Journal of Classical Analysis
Volume 18, Number 2 (2021), 137–148 doi:10.7153/jca-2021-18-10

SHARPENING OF BERNSTEIN AND TURÁN–TYPE INEQUALITIES FOR POLYNOMIALS

THANGJAM BIRKRAMJIT SINGH, MAISNAM TRIVENI DEVI* AND BARCHAND CHANAM

Abstract. Let $p(z)$ be a polynomial of degree n. The polar derivative of $p(z)$ with respect to a real or complex number α is defined by

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

Govil and Mctume [Acta Math. Hungar., 104, 115–126 (2004)] proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for any complex number α with $|\alpha| \geqslant 1+k+k^n$,

$$\max_{|z|=1} |D_\alpha p(z)| \geqslant n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|p(z)| + n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)\min_{|z|=k}|p(z)|.$$

In this paper, we prove an improvement of the above inequality. Further, we prove an improvement of a result due to Govil [Proc. Natl. Acad. Sci., 50, 50–52 (1980)].

1. Introduction and statement of results

If $p(z)$ is a polynomial of degree n. Then, according to a famous well-known classical result due to Bernstein [7], we have

$$\max_{|z|=1}|p'(z)| \leqslant n \max_{|z|=1}|p(z)|. \tag{1}$$

Inequality (1) is sharp and equality holds if $p(z)$ has all its zeros at the origin.

Inequality (1) can be sharpened if the zeros of the polynomial are restricted. In this direction, Erdös conjectured and later Lax [20] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then

$$\max_{|z|=1}|p'(z)| \leqslant \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{2}$$

Inequality (2) is best possible and equality holds for $p(z) = a + bz^n$, where $|a| = |b|$.

It was asked by R. P. Boas that if $p(z)$ is a polynomial of degree n not vanishing in $|z| < k$, $k > 0$, then how large can

$$\left\{\max_{|z|=1}|p'(z)| \Big/ \max_{|z|=1}|p(z)|\right\} \quad \text{be?}$$

Mathematics subject classification (2020): 30A10, 30C10, 30D15.
Keywords and phrases: Polynomials, inequalities, zeros, polar derivative.
* Corresponding author.

Paper JCA-18-10

A partial answer to this problem was given by Malik [21], who proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geqslant 1$, then

$$\max_{|z|=1} \left|p'(z)\right| \leqslant \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{3}$$

In literature, there exist generalizations and improvements of inequality (3) for polynomials having no zeros in $|z| < k$ with $k \geqslant 1$, one can refer to the following for a better insight : Chan and Malik [9], Bidkham and Dewan [8], Qazi [22], Aziz and Zargar [5], Aziz and Shah [4], Chanam and Dewan [10] etc.

For the class of polynomials not vanishing in $|z| < k$, $k \leqslant 1$, the precise estimate for $|p'(z)|$ on $|z| = 1$, in general, does not seem to be easily obtainable. For quite some time, it was believed that if $p(z) \neq 0$ in $|z| < k$, $k \leqslant 1$, then the inequality analogous to (3) should be

$$\max_{|z|=1} \left|p'(z)\right| \leqslant \frac{n}{1+k^n} \max_{|z|=1} |p(z)|, \tag{4}$$

till E. B. Saff gave the example $p(z) = \left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)$ to counter this belief.

Govil [12] obtained inequality (4) for polynomials having no zeros in $|z| < k$, $k \leqslant 1$, with additional hypothesis and proved

THEOREM 1. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having no zeros in* $|z| < k$, $k \leqslant 1$, *such that* $|p'(z)|$ *and* $|\tilde{p}'(z)|$ *attain their maxima at the same point on* $|z| = 1$, *then*

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{1+k^n} \max_{|z|=1} |p(z)|. \tag{5}$$

where $\tilde{p}(z) = z^n \overline{p(\frac{1}{\bar{z}})}$.

On the other hand, in 1939, Turán [25] provided a lower bound estimate of the derivative size of the polynomial by restricting its zeros, and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{6}$$

Aziz and Dawood [2] further refined inequality (6) by involving $\min_{|z|=1} |p(z)|$,

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{2} \left\{ \max_{|z|=1} |p(z)| + \min_{|z|=1} |p(z)| \right\}. \tag{7}$$

Both these inequalities (6) and (7) are best possible and equality holds if $p(z)$ has all its zeros on $|z| = 1$.

