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ALTERNATING EULER SUMS AND BBP–TYPE SERIES

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Abstract. An investigation into a family of definite integrals containing log-polylog functions with negative argument will be undertaken in this paper. It will be shown that Euler sums play an important part in the solution of these integrals and some may be represented as a BBP type formula. In a special case we prove that the corresponding log integral can be represented as a linear combination of the product of zeta functions and the Dirichlet beta function.

1. Introduction, preliminaries and notation

In this paper we investigate a family of integrals with polylogarithmic integrand containing some parameters. It will be shown that the solution of some of these families of integrals may be expressed as a BBP-type representation containing harmonic numbers and include some classical constants such as the Riemann zeta function and the Dirichlet beta function. In particular we investigation a family of integrals of the type

$$I(a,p,q,t) = \int_x \frac{x^a \ln^p(x)}{1+x^2} \mathrm{Li}_t(-x^{4q+2})dx, \tag{1}$$

where $a \geqslant -2$, $p \in \mathbb{N}_0$, $q \in \mathbb{N}$, $t \in \mathbb{N}_0$ and for the domain of $x \in (0,1)$. We also study the integral

$$J(p,q,t) = I(0,p,q,t) = \int_x \frac{\ln^p(x)}{1+x^2} \mathrm{Li}_t(-x^{4q+2})dx \tag{2}$$

on the positive half line $x \geqslant 0$. In general for a mathematical constant K a BBP-type formula has the form

$$K = \sum_{n=0}^{\infty} \frac{1}{\alpha^n} \sum_{j=0}^{k} \frac{\beta_j}{(nk+j)^p}$$

where α, k, p are integers, the base, length and degree of the BBP-type formula and β_j are rational numbers. Recently the following results have been published. In [1], we find the alternating series,

$$\pi = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} \left(\frac{2}{4n+1} + \frac{2}{4n+2} + \frac{1}{4n+3} \right).$$

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In [8], we have

$$G=\frac{3}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\left(\frac{2}{(4n+1)^2}-\frac{2}{(4n+2)^2}+\frac{1}{(4n+3)^2}\right)$$
$$+\frac{1}{32}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{64^n}\left(\frac{8}{(4n+1)^2}+\frac{4}{(4n+2)^2}+\frac{1}{(4n+3)^2}\right).$$

In [9] we find

$$\sqrt[3]{2}=\frac{8}{3}\sum_{n=0}^{\infty}\frac{(-1)^n\binom{3n}{n}}{54^n}\left(\frac{1}{3n-2}-\frac{1}{3n-1}\right),$$

and [23] has published,

$$\frac{\sqrt{10+2\sqrt{5}}}{50}\pi=\sum_{n=0}^{\infty}\frac{(-1)^n}{\phi^{10n}}\left(\frac{1}{\phi^2(5n+2)}+\frac{1}{\phi^4(5n+3)}\right)$$

where the golden ratio $\phi=\left(1+\sqrt{5}\right)/2$. These types of representations are known as BBP-type series and many other papers have recently been published [1], [2], [5], [20], [22], [23] extending and generalizing various aspects of BBP type series. Some other related papers dealing with Euler sums are [3], [10], [12], [13], [14] and the excellent books [19] and [21]. The following special functions will be used in the analysis of the integral (1). The polylogarithm function $\mathrm{Li}_t(z)$ is, for $|z|\leqslant 1$

$$\mathrm{Li}_t(z)=\sum_{m=1}^{\infty}\frac{z^m}{m^t}. \tag{3}$$

The classical Hurwitz zeta function

$$\zeta(p,a)=\sum_{n\geqslant 0}\frac{1}{(n+a)^p}$$

for Re $(p)>1$ and by analytic continuation to other values of $p\neq 1$, where any term of the form $(n+a)=0$ is excluded. The well known result

$$\zeta(z)+\eta(z)=2\lambda(z)$$

connects the zeta function $\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$, with the Dirichet eta function $\eta(z)$ and the Dirichlet lambda function $\lambda(z)$. The zeta function has a simple pole at $z=1$. The Dirichlet beta function, $\beta(z)$ or Dirichlet L function is given by, see Finch [7]

$$\beta(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^z};\quad \mathrm{Re}\,(z)>0 \tag{4}$$

where $\beta(2) = G$ is Catalan's constant. The Dirichlet beta function can be represented in powers of π at positive odd integer values of z, such that

$$\beta(2m+1) = \frac{(-1)^m E(2m)}{2^{2m+2}(2m)!}\pi^{2m+1}$$

where $E(\cdot)$ are the Euler numbers generated by

$$\operatorname{sech}(z) = \frac{2e^z}{e^{2z}+1} = \sum_{n=0}^{\infty} \frac{E(n) z^n}{n!}.$$

The Dirichlet beta function can be analytically extended to the whole complex plane, has no singularities in the complex plane and is given by the functional equation

$$\beta(1-z) = \left(\frac{2}{\pi}\right)^z \sin\left(\frac{\pi z}{2}\right)\Gamma(z)\beta(z).$$

