dx &= -\frac{\cos x}{(2n+2)x^{2n+2}} + \int \frac{\sin x}{-(2n+2)x^{2n+2}}\,dx \\ &= -\frac{\cos x}{(2n+2)x^{2n+2}} + \frac{\sin x}{(2n+2)(2n+1)x^{2n+1}} - \frac{1}{(2n+2)(2n+1)}\int \frac{\cos x}{x^{2n+1}}\,dx.\end{aligned}$$

Assuming that the cosine equation given in the lemma holds for some positive integer n, the coefficient of $\cos x$ in this new integral satisfies

$$\begin{aligned}-\frac{1}{(2n+2)(2n+1)}\cdot\frac{(-1)^n}{(2n)!}\sum_{k=1}^{n}\frac{(-1)^{k+1}(2k-1)!}{x^{2k}} - \frac{1}{(2n+2)x^{2n+2}} \\ = \frac{(-1)^{n+1}}{(2n+2)!}\sum_{k=1}^{n}\frac{(-1)^{k+1}(2k-1)!}{x^{2k}} - \frac{(2n+1)!}{(2n+2)!\,x^{2n+2}} \\ = \frac{(-1)^{n+1}}{(2n+2)!}\sum_{k=1}^{n+1}\frac{(-1)^{k+1}(2k-1)!}{x^{2k}} = \frac{(-1)^{n+1}}{(2n+2)!}U_{n+1}(x),\end{aligned}$$

the coefficient of $\sin x$ satisfies

$$
\begin{aligned}
-\frac{1}{(2n+2)(2n+1)}\cdot\frac{(-1)^n}{(2n)!}\sum_{k=1}^{n}\frac{(-1)^k(2k-2)!}{x^{2k-1}}+\frac{1}{(2n+2)(2n+1)x^{2n+1}}
&=\frac{(-1)^{n+1}}{(2n+2)!}\sum_{k=1}^{n}\frac{(-1)^k(2k-2)!}{x^{2k-1}}+\frac{(2n)!}{(2n+2)!\,x^{2n+1}}\\
&=\frac{(-1)^{n+1}}{(2n+2)!}\sum_{k=1}^{n+1}\frac{(-1)^k(2k-2)!}{x^{2k-1}}=\frac{(-1)^{n+1}}{(2n+2)!}V_{n+1}(x),
\end{aligned}
$$

and the coefficient of $\mathrm{Ci}(x)$ satisfies

$$-\frac{1}{(2n+2)(2n+1)}\cdot\frac{(-1)^n}{(2n)!}=\frac{(-1)^{n+1}}{(2n+2)!};$$

all of which are the correct forms for $n+1$. The cosine result now follows by induction.

The integral involving $\sin x$ then follows from this cosine result. Using the fact that

$$
\begin{aligned}
\frac{-1}{(2n+1)x^{2n+1}}+\frac{(-1)^n}{(2n+1)!}V_n(x)&=\frac{(-1)^n}{(2n+1)!}\left(\frac{(-1)^{n+1}(2n)!}{x^{2n+1}}+V_n(x)\right)\\
&=\frac{(-1)^n}{(2n+1)!}V_{n+1}(x)
\end{aligned}
$$

and integration by parts, we find that

$$
\begin{aligned}
\int\frac{\sin x}{x^{2n+2}}\,dx&=\frac{-\sin x}{(2n+1)x^{2n+1}}+\frac{1}{2n+1}\int\frac{\cos x}{x^{2n+1}}\,dx\\
&=\frac{-\sin x}{(2n+1)x^{2n+1}}+\frac{1}{2n+1}\cdot\frac{(-1)^n}{(2n)!}\Big(U_n(x)\cos x+V_n(x)\sin x+\mathrm{Ci}(x)\Big)\\
&=\frac{(-1)^n}{(2n+1)!}\Big(U_n(x)\cos x+V_{n+1}(x)\sin x+\mathrm{Ci}(x)\Big).
\end{aligned}
$$