Inequalities (6) and (7) have been extended and generalized in different directions (see [4], [6], [13], [14], [15]). For polynomial $p(z)$ having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, Govil [13] proved that

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{1+k^n} \max_{|z|=1} |p(z)|. \tag{8}$$

Govil [14] proved a refinement of inequality (8) and a generalization of (7) taking the same class of polynomials and obtained

$$\max_{|z|=1}|p'(z)| \geqslant \frac{n}{1+k^n}\left\{\max_{|z|=1}|p(z)|+\min_{|z|=k}|p(z)|\right\}. \tag{9}$$

Inequalities (8) and (9) are sharp and equality holds for $p(z)=z^n+k^n$.

The concept of derivative of a polynomial has been generalized to polar derivative of a polynomial as follows.

If $p(z)$ is a polynomial of degree n and α be any real or complex number, the polar derivative of $p(z)$ with respect to α, denoted by $D_\alpha p(z)$, is defined as

$$D_\alpha p(z) = np(z)+(\alpha-z)p'(z).$$

It is easy to see that $D_\alpha p(z)$ is a polynomial of degree at most $(n-1)$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha p(z)}{\alpha}\right] = p'(z).$$

In 1998, Aziz and Rather [3] extended inequality (8) to polar derivative by proving that if $p(z)$ is a polynomial of degree n having all its zeros in $|z|\leqslant k$, $k\geqslant 1$, then for every real or complex number α with $|\alpha|\geqslant k$,

$$\max_{|z|=1}|D_\alpha p(z)| \geqslant n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|p(z)|. \tag{10}$$

Govil and Mctume [16] established the polar derivative extension of inequality (9) and proved

THEOREM 2. *If* $p(z)=\sum_{j=0}^{n}a_jz^j$ *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant k$, $k\geqslant 1$, *then for every real or complex number* α *with* $|\alpha|\geqslant 1+k+k^n$,

$$\max_{|z|=1}|D_\alpha p(z)| \geqslant n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|p(z)| + n\left\{\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right\}\min_{|z|=k}|p(z)|. \tag{11}$$

Recently, inequalities (5) and (10) have been improved by involving certain coefficients of the polynomial in different directions, for a better insight one can refer: Govil and Kumar [15], Kumar [18], Kumar and Dhankar [19] and Rather et al. [24] etc. In this direction, we obtain improved versions of inequalities (5) and (11).

2. Main result

We first begin by presenting the following refinement of Theorem 1 due to Govil [12].

THEOREM 3. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having no zeros in* $|z| < k,\ k \leqslant 1$, *such that* $|p'(z)|$ *and* $|\tilde{p}'(z)|$ *attain their maxima at the same point on* $|z| = 1$, *then*

$$\max_{|z|=1} |p'(z)| \leqslant \left\{ \frac{n}{1+k^n} - \frac{k^n(|a_0| - |a_n|k^n)}{(1+k^n)(|a_n|k^n + |a_0|)} \right\} \max_{|z|=1} |p(z)|, \tag{12}$$

where $\tilde{p}(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

Inequality (12) *is best possible for* $p(z) = z^n + k^n$.

REMARK 1. If $p(z) = \sum_{j=0}^{n} a_j z^j = a_n \prod_{j=0}^{n} (z - z_j)$ where $|z_j| \geqslant k,\ j = 0, 1, 2, \ldots n$ then

$$\prod_{j=0}^{n} |z_j| \geqslant k^n,$$

which implies

$$|a_0| \geqslant k^n |a_n|.$$

This shows that Theorem 3 is a refinement of Theorem 1.

REMARK 2. Taking $k = 1$, Theorem 3 reduces to a refinement of inequality (1.2) of Erdös and Lax [20] for the class of polynomials having no zeros in $|z| < 1$ with the extra assumption that $|p'(z)|$ and $|\tilde{p}'(z)|$ attain their maxima at the same point on $|z| = 1$.