For real values of x, $\psi(x)$ is the digamma (or psi) function defined by

$$\psi(x) := \frac{d}{dx}\{\log\Gamma(x)\} = \frac{\Gamma'(x)}{\Gamma(x)}.$$

We know that for $n \geqslant 1$, $\psi(n+1) - \psi(1) = H_n$ with $\psi(1) = -\gamma$, where γ is the Euler Mascheroni constant $\psi(n)$ is the digamma function and $H_n = \sum\limits_{j=1}^{n} \frac{1}{j}$ is the nth. harmonic number with $n \in \mathbb{N}$. The polygamma function

$$\psi^{(k)}(z) = \frac{d^k}{dz^k}\{\psi(z)\} = (-1)^{k+1} k! \sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}}$$

and has the recurrence

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k k!}{z^{k+1}}.$$

The connection of the polygamma function with harmonic numbers is,

$$\begin{aligned} H_z^{(m+1)} &= \zeta(m+1) + \frac{(-1)^m}{m!}\psi^{(m)}(z+1), \quad z \neq \{-1,-2,-3,\ldots\}. \\ &= \frac{(-1)^m}{m!}\int_0^1 \frac{(1-t^z)}{1-t} \ln^m t \, dt. \end{aligned} \tag{5}$$

The multiplication formula for the polygamma function is

$$\psi^{(p)}(mz) = \delta_{m,0}\ln(m) + \frac{1}{m^{p+1}}\sum_{j=0}^{p-1}\psi^{(p)}\left(z+\frac{j}{m}\right) \tag{6}$$

where $m \in \mathbb{N}$ and $\delta_{m,0}$ is the Kronecker delta. We expect that integrals of the type (1) may be represented by Euler sums and therefore in terms of special functions such as the Riemann zeta function. The paper of [23] gives some examples for the representation BBP and Euler type sums. The following papers [15], [16], and [17] also examined some integrals in terms of Euler sums. Some examples will be given highlighting specific cases of the integrals, some of which are not amenable to a computer mathematical package.

2. Analysis of integrals

Consider the following.

THEOREM 1. *Let* $(p,q,t) \in \mathbb{N}_0$, $a \geqslant -2$, *the following integral,*

$$I(a,p,q,t) = \int_0^1 \frac{x^a \ln^p(x)}{1+x^2} \mathrm{Li}_t(-x^{4q+2}) dx \tag{7}$$

$$= \int_0^{\frac{\pi}{4}} (\tan\theta)^a \ln^p(\tan\theta) \mathrm{Li}_t(-\tan^{4q+2}\theta) d\theta$$

$$= (-1)^p p! \sum_{n \geqslant 1} (-1)^{n+1} H_n^{(t)} \sum_{j=0}^{2q} \frac{(-1)^{j+1}}{((4q+2)n+2j+a+1)^{p+1}} \tag{8}$$

where $H_n^{(t)}$ *are nth. generalized harmonic numbers of order* t.

Proof. For $x \in (0,1)$, from (3) and a Taylor series expansion

$$\mathrm{Li}_t(-x^{4q+2}) = \sum_{n \geqslant 1} \frac{(-1)^n x^{(4q+2)n}}{n^t}, \quad \frac{1}{1+x^2} = \sum_{n \geqslant 0} (-1)^n x^{2n}.$$

It is known that the Cauchy product of two convergent series, see Bromwich and Watson, [4]

$$\left(\sum_{n \geqslant 0} a_n x^n\right)\left(\sum_{n \geqslant 0} b_n x^n\right) = \sum_{n \geqslant 0} c_n x^n$$

where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$, it then follows that

$$\frac{x^a \mathrm{Li}_t(-x^{4q+2})}{1+x^2} = \sum_{n \geqslant 1} (-1)^{n+1} H_n^{(t)} \sum_{j=0}^{2q} (-1)^{j+1} x^{(4q+2)n+2j+a}$$

and therefore

$$\frac{x^a \ln^p(x) \mathrm{Li}_t(-x^{4q+2})}{1+x^2} = \sum_{n \geqslant 1} (-1)^{n+1} H_n^{(t)} \sum_{j=0}^{2q} (-1)^{j+1} x^{(4q+2)n+2j+a} \ln^p(x).$$

Integrating both sides for $x \in (0,1)$, we have, after reversing the order of summation and integration, which is justified by the uniform convergence theorem

$$\int_0^1 \frac{x^a \ln^p(x) \operatorname{Li}_t(-x^{4q+2})}{1+x^2} dx = \sum_{n\geqslant 1} (-1)^{n+1} H_n^{(t)} \sum_{j=0}^{2q} (-1)^{j+1} \int_0^1 x^{(4q+2)n+2j+a} \ln^p(x)\, dx$$

$$= (-1)^p p! \sum_{n\geqslant 1} (-1)^{n+1} H_n^{(t)} \sum_{j=0}^{2q} \frac{(-1)^{j+1}}{((4q+2)n+2j+a+1)^{p+1}}$$

and this is the BBP-type representation for the integral (7) containing harmonic numbers of order t. The second integral in (7) is obtained by the substitution $x = \tan\theta$. □

The next corollary deals with an alternative representation for the integral (7).