This completes the proof. □

For what follows, we make use of the terms $h_k=\sum_{j=1}^{k}1/j$ and $o_k=\sum_{j=1}^{k}1/(2j-1)$ for each positive integer k; the h_k terms are known as harmonic numbers. Note that

$$o_k=\sum_{j=1}^{k}\frac{1}{2j-1}=\sum_{j=1}^{2k}\frac{1}{j}-\frac{1}{2}\sum_{j=1}^{k}\frac{1}{j}=h_{2k}-\frac{1}{2}h_k$$

for all k. It follows that $o_k+\frac{1}{2}h_k=h_{2k}$ and $o_{k+1}+\frac{1}{2}h_k=h_{2k+1}$ for all k.

LEMMA 5. *Let n be a positive integer.*

(a) For each integer k that satisfies $1\leqslant k\leqslant 2n+1$, the coefficient for the constant term in the Laurent series for the function $U_k(x)\cos x$ is $-\frac{1}{2}h_k$.

(b) *For each integer k that satisfies $1 \leqslant k \leqslant 2n+2$, the coefficient for the constant term in the Laurent series for the function $V_k(x)\sin x$ is $-o_k$.*

(c) *For each integer k that satisfies $1 \leqslant k \leqslant 2n+1$, the coefficient for the constant term in the Laurent series for the function $U_k(x)\cos x + V_k(x)\sin x$ is $-h_{2k}$.*

(d) *For each integer k that satisfies $1 \leqslant k \leqslant 2n+1$, the coefficient for the constant term in the Laurent series for the function $U_k(x)\cos x + V_{k+1}(x)\sin x$ is $-h_{2k+1}$.*

Proof. For appropriate values of n and k, the constant term in the Laurent series for the function $U_k(x)\cos x$ satisfies

$$\Big(\sum_{j=1}^{k} \frac{(-1)^{j+1}(2j-1)!}{x^{2j}}\Big)\Big(\sum_{m=0}^{2n+1} \frac{(-1)^m}{(2m)!}x^{2m}\Big) = \sum_{j=1}^{k} \frac{(-1)^{j+1}(2j-1)!(-1)^j}{(2j)!}$$
$$= -\frac{1}{2}\sum_{j=1}^{k}\frac{1}{j} = -\frac{1}{2}h_k.$$

Note that we have just used a finite portion of the Maclaurin series for $\cos x$ as the terms involving higher powers generate polynomial terms in the product. Similarly, the constant term for $V_k(x)\sin x$ satisfies

$$\Big(\sum_{j=1}^{k} \frac{(-1)^{j}(2j-2)!}{x^{2j-1}}\Big)\Big(\sum_{m=0}^{2n+1} \frac{(-1)^m}{(2m+1)!}x^{2m+1}\Big) = \sum_{j=1}^{k} \frac{(-1)^{j}(2j-2)!(-1)^{j-1}}{(2j-1)!}$$
$$= -\sum_{j=1}^{k}\frac{1}{2j-1} = -o_k.$$

Adding the results for parts (a) and (b) then gives parts (c) and (d). We note in passing that it is easily shown that the constant terms that appear in the Laurent series for the functions $U_k(x)\cos x + V_k(x)\sin x$ and $U_k(2x)\cos(2x) + V_k(2x)\sin(2x)$ are equal. □

THEOREM 6. *The value of the integral* $\displaystyle\int_0^\infty \frac{1}{x^{4n+3}}\Big(\cos x - \sum_{k=0}^{n}\frac{(-1)^k}{(2k)!}x^{2k}\Big)^2 dx$ *is*

$$\frac{2}{(4n+2)!}\Big(2^{4n}\ln 2 - 2^{4n}h_{4n+2} + \sum_{k=n+1}^{2n+1}\binom{4n+2}{2k}h_{2k}\Big)$$

for each nonnegative integer n.

