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ON STATISTICAL ω–LIMIT SETS IN A DISCRETE DYNAMICAL SYSTEM

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Abstract. Following the concept of statistical convergence, in this paper we introduce two more subtle notions, viz. statistical ω-limit set and statistical ω-cluster set than the general ω-limit set in a discrete dynamical system of a continuous function and study some properties related to these two points.

1. Introduction

Discrete dynamical systems are used to model many phenomena coming from different subjects, viz. Biology, Economics, Engineering etc. and the idea of ω-limit set is an important and interesting concept defined in a dynamical system. The notion of ω-limit sets have been studied in different aspects in the literature ([1], [2], [3], [4], [5], [6], [14] etc.). It helps us to gain an idea about the behaviour of the dynamical system. Let (X,γ) be a compact metric space and $f : X \longrightarrow X$ is a map. For some point $x \in X$, the forward orbit of x is given by

$$Orb^{+}(x) = \{f^{i}(x) : i \in \mathbb{Z}, i \geqslant 0\}.$$

Here, f° is the identity map on X. Similarly, the backward orbit of x is given by, $Orb^{-}(x) = \{x_{-i}\}_{i\geqslant 0} \subset X$, where $f(x_{-i}) = x_{-i+1}$, for $i > 0$.

For a sequence of points $\{x_n\}$, the ω-limit set is the collection of all accumulation points of $\{x_n\}$. It is given by

$$\omega(\{x_n\}) = \bigcap_{n=0}^{\infty} \overline{\{x_k : k \geqslant n\}}.$$

If $\{x_n\}$ is the orbit of a point $x \in X$ for a map $f : X \to X$, then the ω-limit set is given by,

$$\omega(x,f) = \bigcap_{n=0}^{\infty} \overline{\{f^{k}(x) : k \geqslant n\}}.$$

Our aim is to study the concept of ω-limit points in a more subtle way. For this purpose we adopt the notion of statistical limit points to describe the ω-limit sets in this article. In discrete dynamical system, orbits are some sequences of real numbers. We can say that any orbit spends "almost all" the time in some ε-neighbourhood of an ω-limit

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point. The term “almost all” in general means that only a finite number of elements of the orbit may remain out side the ε-neighbourhood of that point. As a result in some practical problems this property does not hold good. The relaxation of the term “almost all” is more realistic in case of statistical convergence, because in this case an infinite number of elements may remain outside the ε-neighbourhood of a statistical limit point.

The concept of statistical convergence was introduced by Fast [9] and Steinhaus [22] using the idea of natural density [18] and later it was developed by many mathematicians ([7], [8], [10], [12], [13], [16], [17], [19], [21], [23], [24], [25], [26] etc.).

A subset S of the set of natural numbers $\mathbb{N}$ is said to have natural density $\delta(S)$, if

$$\lim_{n\to\infty}\frac{|S(n)|}{n}=\delta(S)$$

, where $S(n)=\{k<n:k\in S\}$ and $|S|$ denotes the cardinality of the set S. A sequence $\{x_n\}$ in X is said to be statistically convergent to some element $\xi\in X$ if for every $\varepsilon>0$, $\delta(\{k\in\mathbb{N}:\gamma(x_k,\xi)\geqslant\varepsilon\})=0$.

Fridy [11] studied the concept of statistical limit points and statistical cluster points and established some interesting properties of these two types of limit points.

According to Fridy [11], a subsequence $\{x_{n_k}\}$ of a sequence $\{x_n\}$ is said to be of density zero or thin subsequence if $\delta(\{n_k:k\in\mathbb{N}\})=0$ and $\{x_{n_k}\}$ is said to be non-thin if either $\delta(\{n_k:k\in\mathbb{N}\})>0$ or density of the set $\{n_k:k\in\mathbb{N}\}$ does not exist.