COROLLARY 1. *For* $(p,t) \in_0$, $a \geqslant -2$, *and* $q \in \mathbb{R} > -\frac{1}{2}$ *then*

$$I(a,p,q,t) = \int_0^1 \frac{x^a \ln^p(x)}{1+x^2} \operatorname{Li}_t(-x^{4q+2}) dx$$
$$= \frac{(-1)^p p!}{2^{2p+2}} \sum_{n\geqslant 1} \frac{(-1)^n}{n^t} \left(H^{(p+1)}_{n\left(q+\frac{1}{2}\right)+\frac{a-1}{4}} - H^{(p+1)}_{n\left(q+\frac{1}{2}\right)+\frac{a-3}{4}} \right), \tag{9}$$

where $H^{(p+1)}_{n\left(q+\frac{1}{2}\right)+\frac{a-1}{4}}$ *are shifted harmonic numbers of order* $p+1$.

Proof. A Taylor series expansion of

$$\operatorname{Li}_t(-x^{4q+2}) = \sum_{n\geqslant 1} \frac{(-1)^n x^{(4q+2)n}}{n^t} \quad \text{and} \quad \frac{1}{1+x^2} = \sum_{j\geqslant 0} (-1)^j x^{2j}$$

allows us to write, after reversing the order of summation and integration, which is justified by the uniform convergence theorem

$$I(a,p,q,t) = \sum_{n\geqslant 1} \frac{(-1)^n}{n^t} \sum_{j\geqslant 0} (-1)^j \int_0^1 x^{(4q+2)n+2j+a} \ln^p(x)\, dx$$

$$= (-1)^p p! \sum_{n\geqslant 1} \frac{1}{n^t} \sum_{j\geqslant 0} \frac{(-1)^j}{((4q+2)n+2j+a+1)^{p+1}}$$

$$= \frac{(-1)^p p!}{2^{2p+2}} \sum_{n\geqslant 1} \frac{(-1)^n}{n^t} \begin{pmatrix} \zeta\left(p+1, \frac{1}{4}\left((4q+2)n+a+1\right)\right) \\ -\zeta\left(p+1, \frac{1}{4}\left((4q+2)n+a+3\right)\right) \end{pmatrix}$$

$$= \frac{(-1)^p p!}{2^{2p+2}} \sum_{n\geqslant 1} \frac{(-1)^{p(-1)^n}}{p! n^t} \begin{pmatrix} \psi^{(p)}\left(\left(q+\frac{1}{2}\right)n + \frac{a+3}{4}\right) \\ -\psi^{(p)}\left(\left(q+\frac{1}{2}\right)n + \frac{a+1}{4}\right) \end{pmatrix}.$$

From the identity (5) we obtain the required identity

$$I(a,p,q,t)=\frac{(-1)^p p!}{2^{2p+2}}\sum_{n\geqslant 1}\frac{(-1)^n}{n^t}\left(H^{(p+1)}_{n\left(q+\frac{1}{2}\right)+\frac{a-1}{4}}-H^{(p+1)}_{n\left(q+\frac{1}{2}\right)+\frac{a-3}{4}}\right). \qquad \square$$

REMARK 1. For $(p,q)\in\mathbb{N}_0$, we see from (8) and (9) the remarkable Euler sum identity

$$\begin{aligned}&\sum_{n\geqslant 1}(-1)^{n+1}H_n^{(t)}\sum_{j=0}^{2q}\frac{(-1)^{j+1}}{((4q+2)n+2j+a+1)^{p+1}}\\&=\frac{1}{2^{2p+2}}\sum_{n\geqslant 1}\frac{(-1)^n}{n^t}\left(H^{(p+1)}_{n\left(q+\frac{1}{2}\right)+\frac{a-1}{4}}-H^{(p+1)}_{n\left(q+\frac{1}{2}\right)+\frac{a-3}{4}}\right).\end{aligned}$$

The next corollary deals with some special significant cases of the integral (7).

REMARK 2. For the special case $q=0$, $(p,t)\in\mathbb{N}_0$ we have

$$\begin{aligned}I(a,p,0,t)&=\int_0^1\frac{x^a\ln^p(x)}{1+x^2}\mathrm{Li}_t(-x^2)dx\\&=\frac{(-1)^p p!}{2^{2p+2}}\sum_{n\geqslant 1}\frac{(-1)^n}{n^t}\left(H^{(p+1)}_{\frac{n}{2}+\frac{a-1}{4}}-H^{(p+1)}_{\frac{n}{2}+\frac{a-3}{4}}\right)\\&=(-1)^p p!\sum_{n\geqslant 1}\frac{(-1)^n H_n^{(t)}}{(2n+a+1)^{p+1}}\end{aligned}\tag{10}$$

and if we choose a as an odd integer $a=2m-1$, $m\in\mathbb{N}$, say $a=1$, we can obtain the representation

$$I(1,p,0,t)=\frac{(-1)^p p!}{2^{2p+2}}\sum_{n\geqslant 1}\frac{(-1)^n}{n^t}\left(H^{(p+1)}_{\frac{n}{2}}-H^{(p+1)}_{\frac{n}{2}-\frac{1}{2}}\right)$$

and using the multiplication formula (6)

$$H_n^{(p+1)}=\eta(p+1)+\frac{1}{2^{p+1}}H^{(p+1)}_{\frac{n}{2}}+\frac{1}{2^{p+1}}H^{(p+1)}_{\frac{n}{2}-\frac{1}{2}}$$

we have the simplification

$$I(1,p,0,t)=\frac{(-1)^p p!}{2^{2p+2}}\sum_{n\geqslant 1}\frac{(-1)^n}{n^t}\left(\begin{array}{c}2H^{(p+1)}_{\frac{n}{2}}-2^{p+1}H_n^{(p+1)}\\ \\ +2^{p+1}(-1)^p p!\eta(p+1)\end{array}\right).\tag{11}$$