COROLLARY 1. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having no zeros in* $|z| < 1$, *such that* $|p'(z)|$ *and* $|\tilde{p}'(z)|$ *attain their maxima at the same point on* $|z| = 1$, *then*

$$\max_{|z|=1} |p'(z)| \leqslant \frac{1}{2} \left\{ n + \frac{|a_0| - |a_n|}{|a_0| + |a_n|} \right\} \max_{|z|=1} |p(z)|.$$

Our next result is an improvement of Theorem 2 due to Govil and Mctume [16].

THEOREM 4. *If* $p(z)=\sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant k,\ k\geqslant 1$*, then for any real or complex number* α *with* $|\alpha|\geqslant 1+k+k^n$*,*

$$\begin{aligned}
&\max_{|z|=1}|D_\alpha p(z)| \\
&\geqslant \frac{(|\alpha|-k)}{1+k^n}\left\{n+\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+|a_0+e^{i\theta_0}m|}\right\}\max_{|z|=1}|p(z)| \\
&\quad+\left\{n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)+\frac{(|\alpha|-k)}{1+k^n}\left(\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+|a_0+e^{i\theta_0}m|}\right)\right\}m, \qquad (13)
\end{aligned}$$

where $m=\min_{|z|=k}|p(z)|$ *and* $\theta_0=\arg\{p(e^{i\phi_0})\}$ *such that* $|p(e^{i\phi_0})|=\max_{|z|=1}|p(z)|$.

REMARK 3. If $p(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z|\leqslant k$, $k\geqslant 1$, then for any complex number $|\lambda|e^{i\theta_0}$ with $|\lambda|<1$, by Rouche's theorem it follows that the polynomial $p(z)+|\lambda|e^{i\theta_0}m=(a_0+|\lambda|e^{i\theta_0}m)+a_1z+\cdots+c_nz^n$ has all its zeros in $|z|\leqslant k$, where $m=\min_{|z|=k}|p(z)|$, then

$$k^n\geqslant\left|\frac{a_0+|\lambda|e^{i\theta_0}m}{a_n}\right|,$$

which implies that

$$k^n|a_n|\geqslant|a_0+|\lambda|e^{i\theta_0}m|.$$

Taking $|\lambda|\to 1$, we get

$$k^n|a_n|\geqslant|a_0+e^{i\theta_0}m|.$$

From which we conclude that Theorem 4 is a refinement of Theorem 2.

REMARK 4. Dividing both sides of (13) by $|\alpha|$ and taking limit as $|\alpha|\to\infty$, we have the following refinement of inequality (9) due to Govil [14].

COROLLARY 2. *If* $p(z)=\sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant k,\ k\geqslant 1$*, then*

$$\begin{aligned}
\max_{|z|=1}\left|p'(z)\right| \geqslant \frac{1}{1+k^n}&\left\{n+\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+|a_0+e^{i\theta_0}m|}\right\}\max_{|z|=1}|p(z)| \\
&+\left\{\frac{n}{1+k^n}+\frac{1}{1+k^n}\left(\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+|a_0+e^{i\theta_0}m|}\right)\right\}m, \qquad (14)
\end{aligned}$$

where $m=\min_{|z|=k}p(z)$ *and* $\theta_0=\arg\{p(e^{i\phi_0})\}$ *such that* $|p(e^{i\phi_0})|=\max_{|z|=1}|p(z)|$.

Inequality (14) is best possible for $p(z)=z^n+k^n$.

REMARK 5. Taking $k=1$ in Corollary 2, inequality (14) reduces to a refinement of inequality (7) due to Aziz and Dawood [2].

COROLLARY 3. *If* $p(z)=\sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant 1$*, then*

$$\max_{|z|=1}|p'(z)| \geqslant \frac{1}{2}\left\{n+\frac{|a_n|-|a_0+e^{i\theta_0}m|}{|a_n|+|a_0+e^{i\theta_0}m|}\right\}\max_{|z|=1}|p(z)| + \frac{1}{2}\left\{n+\left(\frac{|a_n|-|a_0+e^{i\theta_0}m|}{|a_n|+|a_0+e^{i\theta_0}m|}\right)\right\}m, \tag{15}$$

where $m=\min_{|z|=1}|p(z)|$ *and* $\theta_0=\arg\left\{p(e^{i\phi_0})\right\}$ *such that* $|p(e^{i\phi_0})|=\max_{|z|=1}|p(z)|$.

Further, we will extend Theorem 3 to the polar derivative version as follows.