Proof. Fix a nonnegative integer n. Since the integrand behaves like $1/x^3$ for large values of x and like x for values of x near 0, the improper integral is guaranteed to exist. Letting $A(x)$ represent an antiderivative of the integrand (where the constant of integration is 0), we want to compute

$$\lim_{b\to\infty} A(b) - \lim_{a\to 0^+} A(a) = -\lim_{a\to 0^+} A(a).$$

We first obtain the following three antiderivatives:

$$\int \frac{\cos^2 x}{x^{4n+3}}\,dx, \qquad \int \frac{-2\cos x}{x^{4n+3}} \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k}\,dx, \qquad \int \frac{1}{x^{4n+3}} \Big(\sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k}\Big)^2 dx.$$

For the first integral, we have

$$\int \frac{\cos^2 x}{x^{4n+3}}\,dx = \frac{1}{2}\int \frac{1+\cos(2x)}{x^{4n+3}}\,dx = \frac{-1}{2(4n+2)x^{4n+2}} + \int \frac{\cos(2x)}{2x^{4n+3}}\,dx$$

and, using Lemma 4, we obtain

$$\begin{aligned}
\int \frac{\cos(2x)}{2x^{4n+3}}\,dx &= \int \frac{\cos t}{2(t/2)^{4n+3}} \cdot \frac{dt}{2} \\
&= 2^{4n+1} \int \frac{\cos t}{t^{4n+3}}\,dt \\
&= 2^{4n+1} \cdot \frac{(-1)^{2n+1}}{(4n+2)!} \Big(U_{2n+1}(t)\cos t + V_{2n+1}(t)\sin t + \mathrm{Ci}(t)\Big) \\
&= \frac{-2^{4n+1}}{(4n+2)!} \Big(U_{2n+1}(2x)\cos(2x) + V_{2n+1}(2x)\sin(2x) + \mathrm{Ci}(2x)\Big).
\end{aligned}$$

Combining the first part of the $\cos^2 x$ integral with the third integral, we have

$$\frac{-1}{2(4n+2)x^{4n+2}} + \int \frac{1}{x^{4n+3}} \Big(\sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k}\Big)^2 dx = \frac{P_n(x)}{x^{4n+2}},$$

where $P_n(x)$ is a polynomial of degree $4n$. For the middle integral, we find that

$$\begin{aligned}
\int \frac{\cos x}{x^{4n+3}} \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k}\,dx &= \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} \int \frac{\cos x}{x^{4n-2k+3}}\,dx \\
&= \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} \cdot \frac{(-1)^{2n-k+1}}{(4n-2k+2)!} \Big(U_{2n-k+1}(x)\cos x + V_{2n-k+1}(x)\sin x + \mathrm{Ci}(x)\Big) \\
&= \frac{-1}{(4n+2)!} \sum_{k=0}^{n} \binom{4n+2}{2(2n-k+1)} \Big(U_{2n-k+1}(x)\cos x + V_{2n-k+1}(x)\sin x + \mathrm{Ci}(x)\Big) \\
&= \frac{-1}{(4n+2)!} \sum_{k=n+1}^{2n+1} \binom{4n+2}{2k} \Big(U_k(x)\cos x + V_k(x)\sin x + \mathrm{Ci}(x)\Big).
\end{aligned}$$

It follows that $A(x)$ has the form

$$\begin{aligned}
A(x) = \frac{-2^{4n+1}}{(4n+2)!} &\Big(U_{2n+1}(2x)\cos(2x) + V_{2n+1}(2x)\sin(2x) + \mathrm{Ci}(2x)\Big) + \frac{P_n(x)}{x^{4n+2}} \\
&+ \frac{2}{(4n+2)!} \sum_{k=n+1}^{2n+1} \binom{4n+2}{2k} \Big(U_k(x)\cos x + V_k(x)\sin x + \mathrm{Ci}(x)\Big).
\end{aligned}$$