From (10) and (11) we have for $t\in\mathbb{N}\geqslant 2$, $p\in\mathbb{N}_0$

$$\eta(t)\eta(p+1)=\frac{1}{2^p}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^t}\left(2^p H_n^{(p+1)}-H^{(p+1)}_{\frac{n}{2}}\right)+\sum_{n\geqslant 1}\frac{(-1)^{n+1}H_n^{(t)}}{(n+1)^{p+1}},$$

re-ordering the counter in the third sum yields

$$\eta(t)\,\eta(p+1)-\eta(p+t+1)=\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^t}\left(H_n^{(p+1)}-\frac{1}{2^p}H_{\frac{n}{2}}^{(p+1)}\right)\\ -\sum_{n\geqslant 1}\frac{(-1)^{n+1}H_n^{(t)}}{n^{p+1}}$$

where $\eta(0)=\frac{1}{2}$ and $\eta(1)=\ln 2$. In the case $p+1=t$ we find the new Euler sum identity, for $t\in\mathbb{N}$

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}H_{\frac{n}{2}}^{(t)}}{n^t}=2^{t-1}\left(\eta(2t)-\eta^2(t)\right),$$

since we know, for integer $t\geqslant 2$

$$\sum_{n\geqslant 1}\frac{H_n^{(t)}}{n^t}=\frac{1}{2}\left(\zeta(2t)+\zeta^2(t)\right)$$

it follows that

$$\sum_{n\geqslant 1}\frac{H_{\frac{n}{2}}^{(t)}}{n^t}=2^{t-1}\left(\eta(2t)-\eta^2(t)\right)+\frac{1}{2^t}\left(\zeta(2t)+\zeta^2(t)\right).$$

For the case $a=0,\ q=-\frac{1}{4}$ we have

$$I\left(0,p,-\frac{1}{4},t\right)=\frac{(-1)^p\,p!}{2^{2p+1}}\sum_{n\geqslant 1}\frac{(-1)^n}{n^t}\left(H_{\frac{n}{4}-\frac{1}{4}}^{(p+1)}-H_{\frac{n}{4}-\frac{3}{4}}^{(p+1)}\right)$$

and for $p=1,\ t=4$

$$I\left(0,1,-\frac{1}{4},4\right)=\int_0^1\frac{\ln(x)}{1+x^2}\mathrm{Li}_4(-x)dx\\ =\frac{7}{8}\zeta(4)\,G+\frac{9}{8}\zeta(2)\,\beta(4)-\frac{5}{2}\beta(6).$$

In the next corollary we give two more special cases.

COROLLARY 2. *For $p\in\mathbb{N}$, $t=1$, $a\geqslant -2$, and $q\in\mathbb{N}$ then*

$$I(a,p,q,1)=-\int_0^1\frac{x^a\ln^p(x)}{1+x^2}\ln(1+x^{4q+2})dx$$

$$=(-1)^{p+1}p!\sum_{n\geqslant 1}(-1)^{n+1}H_n\sum_{j=0}^{2q}\frac{(-1)^{j+1}}{((4q+2)n+2j+a-1)^{p+1}}. \qquad (12)$$

For $p\in\mathbb{N}$, $t=0$, $a\geqslant -2$, *and* $q\in\mathbb{N}$ *then*

$$I(a,p,q,0)=-\int_0^1 \frac{x^{a+4q+2}\ln^p(x)}{(1+x^2)(1+x^{4q+2})}dx$$

$$=\sum_{j=0}^{2q}(-1)^{j+1}\left(\begin{array}{c}\frac{p}{2^p(4q+2)^{p+1}}\left(\psi^{(p-1)}\left(\frac{2j+a+1}{2(4q+2)}+\frac{1}{2}\right)-\psi^{(p-1)}\left(\frac{2j+a+1}{2(4q+2)}+1\right)\right)\\ +\frac{2j+a+1}{2^{p+1}(4q+2)^{p+2}}\left(\psi^{(p)}\left(\frac{2j+a+1}{2(4q+2)}+\frac{1}{2}\right)-\psi^{(p)}\left(\frac{2j+a+1}{2(4q+2)}+1\right)\right)\end{array}\right). \tag{13}$$

Proof. For the case $t=1$, we notice that $\mathrm{Li}_1(-x^{4q+2})=-\ln(1+x^{4q+2})$ and (12) follows from (8). For the case $t=0$, we notice that $\mathrm{Li}_0(-x^{4q+2})=-\frac{x^{4q+2}}{(1+x^{4q+2})}$. A Taylor series expansion produces

$$\frac{x^{4q+2}}{(1+x^2)(1+x^{4q+2})}=\sum_{n\geqslant 1}(-1)^{n+1}n\sum_{j=0}^{2q}(-1)^j x^{(4q+2)n+2j}$$

in which case

$$I(a,p,q,0)=(-1)^p p!\sum_{j=0}^{2q}(-1)^{j+1}\sum_{n\geqslant 1}\frac{(-1)^{n+1}n}{((4q+2)n+2j+a+1)^{p+1}}.$$

A partial fraction decomposition and simplification leads to (13). □

Some examples follow.