THEOREM 5. *If* $p(z)=\sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having no zeros in* $|z|<k$, $k\leqslant 1$*, such that* $|p'(z)|$ *and* $|\tilde{p}'(z)|$ *attain their maxima at the same point on* $|z|=1$*, then for any real or complex number* α *with* $|\alpha|>1$,

$$\max_{|z|=1}|D_\alpha p(z)| \leqslant \left\{\frac{n(|\alpha|+k^n)}{1+k^n}-\frac{(|\alpha|-1)k^n(|a_0|-|a_n|k^n)}{(1+k^n)(|a_n|k^n+|a_0|)}\right\}\max_{|z|=1}|p(z)|. \tag{16}$$

3. Lemmas

For the proof of our theorems, we need the following lemmas.

LEMMA 1. *If* $p(z)$ *is a polynomial of degree* n*, then on* $|z|=1$,

$$|p'(z)|+|\tilde{p}'(z)| \leqslant n\max_{|z|=1}|p(z)|,$$

where $\tilde{p}(z)=z^n\overline{p(\frac{1}{\bar{z}})}$.

This result is a special case of a result due to Govil and Rahman [17].

LEMMA 2. *If* $\{z_i\}_{j=1}^{n}$ *is a finite collection of real numbers such that* $0\leqslant z_j\leqslant 1$, $j=1,2,\cdots,n$*, then*

$$\sum_{j=1}^{n}\frac{1-z_j}{1+z_j} \geqslant \frac{1-\prod_{j=1}^{n} z_j}{1+\prod_{j=1}^{n} z_j},$$

The above Lemma is due to Rather et al. [23].

LEMMA 3. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then*

$$\max_{|z|=k} |p(z)| \geqslant \frac{2k^n}{1+k^n} \max_{|z|=1} |p(z)|.$$

The above result appears in Aziz [1].

LEMMA 4. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for $|z| = 1$ in which $p(z) \neq 0$, we have*

$$Re\left(\frac{p'(z)}{p(z)}\right) \geqslant \frac{1}{2}\left\{n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right\}.$$

The above result was proved by Dubinin [11] where the author made use of famous Boundary Schwartz Lemma. Here we present an alternative proof by using Lemma 2.

Proof of Lemma 4. Let $p(z) = \sum_{j=0}^{n} a_j z^j = a_n \prod_{j=1}^{n} (z - z_j)$, where $|z_j| \leqslant 1$, $j = 1, 2, \cdots, n$. Then for $|z| = 1$, in which $p(z) \neq 0$,

$$\Re\left(\frac{zp'(z)}{p(z)}\right) = \sum_{j=1}^{n} \Re\left(\frac{z}{z - z_j}\right). \tag{17}$$

For $|z| = 1$ and $|z_j| \leqslant 1$, $j = 1, 2, \ldots, n$ some straight forward calculations give

$$\Re\left(\frac{z}{z - z_j}\right) \geqslant \frac{1}{1 + |z_j|}. \tag{18}$$

Combining inequalities (17) and (18), we get

$$\Re\left(\frac{zp'(z)}{p(z)}\right) \geqslant \sum_{j=1}^{n} \frac{1}{1 + |z_j|} = \frac{1}{2}\left\{n + \sum_{j=1}^{n} \frac{1 - |z_j|}{1 + |z_j|}\right\}. \tag{19}$$

Applying Lemma 2 in inequality (19), we get

$$\Re\left(\frac{zp'(z)}{p(z)}\right) \geqslant \frac{1}{2}\left\{n + \frac{1 - \prod_{j=1}^{n} |z_j|}{1 + \prod_{j=1}^{n} |z_j|}\right\} = \frac{1}{2}\left\{n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right\}, \tag{20}$$

for $|z| = 1$ on which $p(z) \neq 0$. □

REMARK 6. Govil and Kumar [15] also proved Lemma 4 by using the Principle of Mathematical Induction. As a consequence of Lemma 4 we obtain the next Lemma which is a special case of a result proved by Govil and Kumar [15].