A number λ is called a statistical limit point of a sequence $\{x_n\}$ if there is a non-thin subsequence of $\{x_n\}$ that converges to λ and a number μ is said to be a statistical cluster point of $\{x_n\}$ if for every $\varepsilon>0$, $\{k\in\mathbb{N}:\gamma(x_k,\mu)\geqslant\varepsilon\}$ does not have density zero.

It is shown that the set of statistical cluster points of a bounded sequence is non-empty and if this set contains only one point then the sequence is statistically convergence to that point. These ideas motivate us to study ω-limit sets of a dynamical system using statistical convergence as a tool. We also study the concept of invariant sets [6] which are central to control theory and to validation of systems, such as programs , physical systems or hybrid systems. In this connection we recall that a set $A\subset X$ is said to be invariant under a map $f:X\to X$ if $f(A)\subset A$ and is strongly invariant (i.e., s-invariant) if $f(A)=A$.

2. Statistical ω-limit set

In this section we introduce two notions; viz. statistical ω-limit set and statistical ω-cluster set according to the concepts of statistical limit point and statistical cluster point and study some properties of the two sets.

DEFINITION 1. If $\{x_n\}$ is the orbit of a point $x\in X$ for a map $f:X\to X$, then the collection of all statistical limit points of $\{x_n\}$ is said to be the statistical ω-limit set of $\{x_n\}$. Accordingly, the collection of all statistical cluster points of $\{x_n\}$ is the statistical ω-cluster set of $\{x_n\}$.

We denote the statistical ω-limit set and statistical ω-cluster set of $\{x_n\}$ by $\omega_l(x,f)$ and $\omega_c(x,f)$ respectively. Also $\omega(x,f)$ denotes the ordinary ω-limit set.

Immediately the following results follows from [11].

THEOREM 1. *If $\{x_n\}$ is the orbit of a point $x \in X$ for a map $f : X \to X$, then*
(a) $\omega_l(x,f) \subset \omega(x,f)$.
(b) $\omega_c(x,f) \subset \omega(x,f)$.
(c) $\omega_l(x,f) \subset \omega_c(x,f)$.
(d) $\omega_l(x,f)$ need not be a closed set.
(e) $\omega_c(x,f)$ is a closed point set and hence compact in X.
(f) If $\{x_n\}$ and $\{y_n\}$ are orbits of two points x and y such that $x_k = y_k$ for almost all $k \in \mathbb{N}$, then

$$\omega_l(x,f) = \omega_l(y,f) \text{ and } \omega_c(x,f) = \omega_c(y,f).$$

(g) For the orbit $\{x_n\}$ of any point $x \in X$ there exists an orbit $\{y_n\}$ of some point $y \in X$ such that
(i) $\omega(y,f) = \omega_c(x,f)$ and $x_k = y_k$ for almost all k and
(ii) $\{y_n : n \in \mathbb{N}\} \subset Orb^+(x)$.
(h) If the orbit of some point $x \in X$ for a function $f : X \to X$ is bounded, then $\omega_c(x,f) \neq \phi$.

The following result can be verified from [15].

THEOREM 2. *If $\{x_n\}$ is the orbit of a point $x \in X$ for a map $f : X \to X$, then $\omega_l(x,f)$ is an F_σ set.*

In the following we study the invariant property for the sets $\omega_c(x,f)$ and $\omega_l(x,f)$.

THEOREM 3. *For a point $x_0 \in X$ and a continuous function $f : X \to X$, $\omega_c(x_0,f)$ is s-invariant.*

Proof. Let $A = \omega_c(x_0,f)$.
Case I: If $A \cap Orb^+(x_0) = \phi$, then the result is trivially true.
Case II: Let $x \in A \cap Orb^+(x_0)$ and choose $\varepsilon > 0$. Since f is continuous, there exists a real number δ with $0 < \delta < \varepsilon$ for which

$$u \in \gamma(x,\delta) \text{ implies that } f(u) \in \gamma(f(x),\varepsilon). \tag{1}$$

Again, x is a statistical cluster point of $Orb^+(x_0)$. So, for $T = \{k \in \mathbb{N} : f^{k-1}(x_0) \in \gamma(x,\delta)\}$ we have $\delta(T) \neq 0$.