EXAMPLE 1. Let $(a,p,q,t)=(1,5,0,0)$

$$I(1,5,0,0)=\frac{465}{256}\zeta(6)-\frac{225}{128}\zeta(5).$$

Let $(a,p,q,t)=(2,2,1,0)$

$$I(2,2,1,0)=\frac{1249}{3675}-\frac{1}{3}G-\frac{\pi^3}{864}+\frac{5}{10368}\left(\psi^{(2)}\left(\frac{11}{12}\right)-\psi^{(2)}\left(\frac{17}{12}\right)\right)$$
$$+\frac{7}{10368}\left(\psi^{(2)}\left(\frac{19}{12}\right)-\psi^{(2)}\left(\frac{13}{12}\right)\right)$$
$$=2\sum_{n\geqslant 1}(-1)^n n\left(\frac{1}{(6n+3)^3}-\frac{1}{(6n+5)^3}+\frac{1}{(6n+7)^3}\right).$$

Let $(a,p,q,t)=\left(0,5,-\frac{1}{4},2\right)$

$$I\left(0,5,-\frac{1}{4},2\right)=840\beta(8)-40\pi^2\beta(6)-\frac{7}{2}\pi^4\beta(4)-\frac{31}{256}\pi^6 G$$
$$=\frac{5!}{2^{12}}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^2}\left(H^{(6)}_{\frac{n}{4}-\frac{1}{4}}-H^{(6)}_{\frac{n}{4}-\frac{3}{4}}\right).$$
$$=\int_0^{\frac{\pi}{4}}\ln^5(\tan\theta)\operatorname{Li}_2(-\tan\theta)d\theta=\int_0^1\frac{\ln^5 x\operatorname{Li}_2(-x)}{1+x^2}dx$$

where $G=\sum_{n\geqslant 0}\frac{(-1)^{n+1}}{(2n+1)^2}.$ is Catalan's constant, $\beta(\cdot)$ is the Dirichlet beta function.

Let $(a,p,q,t)=(0,0,0,2)$,

$$I(0,0,0,2)=\int_0^1\frac{\operatorname{Li}_2(-x^2)}{1+x^2}dx=\sum_{n\geqslant 1}\frac{(-1)^n H_n^{(2)}}{(2n+1)}$$
$$=\frac{1}{4}\sum_{n\geqslant 1}\frac{(-1)^n}{n^2}\left(H_{\frac{n}{2}-\frac{1}{4}}-H_{\frac{n}{2}-\frac{3}{4}}\right)$$

$$=2G\ln 2-\frac{11\pi^3}{96}-\frac{\pi}{8}\ln^2 2-i\left(4\operatorname{Li}_3\left(\frac{1+i}{2}\right)+\frac{5\pi^2}{48}\ln 2-\frac{1}{12}\ln^3 2-\frac{35}{16}\zeta(3)\right).$$

From Lewin ([11], p. 164, 296) we have that

$$\operatorname{Re}\left(\operatorname{Li}_3\left(\frac{1+i}{2}\right)\right)=\frac{1}{48}\ln^3 2+\frac{35}{64}\zeta(3)-\frac{5\pi^2}{192}\ln 2$$

and therefore

$$I(0,0,1,2)=24\operatorname{Im}\left(\operatorname{Li}_3\left(\frac{1+i}{2}\right)\right)+12G\ln 2-\frac{31\pi^3}{48}-\frac{3\pi}{4}\ln^2 2.$$

Sofo and Nimbran [18] have shown that the imaginary part of the trilogarithm:

$$W(3):=\operatorname{Im}\left(\operatorname{Li}_3\left(\frac{1\pm i}{2}\right)\right)=\sum_{n\geqslant 1}\frac{\sin\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}n^3}$$
$$=\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{2^{2n}}\left(\frac{2}{(4n-3)^3}+\frac{2}{(4n-2)^3}+\frac{1}{(4n-1)^3}\right),$$

and finally we have

$$I(0,0,0,2)=4W(3)+2G\ln 2-\frac{11\pi^3}{96}-\frac{\pi}{8}\ln^2 2.$$

Let $(a,p,q,t)=(0,0,2,1)$

$$I(0,0,2,1)=\int_0^1 \frac{\mathrm{Li}_1(-x^{10})}{1+x^2}dx = 5G-\frac{\pi}{4}\ln\left(\frac{20}{5+8\alpha}\right)$$

$$=\sum_{n\geqslant 1}(-1)^n H_n\left(\frac{1}{10n+1}-\frac{1}{10n+3}+\frac{1}{10n+5}-\frac{1}{10n+7}+\frac{1}{10n+9}\right)$$

where $\alpha=\left(1-\sqrt{5}\right)/2$.