LEMMA 5. *If* $p(z)=\sum_{j=0}^{n}a_jz^j$ *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant k$, $k\geqslant 1$, *then for any real or complex number* α *with* $|\alpha|\geqslant k$,

$$\max_{|z|=1}|D_\alpha p(z)|\geqslant(|\alpha|-k)\left\{\frac{n}{1+k^n}+\frac{(k^n|a_n|-|a_0|)}{(1+k^n)(k^n|a_n|+|a_0|)}\right\}\max_{|z|=1}|p(z)|. \tag{21}$$

Proof of Lemma 5. Since $p(z)$ is a polynomial of degree n having all its zeros in $|z|\leqslant k$, $|k|\geqslant 1$, hence all the zeros of $p(kz)$ lie in $|z|\leqslant 1$. Now taking $g(z)=p(kz)$, and using Lemma 4, we get

$$\Re\left(\frac{zg'(z)}{g(z)}\right)\geqslant\frac{1}{2}\left\{n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right\},\quad\text{for } |z|=1,\ p(z)\neq 0, \tag{22}$$

which implies

$$\left|\frac{g'(z)}{g(z)}\right|\geqslant\Re\left(\frac{zp'(z)}{p(z)}\right)\geqslant\frac{1}{2}\left\{n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right\},$$

which is equivalent to

$$|g'(z)|\geqslant\frac{1}{2}\left\{n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right\}|g(z)|,\quad\text{for } |z|=1,\ p(z)\neq 0.$$

Since the above inequality is trivially satisfied for $p(z)=0$, we have

$$|g'(z)|\geqslant\frac{1}{2}\left\{n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right\}|g(z)|,\quad\text{for } |z|=1. \tag{23}$$

Since $g(z)$ has all its zeros in $|z|\leqslant 1$, then using the fact that $|g'(z)|\geqslant|\widetilde{g}'(z)|$ on $|z|=1$, we have for $|\frac{\alpha}{k}|\geqslant 1$ and $|z|=1$,

$$\begin{aligned}|D_{\frac{\alpha}{k}}g(z)| &= |ng(z)+\left(\frac{\alpha}{k}-z\right)g'(z)|\\ &\geqslant\left|\frac{\alpha}{k}\right||g'(z)|-|ng(z)-zg'(z)|\\ &=\left|\frac{\alpha}{k}\right||g'(z)|-|\widetilde{g}'(z)|\\ &\geqslant\left(\left|\frac{\alpha}{k}\right|-1\right)|g'(z)|.\end{aligned} \tag{24}$$

Combining (23) and (24), we get

$$\max_{|z|=1}|D_{\frac{\alpha}{k}}g(z)|\geqslant\frac{(|\alpha|-k)}{2k}\left\{n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right\}\max_{|z|=1}|g(z)|,$$

which implies

$$\max_{|z|=1}|np(kz)+\left(\frac{\alpha}{k}-z\right)kp'(kz)|\geqslant\frac{(|\alpha|-k)}{2k}\left\{n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right\}\max_{|z|=1}|p(kz)|. \tag{25}$$

Using Lemma 3 and the fact that

$$\max_{|z|=1} |np(kz) + \left(\frac{\alpha}{k} - z\right) kp'(kz)| = \max_{|z|=k} |D_\alpha p(z)|,$$

inequality (25) gives

$$\max_{|z|=k} |D_\alpha p(z)| \geqslant \frac{(|\alpha|-k)}{k} \left\{ n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|} \right\} \frac{k^n}{1+k^n} \max_{|z|=1} |p(z)|. \tag{26}$$

As we can see $D_\alpha p(z)$ is a polynomial of degree at most $(n-1)$ and $k \geqslant 1$, hence by famous Bernstein inequality, we have

$$\max_{|z|=k} |D_\alpha p(z)| \leqslant k^{n-1} \max_{|z|=1} |D_\alpha p(z)|. \tag{27}$$

Combining (26) and (27), we get

$$\max_{|z|=1} |D_\alpha p(z)| \geqslant \frac{(|\alpha|-k)}{1+k^n} \left\{ n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|} \right\} \max_{|z|=1} |p(z)|.$$

This completes the proof of Lemma 5. □

REMARK 7. Dividing both sides of inequality (21) by $|\alpha|$ and taking limit as $|\alpha| \to \infty$, we get

LEMMA 6. *If* $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$, $k \geqslant 1$, *then*

$$\max_{|z|=1} |p'(z)| \geqslant \left\{ \frac{n}{1+k^n} + \frac{(|a_n|k^n - |a_0|)}{(1+k^n)(|a_n|k^n + |a_0|)} \right\} \max_{|z|=1} |p(z)|.$$

4. Proofs of Theorems

Proof of Theorem 3. Since $p(z)$ is a polynomial of degree n having no zeros in $|z| < k$, $k \leqslant 1$, then $\tilde{p}(z) = z^n \overline{p(\frac{1}{\bar{z}})}$ has all its zeros in $|z| \leqslant \frac{1}{k}$, $\frac{1}{k} \geqslant 1$.