Consider $S = \{k \in \mathbb{N} : f^k(x_0) \in \gamma(f(x),\varepsilon)\}$. If $k \in T$ then $f^{k-1}(x_0) \in \gamma(x,\delta)$. Using (1) we have, $f^k(x_0) \in \gamma(f(x),\varepsilon)$. Clearly, $k \in S$ and consequently $T \subset S$. So, $\delta(T) \neq 0$ implies that $\delta(S) \neq 0$. i.e., $f(x) \in \omega_c(x,f) = A$ and thus $f(A) \subset A$.

We now prove $A \subset f(A)$. Let $y \in A$ and ε be arbitrary. Then

$$\delta\{n \in \mathbb{N} : f^n(x_0) \in \gamma(y,\varepsilon)\} \neq 0.$$

Let $U = \{n \in \mathbb{N} : f^n(x_0) \in \gamma(y, \varepsilon)\} = \{n_1, n_2, \ldots\}$. Then $f^{n_k}(x_0) \in \gamma(y, \varepsilon)$ for all $k \in \mathbb{N}$.

Since A is compact $\{f^{n_k - 1}(x_0)\}$ has a statistical cluster point z, say in A ([11], Theorem 3, page: 1191). Since f is continuous at z, there exists a real number ε' with $0 < \varepsilon' < \varepsilon$ for which

$$u \in \gamma(z, \varepsilon') \text{ implies that } f(u) \in \gamma(f(x), \varepsilon). \qquad (2)$$

Let $V = \{n_k - 1 : f^{n_k - 1}(x_0) \in \gamma(z, \varepsilon')\}$. Then $\delta(V) \neq 0$ and $p \in V$ implies that $p + 1 \in U$.

If $p \in V$, then $f^p(x_0) \in \gamma(z, \varepsilon')$. Using (2) we have $f^{p+1}(x_0) \in \gamma(f(z), \varepsilon)$ for some $p + 1 \in U$.

Since, $\delta(V) \neq 0$, $f^{p+1}(x_0) \in \gamma(f(z), \varepsilon) \cap \gamma(y, \varepsilon)$ for infinitely many $p + 1 \in U$. This implies that $f(z) = y$ and thus $y = f(z) \in f(A)$. So, $A \subset f(A)$ and hence $A = f(A)$. Therefore, we conclude that $\omega_c(x, f)$ is s-invariant. □

The following Corollary follows from the first part of the proof of the previous theorem.

COROLLARY 1. *For a point $x_0 \in X$ and a continuous function $f : X \to X$, $\omega_l(x_0, f)$ is invariant.*

REMARK 1. It is a fact that $\omega_l(x, f)$ may not be s-invariant. The claim can be justified by the following example described in [11].

EXAMPLE 1. Let $\{x_k\}_{k \in \mathbb{N}}$ be an orbit of some point x under a map f where, $x_k = \dfrac{1}{p}$ for $k = 2^{p-1}(2q + 1)$. For this orbit,

$$\omega_l(x, f) = \{\frac{1}{p} : p \in \mathbb{N}\}.$$

Here, $x_k = 1$ if and only if k is an odd integer. So, if $y \in \omega_l(x, f)$ and $y \neq 1$, then $y = x_{2k}$ for some positive integer k. This implies that $f(y) = 1$ and consequently $f(\omega_l(x, f))$ consists of only two points. Hence $\omega_l(x, f) \nsubseteq f(\omega_l(x, f))$.

THEOREM 4. *For some point $x_0 \in X$ and for a function $f : X \to X$ if $y_0 \in \omega_c(x_0, f)$, then $\omega_c(y_0, f) \subset \omega_c(x_0, f)$.*

Proof. Let $\omega_c(x_0, f) = A$ and $y_0 \in A$. Since $f(A) \subset A$ then $Orb^+(y_0) \subset \omega_c(x_0, f)$. Again $\omega_c(x_0, f)$ is closed. So, $\omega_c(y_0, f) \subset \omega_c(x_0, f)$. □

The property of *weak incompressibility* was first observed by Sarkovski. He proved in [20] that it is an inherent property of ω-limit sets. It was originally stated as a property of invariant sets, but Barwell et al. [4] modified the definition slightly to remove the necessity of invariance.