Let $(a,p,q,t)=(-2,1,0,4)$

$$\int_0^1 \frac{x^{-2}\ln(x)\,\mathrm{Li}_4(-x^2)}{1+x^2}dx=\int_0^{\frac{\pi}{4}} \cot^2(x)\ln(\tan\theta)\,\mathrm{Li}_4(-\tan^2\theta)d\theta$$

$$=\sum_{n\geqslant 1}(-1)^{n+1}\frac{H_n^{(4)}}{(2n-1)^2}=\frac{1}{2^4}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^4}\left(H^{(2)}_{\frac{n}{2}-\frac{3}{4}}-H^{(2)}_{\frac{n}{2}-\frac{5}{4}}\right)$$

$$=16G-16\pi+6\zeta(2)+3\zeta(3)+\frac{7}{8}\zeta(4)+32\ln 2+\frac{31}{2}\pi\zeta(5)-\frac{7}{8}\zeta(4)G$$
$$-6\zeta(2)\beta(4)-40\beta(6).$$

In the next theorem we consider the integral (7) on the positive half line $x\geqslant 0$.

THEOREM 2. *For $(p,t)\in\mathbb{N}$, $q>0$*

$$J(p,q,t)=\int_0^\infty \frac{\ln^p(x)\,\mathrm{Li}_t(-x^{4q+2})}{1+x^2}dx=\int_0^\infty g(x;p,q,t)\,dx \tag{14}$$

$$=\int_0^{\frac{\pi}{2}} \ln^p(\tan\theta)\,\mathrm{Li}_t(-\tan^{4q+2}\theta)d\theta$$

$$=\left(1+(-1)^{p+t+1}\right)I(0,p,q,t) \tag{15}$$

$$+2(-1)^{p+t+1}\sum_{j=0}^{\lfloor\frac{t}{2}\rfloor}\frac{(4q+2)^{t-2j}}{(t-2j)!}\eta(2j)\int_0^1 \frac{\ln^{p+t-2j}(x)}{1+x^2}dx$$

where

$$g(x;p,q,t)=\frac{\ln^p(x)\,\mathrm{Li}_t(-x^{4q+2})}{1+x^2}, \tag{16}$$

$I(0,p,q,t)$ is given by (8) or (9) $\eta(2j)$ is the Dirichlet Eta function and $\lfloor\frac{t}{2}\rfloor$ is the Floor function.

Proof. We begin with

$$J(p,q,t) = \int_0^\infty \frac{\ln^p(x)\,\mathrm{Li}_t(-x^{4q+2})}{1+x^2}dx = \int_0^\infty g(x;p,q,t)\,dx$$

and put

$$J(p,q,t) = \int_0^\infty g(x;p,q,t)\,dx = \int_0^1 g(x;p,q,t)\,dx + \int_1^\infty g(x;p,q,t)\,dx.$$

We notice that $g(x;p,q,t)$ is continuous, bounded and differentiable on the interval $x \in (0,1]$, with $\lim_{x\to 0^+} g(x;p,q,t) = \lim_{x\to 1} g(x;p,q,t) = 0$. Now we make the transformation $xy = 1$ in the third integral so that

$$\int_0^\infty g(x;p,q,t)\,dx = \int_0^1 g(x;p,q,t)\,dx + (-1)^p \int_0^1 \frac{\ln^p(y)}{1+y^2}\mathrm{Li}_t(-y^{-(4q+2)})dy. \tag{17}$$

From Lewin ([11], p. 299), Jonquiĕre's relation states

$$\begin{aligned}\mathrm{Li}_s(-z) + (-1)^t \,\mathrm{Li}_s\left(-\frac{1}{z}\right) &= -2\sum_{j=0}^{\lfloor \frac{t}{2}\rfloor} \frac{(\ln z)^{t-2j}}{(t-2j)!}\eta(2j) \\ &= 2\sum_{j=0}^{\lfloor \frac{t}{2}\rfloor} \frac{(\ln z)^{t-2j}}{(t-2j)!}\mathrm{Li}_{2j}(-1)\end{aligned} \tag{18}$$

where $\mathrm{Li}_s(z)$ is a polylogarithm, and this formula corrects a minor misprint in Lewin's book. The relation (18) can also be written in terms of Bernoulli numbers so that

$$\mathrm{Li}_t(-z) + (-1)^t \,\mathrm{Li}_t\left(-\frac{1}{z}\right) = \frac{1}{t!}\sum_{j=0}^{t}\left(1-2^{1-j}\right)\binom{t}{j} B_j\,(2\pi i)^j\,(\ln z)^{t-2j} \tag{19}$$

where B_j are the Bernoulli numbers. Now we can substitute (18) into (17), so that

$$\begin{aligned}\int_0^\infty g(x;p,q,t)\,dx &= \left(1+(-1)^{p+t+1}\right)\int_0^1 g(x;p,q,t)\,dx \\ &\quad +2(-1)^{p+t+1}\sum_{j=0}^{\lfloor \frac{t}{2}\rfloor}\frac{(4q+2)^{t-2j}}{(t-2j)!}\eta(2j)\int_0^1 \frac{\ln^{p+t-2j}(x)}{1+x^2}dx.\end{aligned}$$