It is easy to see that $\lim_{b\to\infty} A(b) = 0$. Using basic properties of binomial coefficients, we note that

$$\frac{2}{(4n+2)!}\sum_{k=n+1}^{2n+1}\binom{4n+2}{2k} = \frac{1}{(4n+2)!}\sum_{k=0}^{2n+1}\binom{4n+2}{2k} = \frac{2^{4n+1}}{(4n+2)!},$$

which allows us to express $A(x)$ as

$$\begin{aligned}A(x) = {}&\frac{-2^{4n+1}}{(4n+2)!}\Big(U_{2n+1}(2x)\cos(2x)+V_{2n+1}(2x)\sin(2x)\Big)+\frac{P_n(x)}{x^{4n+2}}\\&+\frac{2^{4n+1}}{(4n+2)!}\Big(\mathrm{Ci}(x)-\mathrm{Ci}(2x)\Big)\\&+\frac{2}{(4n+2)!}\sum_{k=n+1}^{2n+1}\binom{4n+2}{2k}\Big(U_k(x)\cos x+V_k(x)\sin x\Big).\end{aligned}$$

The terms in the definite integral involving $\ln 2$ depend solely on the parts of $A(x)$ that contain the Ci function terms. This value is given by

$$\frac{2^{4n+1}}{(4n+2)!}\Big(\mathrm{Ci}(x)-\mathrm{Ci}(2x)\Big)\Big|_0^\infty = 0+\frac{2^{4n+1}}{(4n+2)!}\lim_{a\to 0^+}\Big(\mathrm{Ci}(2a)-\mathrm{Ci}(a)\Big) = \frac{2^{4n+1}}{(4n+2)!}\ln 2.$$

This verifies the part of Theorem 6 giving the coefficient of $\ln 2$.

Omitting the Ci terms from the expression for $A(x)$, we can represent each of the remaining terms as a Laurent series about the origin. Since the limit at 0 is known to exist, all of the terms with negative exponents must cancel and the desired limit at 0 is the constant term. Returning to the expression for $A(x)$ and using Lemma 5, the constants corresponding to the first and last terms are

$$\frac{2^{4n+1}}{(4n+2)!}h_{4n+2} \qquad \text{and} \qquad \frac{-2}{(4n+2)!}\sum_{k=n+1}^{2n+1}\binom{4n+2}{2k}h_{2k},$$

respectively. Adding these two results and multiplying by -1 yields

$$\frac{2}{(4n+2)!}\sum_{k=n+1}^{2n+1}\binom{4n+2}{2k}h_{2k}-\frac{2^{4n+1}}{(4n+2)!}h_{4n+2}$$

as the value of the rational part for $-\lim_{a\to 0^+} A(a)$. Noting that

$$2^{4n}h_{4n+2}-\sum_{k=n+1}^{2n+1}\binom{4n+2}{2k}h_{2k} = \sum_{k=n+1}^{2n+1}\binom{4n+2}{2k}h_{4n+2}-\sum_{k=n+1}^{2n+1}\binom{4n+2}{2k}h_{2k},$$

we see that the rational part of the integral is negative for all $n \geqslant 1$. $\square$

THEOREM 7. *The value of the integral* $\int_0^\infty \frac{1}{x^{4n+5}}\left(\sin x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}x^{2k+1}\right)^2 dx$ *is*

$$\frac{2}{(4n+4)!}\left(2^{4n+2}\ln 2 - 2^{4n+2}h_{4n+4} + \sum_{k=n+1}^{2n+1}\binom{4n+4}{2k+1}h_{2k+1}\right)$$

for each nonnegative integer n.