DEFINITION 2. [4] A set $A \subset X$ is said to have weak incompressibility if for any proper non-empty open subset U in A, $\overline{f(U)} \cap (A-U) \neq \phi$. Equivalently, if for any proper non-empty closed subset D in A we have $D \cap \overline{f(A-D)} \neq \phi$.

Here we verify this property for ω-cluster point sets.

THEOREM 5. *For $x_0 \in X$, $\omega_c(x_0, f)$ has weak incompressibility.*

Proof. Let $A = \omega_c(x_0, f)$ and D be a non-empty proper closed subset of A. If possible let $D \cap \overline{f(A-D)} = \phi$.

Then, there exists two open sets U and V such that $\overline{U} \cap \overline{V} = \phi$ with $D \subset U$ and $\overline{f(A-D)} \subset V$. This implies that $A - D \subset f^{-1}(V)$.

Since f is continuous, $f^{-1}(V) = W$ (say), is open in X. So, $f(\overline{W}) = \overline{f(W)} = \overline{V}$ and consequently $\overline{f(W)} \cap \overline{U} = \phi$.

As $D \subset U$ and $A - D \subset W$ we have $A \subset U \cup W$ with $A \cap U \neq \phi$ and $A \cap W \neq \phi$.

Since $f(A) \subset A$, there exists some natural number p such that $f^n(x_0) \in U \cap W$ for all $n \geqslant p$. Clearly, $f^n(x_0) \in W$ for infinitely many $n \geqslant p$. Also $f^n(x_0) \in U$ for infinitely many $n \geqslant p$. Then for infinitely many $n \geqslant p$, $f^n(x_0) \in W$ implies that $f^{n+1}(x_0) \in U$.

For $S = \{n \in \mathbb{N} : f^n(x_0) \in W\}$ we have $\delta(S) \neq 0$ otherwise, $A \cap W = \phi$ and this will lead to a contradiction.

Similarly, for $T = \{n \in \mathbb{N} : f^{n+1}(x_0) \in U\}$ we have $\delta(T) \neq 0$.

We know that if a number sequence has a bounded non-thin subsequence, then the sequence has an statistical cluster point ([11], Theorem 3, page: 1191). So, W has a statistical cluster point α, say. Clearly, $\alpha \in \overline{W}$ and this implies that $f(\alpha) \in f(\overline{W})$.

Let $\varepsilon > 0$. Since f is continuous at $\alpha \in X$, there exists a $\eta > 0$ such that $x \in N(c, \eta)$ implies that $f(x) \in \gamma(f(\alpha), \varepsilon)$. As, α is a cluster point of W, $\delta\{n \in \mathbb{N} : \gamma(f^n(x_0), \alpha) < \eta\} \neq 0$.

Again, by continuity of f, $\gamma(f^n(x_0), \alpha) < \eta$ implies that $\gamma(f^{n+1}(x_0), f(\alpha)) < \varepsilon$. So, $\{n \in \mathbb{N} : \gamma(f^n(x_0), \alpha) < \eta\} \subset \{n \in \mathbb{N} : \gamma(f^{n+1}(x_0), f(\alpha)) < \varepsilon\}$ and consequently, $\delta\{n \in \mathbb{N} : \gamma(f^{n+1}(x_0), f(\alpha)) < \varepsilon\} \neq 0$. Thus, $f(\alpha)$ is a cluster point of U and $f(\alpha) \in \overline{U}$. This gives a contradiction. So, we can conclude that $D \cap \overline{f(A-D)} \neq \phi$ and hence $\omega_c(x_0, f)$ has weak incompressibility. □

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