The integral

$$I(0,p,q,t) = \int_0^1 \frac{\ln^p(x)\,\mathrm{Li}_t(-x^{4q+2})}{1+x^2}dx$$

has been evaluated in Theorem 1 and therefore

$$J(p,q,t) = \left(1+(-1)^{p+t+1}\right) I(0,p,q,t)$$
$$+2(-1)^{p+t+1} \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(4q+2)^{t-2j}}{(t-2j)!} \eta(2j) \int_0^1 \frac{\ln^{p+t-2j}(x)}{1+x^2} dx$$

and the proof is finished. Note that the integral $I(0,p,q,t)$ does not contribute to $J(p,q,t)$ in the case when $p+t+1$ is an odd integer. The third integral in (14) is obtained by the substitution $x = \tan\theta$. □

REMARK 3. It can be noted, from Jonquière's relation (19) that we are able to determine the value of the integral

$$\int_0^1 \frac{\ln^p(x)\,\mathrm{Li}_t(-x^{-(4q+2)})}{1+x^2} dx = (-1)^{t+1} I(0,p,q,t) \tag{20}$$
$$+2(-1)^{t+1} \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(4q+2)^{t-2j}}{(t-2j)!} \eta(2j) \int_0^1 \frac{\ln^{p+t-2j}(x)}{1+x^2} dx.$$

Using the idea that the degree $(p+t+1)$, in Theorem 2, can be either odd or even we extract the following two special cases.

COROLLARY 3. *Let $p = t$, then*

$$J(t,q,t) = \int_0^\infty \frac{\ln^t(x)\,\mathrm{Li}_t(-x^{4q+2})}{1+x^2} dx$$
$$= \frac{(-1)^t \pi^{2t+1}}{t!} \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(4q+2)^{t-2j}}{2^{2(t-2j)+2}} \binom{t}{2j} \frac{\eta(2j)}{\zeta(2j)} B_{2j} E_{2(t-j)}$$

where B_{2j} are the Bernoulli numbers and $E_{2(t-j)}$ are the Euler numbers.

Let $p+1 = t$, then

$$J(t,q,t) = \int_0^\infty \frac{\ln^{t-1}(x)\,\mathrm{Li}_t(-x^{4q+2})}{1+x^2} dx$$
$$= 2(-1)^{t-1}(t-1)! \sum_{n\geqslant 1} (-1)^{n+1} H_n^{(t)} \sum_{j=0}^{2q} \frac{(-1)^{j+1}}{((4q+2)n+2j+1)^t}$$
$$-(t-1)! \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} (4q+2)^{t-2j} \binom{2t-2j-1}{t-2j} \eta(2j)\beta(2t-2j)$$

where $\beta(2t-2j)$ is the Dirichlet beta functions and $\eta(2j)$ is the Dirichlet Eta function.

Proof. The proof follows from (15) and in the evaluation of the integral when $p=t$,

$$\int_0^1 \frac{\ln^{2t-2j}(x)}{1+x^2}dx = \frac{2(-1)^{t-2j}\pi^{2(t-j)+1}}{2^{2(t-j)+2}}\binom{t}{2j}E_{2(t-j)}$$

and when $p+1=t$,

$$\int_0^1 \frac{\ln^{2t-2j-1}(x)}{1+x^2}dx = -(2t-2j-1)!\beta(2t-2j). \quad \square$$

Some examples follow.

EXAMPLE 2. Let $(p,q,t)=(3,4,5)$

$$\begin{aligned} J(3,4,5) &= \int_0^\infty \frac{\ln^3(x)\operatorname{Li}_5(-x^{18})}{1+x^2}dx = \int_0^{\frac{\pi}{2}} \ln^3(\tan\theta)\operatorname{Li}_5(-\tan^{18}\theta)d\theta \\ &= -2\sum_{j=0}^{\lfloor\frac{5}{2}\rfloor} \frac{(18)^{5-2j}}{(5-2j)!}\eta(2j)\int_0^1 \frac{\ln^{5-2j}(x)}{1+x^2}dx \\ &= -\frac{85345\pi^9}{4}. \end{aligned}$$

Let $(p,q,t)=\left(2,\frac{1}{2},3\right)$

$$\begin{aligned} J\left(2,\frac{1}{2},3\right) &= \int_0^\infty \frac{\ln^2(x)\operatorname{Li}_3(-x^4)}{1+x^2}dx = \int_0^{\frac{\pi}{2}} \ln^2(\tan\theta)\operatorname{Li}_3(-\tan^4\theta)d\theta \\ &= -\frac{3\pi^3}{512\sqrt{2}}\left(\psi^{(2)}\left(\frac{1}{8}\right)-\psi^{(2)}\left(\frac{3}{8}\right)-\psi^{(2)}\left(\frac{5}{8}\right)+\psi^{(2)}\left(\frac{7}{8}\right)\right) \\ &\quad +\frac{\pi^2}{512\sqrt{2}}\left(\psi^{(3)}\left(\frac{1}{8}\right)-\psi^{(3)}\left(\frac{3}{8}\right)+\psi^{(3)}\left(\frac{5}{8}\right)+\psi^{(3)}\left(\frac{7}{8}\right)\right) \\ &\quad +\frac{\pi}{2048\sqrt{2}}\left(\psi^{(4)}\left(\frac{1}{8}\right)-\psi^{(4)}\left(\frac{3}{8}\right)-\psi^{(4)}\left(\frac{5}{8}\right)+\psi^{(4)}\left(\frac{7}{8}\right)\right) \\ &\quad -\frac{3}{32}\pi^3\zeta(3). \end{aligned}$$