Proof. The proof is similar to the proof of Theorem 6, but we include the details for completeness. Fix a nonnegative integer n. Since the integrand behaves like $1/x^3$ for large values of x and like x for values of x near 0, the improper integral is guaranteed to exist. Letting $B(x)$ represent an antiderivative of the integrand (where the constant of integration is 0), we want to compute

$$\lim_{b\to\infty} B(b) - \lim_{a\to 0^+} B(a) = -\lim_{a\to 0^+} B(a).$$

We first obtain the following three antiderivatives:

$$\int \frac{\sin^2 x}{x^{4n+5}}\,dx, \quad \int \frac{-2\sin x}{x^{4n+5}}\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}x^{2k+1}\,dx, \quad \int \frac{1}{x^{4n+5}}\left(\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}x^{2k+1}\right)^2 dx.$$

For the first integral, we have

$$\int \frac{\sin^2 x}{x^{4n+5}}\,dx = \frac{1}{2}\int \frac{1-\cos(2x)}{x^{4n+5}}\,dx = \frac{-1}{2(4n+4)x^{4n+4}} - \int \frac{\cos(2x)}{2x^{4n+5}}\,dx$$

and, using Lemma 4, we obtain

$$\begin{aligned}
-\int \frac{\cos(2x)}{2x^{4n+5}}\,dx &= -\int \frac{\cos t}{2(t/2)^{4n+5}}\cdot\frac{dt}{2} \\
&= -2^{4n+3}\int \frac{\cos t}{t^{4n+5}}\,dt \\
&= -2^{4n+3}\cdot\frac{(-1)^{2n+2}}{(4n+4)!}\Big(U_{2n+2}(t)\cos t + V_{2n+2}(t)\sin t + \mathrm{Ci}(t)\Big) \\
&= \frac{-2^{4n+3}}{(4n+4)!}\Big(U_{2n+2}(2x)\cos(2x) + V_{2n+2}(2x)\sin(2x) + \mathrm{Ci}(2x)\Big).
\end{aligned}$$

Combining the first part of the $\sin^2 x$ integral with the third integral, we have

$$\frac{-1}{2(4n+4)x^{4n+4}} + \int \frac{1}{x^{4n+5}}\left(\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}x^{2k+1}\right)^2 dx = \frac{Q_n(x)}{x^{4n+4}},$$

where $Q_n(x)$ is a polynomial of degree $4n+2$. For the middle integral, we find that

$$\int \frac{\sin x}{x^{4n+5}} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}\, dx = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} \int \frac{\sin x}{x^{4n-2k+4}}\, dx$$
$$= \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} \cdot \frac{(-1)^{2n-k+1}}{(4n-2k+3)!} \Big(U_{2n-k+1}(x)\cos x + V_{2n-k+2}(x)\sin x + \operatorname{Ci}(x) \Big)$$
$$= \frac{-1}{(4n+4)!} \sum_{k=0}^{n} \binom{4n+4}{2(2n-k+1)+1} \Big(U_{2n-k+1}(x)\cos x + V_{2n-k+2}(x)\sin x + \operatorname{Ci}(x) \Big)$$
$$= \frac{-1}{(4n+4)!} \sum_{k=n+1}^{2n+1} \binom{4n+4}{2k+1} \Big(U_k(x)\cos x + V_{k+1}(x)\sin x + \operatorname{Ci}(x) \Big).$$

It follows that $B(x)$ has the form

$$B(x) = \frac{-2^{4n+3}}{(4n+4)!} \Big(U_{2n+2}(2x)\cos(2x) + V_{2n+2}(2x)\sin(2x) + \operatorname{Ci}(2x) \Big) + \frac{Q_n(x)}{x^{4n+4}}$$
$$+ \frac{2}{(4n+4)!} \sum_{k=n+1}^{2n+1} \binom{4n+4}{2k+1} \Big(U_k(x)\cos x + V_{k+1}(x)\sin x + \operatorname{Ci}(x) \Big).$$