Hence, by Lemma 6, and using the fact that $\max_{|z|=1} |\tilde{p}(z)| = \max_{|z|=1} |p(z)|$, we have

$$\max_{|z|=1} |\tilde{p}'(z)| \geqslant \left\{ \frac{nk^n}{1+k^n} + \frac{k^n(|a_0| - |a_n|k^n)}{(1+k^n)(|a_0| + |a_n|k^n)} \right\} \max_{|z|=1} |p(z)|. \tag{28}$$

By Lemma 1, we have on $|z| = 1$

$$|p'(z)| + |\tilde{p}'(z)| \leqslant n \max_{|z|=1} |p(z)|. \tag{29}$$

Since $|p'(z)|$ and $|\tilde{p}'(z)|$ attain their maxima at the same point on $|z| = 1$, we have

$$\max_{|z|=1} \{|p'(z)| + |\tilde{p}'(z)|\} = \max_{|z|=1} |p'(z)| + \max_{|z|=1} |\tilde{p}'(z)|. \tag{30}$$

Therefore, from (28), (29) and (30), we have

$$\max_{|z|=1}|p'(z)|+\left\{\frac{nk^n}{1+k^n}+\frac{k^n(|a_0|-|a_n|k^n)}{(1+k^n)(|a_0|+|a_n|k^n)}\right\}\max_{|z|=1}|p(z)|\leqslant n\max_{|z|=1}|p(z)|,$$

which after proper rearrangement of terms gives

$$\max_{|z|=1}|p'(z)|\leqslant\left\{\frac{n}{1+k^n}-\frac{k^n(|a_0|-|a_n|k^n)}{(1+k^n)(|a_0|+|a_n|k^n)}\right\}\max_{|z|=1}|p(z)|.$$

Thus the proof of Theorem 3 is complete. □

Proof of Theorem 4. If $p(z)$ is a polynomial of degree n having atleast one zero on $|z|=k$, then $m=0$ and the result follows trivially from Lemma 5. So, without loss of generality, let us assume that $p(z)$ has all its zeros in $|z|<k$, $k\geqslant 1$, then it follows by Rouche's theorem that for any complex number λ with $|\lambda|<1$, the polynomial $p(z)+\lambda m=(a_0+\lambda m)+a_1z+\cdots+a_nz^n$ also has all its zeros in $|z|<k$, $k\geqslant 1$. Therefore, applying Lemma 5 to $p(z)+\lambda m$, we get for $|\alpha|\geqslant 1+k+k^n$,

$$\begin{aligned}&\max_{|z|=1}|D_\alpha(p(z)+\lambda m)|\\ &\geqslant(|\alpha|-k)\left\{\frac{n}{1+k^n}+\frac{k^n|a_n|-|a_0+\lambda m|}{(1+k^n)(k^n|a_n|+|a_0+\lambda m|)}\right\}\max_{|z|=1}|p(z)+\lambda m|.\end{aligned}\tag{31}$$

Let $0\leqslant\phi_0<2\pi$, be such that $\left|p(e^{i\phi_0})\right|=\max_{|z|=1}|p(z)|$. Then inequality (31) gives

$$\begin{aligned}&\max_{|z|=1}|D_\alpha p(z)+n\lambda m|\\ &\geqslant(|\alpha|-k)\left\{\frac{n}{1+k^n}+\frac{k^n|a_n|-|a_0+\lambda m|}{(1+k^n)(k^n|a_n|+|a_0+\lambda m|)}\right\}|p(e^{i\phi_0})+\lambda m|.\end{aligned}\tag{32}$$