It is easy to see that $\lim_{b\to\infty} B(b) = 0$. Using the fact that

$$\frac{2}{(4n+4)!} \sum_{k=n+1}^{2n+1} \binom{4n+4}{2k+1} = \frac{1}{(4n+4)!} \sum_{k=0}^{2n+1} \binom{4n+4}{2k+1} = \frac{2^{4n+3}}{(4n+4)!},$$

we can express $B(x)$ as

$$B(x) = \frac{-2^{4n+3}}{(4n+4)!} \Big(U_{2n+2}(2x)\cos(2x) + V_{2n+2}(2x)\sin(2x) \Big) + \frac{Q_n(x)}{x^{4n+4}}$$
$$+ \frac{2^{4n+3}}{(4n+4)!} \Big(\operatorname{Ci}(x) - \operatorname{Ci}(2x) \Big)$$
$$+ \frac{2}{(4n+4)!} \sum_{k=n+1}^{2n+1} \binom{4n+4}{2k+1} \Big(U_k(x)\cos x + V_{k+1}(x)\sin x \Big).$$

The terms in the definite integral involving $\ln 2$ depend solely on the parts of $B(x)$ that contain the Ci function terms. We thus find that

$$\frac{2^{4n+3}}{(4n+4)!} \Big(\operatorname{Ci}(x) - \operatorname{Ci}(2x) \Big) \Big|_0^{\infty} = 0 + \frac{2^{4n+3}}{(4n+4)!} \lim_{a\to 0^+} \Big(\operatorname{Ci}(2a) - \operatorname{Ci}(a) \Big) = \frac{2^{4n+3}}{(4n+4)!} \ln 2$$

gives the coefficient of $\ln 2$ for our integral.

Returning to the expression for $B(x)$ and omitting the Ci terms, we can represent each of the remaining terms as a Laurent series about the origin. Since the limit at 0 is known to exist, all of the terms with negative exponents must cancel and the desired limit at 0 is the constant term. By Lemma 5, we find that the constants corresponding to the first and last terms are

$$\frac{2^{4n+3}}{(4n+4)!}h_{4n+4} \qquad \text{and} \qquad \frac{-2}{(4n+4)!}\sum_{k=n+1}^{2n+1}\binom{4n+4}{2k+1}h_{2k+1},$$

respectively. Adding these two results and multiplying by -1 yields

$$\frac{2}{(4n+4)!}\sum_{k=n+1}^{2n+1}\binom{4n+4}{2k+1}h_{2k+1}-\frac{2^{4n+3}}{(4n+4)!}h_{4n+4}$$

for the portion of $-\lim_{a\to0^+}B(a)$ that does not involve $\ln 2$. A proof that this number is negative (see the end of the proof of Theorem 6) is left to the reader. □

Since computer algebra systems such as Maple or Wolfram Alpha generate the exact values of these integrals for small values of n, it is possible to check the equations given in Theorems 6 and 7 for such cases. As expected, the results agree with the formulas given for the integrals.

Given the nature of the proofs for Theorems 6 and 7, it is natural to ask if the same approach will work for the four integrals listed at the beginning of the paper. The answer is yes. As might be expected the details are a bit tedious but not quite as tedious as those found in the proofs of Theorems 6 and 7. We offer some observations for the reader interested in pursuing this path. It should come as no surprise that the sine integral function $\mathrm{Si}(x)$ appears. It is defined by (see [1])

$$\mathrm{Si}(x)=\int_0^x\frac{\sin t}{t}\,dt \qquad \text{and satisfies} \qquad \lim_{a\to0^+}\mathrm{Si}(a)=0, \quad \lim_{b\to\infty}\mathrm{Si}(b)=\frac{\pi}{2}.$$

The last fact explains why π appears in the values of these integrals; it does take some effort to verify the appropriate coefficient using this method. Since the integrands in every one of the four cases are even functions, the antiderivatives (using zero for the constant of integration) are odd functions. This means that the coefficient of the constant term in the Laurent series must be 0. Hence, there are no rational terms associated with the values of these integrals.

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