Now

$$\left|p(e^{i\phi_0})+\lambda m\right|=\left||p(e^{i\phi_0})|e^{i\theta_0}+|\lambda|e^{i\phi}m\right|=\left||p(e^{i\phi_0})|+|\lambda|e^{i(\phi-\theta_0)}m\right|.$$

Setting the argument ϕ such that $\phi=\theta_0$, then $\left|p(e^{i\phi_0})+\lambda m\right|=\left|p(e^{i\phi_0})\right|+|\lambda|m$, then it follows from inequality (32) that

$$\begin{aligned}&\max_{|z|=1}|D_\alpha p(z)|+n|\lambda|m\\ &\geqslant(|\alpha|-k)\left\{\frac{n}{1+k^n}+\frac{k^n|a_n|-|a_0+|\lambda|e^{i\theta_0}m|}{(1+k^n)(k^n|a_n|+|a_0+|\lambda|e^{i\theta_0}m|)}\right\}\left(\left|p(e^{i\phi_0})\right|+|\lambda|m\right),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\max_{|z|=1}|D_\alpha p(z)|\\ &\geqslant\frac{(|\alpha|-k)}{1+k^n}\left\{n+\frac{k^n|a_n|-|a_0+|\lambda|e^{i\theta_0}m|}{k^n|a_n|+|a_0+|\lambda|e^{i\theta_0}m|}\right\}\max_{|z|=1}|p(z)|\\ &\quad+|\lambda|\left\{n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)+\frac{(|\alpha|-k)}{1+k^n}\left(\frac{k^n|a_n|-|a_0+|\lambda|e^{i\theta_0}m|}{k^n|a_n|+|a_0+|\lambda|e^{i\theta_0}m|}\right)\right\}m.\end{aligned}$$

Taking $|\lambda| \to 1$, the above inequality reduces to

$$\max_{|z|=1}|D_\alpha p(z)| \geqslant \frac{(|\alpha|-k)}{1+k^n}\left\{n+\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+|a_0+e^{i\theta_0}m|}\right\}\max_{|z|=1}|p(z)| + \left\{n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)+\frac{(|\alpha|-k)}{1+k^n}\left(\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+|a_0+e^{i\theta_0}m|}\right)\right\}m.$$

This completes the proof of Theorem 4. □

Proof of Theorem 5. Using Lemma 1 we have on $|z|=1$ and for any real or complex number α with $|\alpha| \geqslant 1$,

$$\begin{aligned} |D_\alpha p(z)| &= \left|np(z)+(\alpha-z)p'(z)\right| \\ &\leqslant \left|np(z)-zp'(z)\right|+|\alpha|\left|p'(z)\right| \\ &= \left|\tilde{p}'(z)\right|+|\alpha|\left|p'(z)\right| \\ &\leqslant n\max_{|z|=1}|p(z)|+(|\alpha|-1)\left|p'(z)\right|. \end{aligned} \tag{33}$$

Applying Theorem 3 to inequality (33), we get for $|z|=1$

$$|D_\alpha p(z)| \leqslant n\max_{|z|=1}|p(z)|+(|\alpha|-1)\left\{\frac{n}{1+k^n}-\frac{k^n(|a_0|-|a_n|k^n)}{(1+k^n)(|a_n|k^n+|a_0|)}\right\}\max_{|z|=1}|p(z)|,$$

which gives

$$\max_{|z|=1}|D_\alpha p(z)| \leqslant \left\{\frac{n(|\alpha|+k^n)}{1+k^n}-\frac{(|\alpha|-1)k^n(|a_0|-|a_n|k^n)}{(1+k^n)(|a_n|k^n+|a_0|)}\right\}\max_{|z|=1}|p(z)|.$$

Thus the proof of Theorem 5 is complete. □

Acknowledgement. The authors are very grateful to the Editor for his valuable suggestions in upgrading the paper in the present form.

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(Received June 30, 2021)

Thangjam Birkramjit Singh
Department of Mathematics
National Institute of Technology Manipur
Manipur-795004, India
e-mail: birkramth@gmail.com

Maisnam Triveni Devi
Department of Mathematics
National Institute of Technology Manipur
Manipur-795004, India
e-mail: trivenimaisnam@gmail.com

Barchand Chanam
Department of Mathematics
National Institute of Technology Manipur
Manipur-795004, India
e-mail: barchand_2004@yahoo.co.in