This allows us to also calculate, from (15)

$$\int_0^1 \frac{\ln^2(x)\operatorname{Li}_3(-x^4)}{1+x^2}dx = 4\pi^2\beta(4)+1280\beta(6)-\frac{3}{32}\pi^3\zeta(3)$$
$$+\frac{3\pi^3}{512\sqrt{2}}\left(\psi^{(2)}\left(\frac{1}{8}\right)-\psi^{(2)}\left(\frac{3}{8}\right)-\psi^{(2)}\left(\frac{5}{8}\right)+\psi^{(2)}\left(\frac{7}{8}\right)\right)$$

$$+\frac{\pi^2}{512\sqrt{2}}\left(\psi^{(3)}\left(\frac{1}{8}\right)-\psi^{(3)}\left(\frac{3}{8}\right)+\psi^{(3)}\left(\frac{5}{8}\right)+\psi^{(3)}\left(\frac{7}{8}\right)\right)$$
$$+\frac{\pi}{2048\sqrt{2}}\left(\psi^{(4)}\left(\frac{1}{8}\right)-\psi^{(4)}\left(\frac{3}{8}\right)-\psi^{(4)}\left(\frac{5}{8}\right)+\psi^{(4)}\left(\frac{7}{8}\right)\right),$$

and from (20),

$$\int_0^1 \frac{\ln^2(x)\,\mathrm{Li}_3(-x^{-4})}{1+x^2}dx = -2\pi^2\beta(4)-640\beta(6)-\frac{3}{64}\pi^3\zeta(3)$$
$$+\frac{3\pi^3}{1024\sqrt{2}}\left(\psi^{(2)}\left(\frac{1}{8}\right)-\psi^{(2)}\left(\frac{3}{8}\right)-\psi^{(2)}\left(\frac{5}{8}\right)+\psi^{(2)}\left(\frac{7}{8}\right)\right)$$
$$+\frac{\pi^2}{1024\sqrt{2}}\left(\psi^{(3)}\left(\frac{1}{8}\right)-\psi^{(3)}\left(\frac{3}{8}\right)+\psi^{(3)}\left(\frac{5}{8}\right)+\psi^{(3)}\left(\frac{7}{8}\right)\right)$$
$$+\frac{\pi}{4096\sqrt{2}}\left(\psi^{(4)}\left(\frac{1}{8}\right)-\psi^{(4)}\left(\frac{3}{8}\right)-\psi^{(4)}\left(\frac{5}{8}\right)+\psi^{(4)}\left(\frac{7}{8}\right)\right).$$

Let $(p,q,t)=(2,1,3)$

$$J(2,1,3)=\int_0^\infty \frac{\ln^2(x)\,\mathrm{Li}_3(-x^6)}{1+x^2}dx=\int_0^{\frac{\pi}{2}}\ln^2(\tan\theta)\,\mathrm{Li}_3(-\tan^6\theta)d\theta$$
$$=\frac{\pi^2}{36\sqrt{3}}\left(\psi^{(3)}\left(\frac{5}{6}\right)-\psi^{(3)}\left(\frac{1}{6}\right)\right)-\frac{419}{36}\pi^3\zeta(3)-\frac{4991}{6}\pi\zeta(5)$$

also

$$\int_0^1 \frac{\ln^2(x)\,\mathrm{Li}_3(-x^6)}{1+x^2}dx=\frac{\pi^2}{72\sqrt{3}}\left(\psi^{(3)}\left(\frac{5}{6}\right)-\psi^{(3)}\left(\frac{1}{6}\right)\right)-\frac{419}{72}\pi^3\zeta(3)$$
$$+3\pi^2\beta(4)+2160\beta(6)-\frac{4991}{12}\pi\zeta(5)$$
$$=2\sum_{n\geqslant 1}(-1)^n H_n^{(3)}\left(\frac{1}{(6n+1)^3}-\frac{1}{(6n+3)^3}+\frac{1}{(6n+5)^3}\right)$$

and from (20),

$$\int_0^1 \frac{\ln^2(x)\,\mathrm{Li}_3(-x^{-6})}{1+x^2}dx=\frac{\pi^2}{72\sqrt{3}}\left(\psi^{(3)}\left(\frac{5}{6}\right)-\psi^{(3)}\left(\frac{1}{6}\right)\right)-\frac{419}{72}\pi^3\zeta(3)$$
$$-3\pi^2\beta(4)-2160\beta(6)-\frac{4991}{12}\pi\zeta(5).$$

Finally we give the example, $(p,q,t) = (0,2,3)$

$$\int_0^\infty \frac{\mathrm{Li}_3(-x^{10})}{1+x^2} = \frac{\pi}{10}\left(1+\frac{1}{\phi}\right)\left(\psi^{(2)}\left(\frac{1}{10}\right)+\psi^{(2)}\left(\frac{9}{10}\right)\right) - \frac{\pi}{10\phi}\left(\psi^{(2)}\left(\frac{3}{10}\right)+\psi^{(2)}\left(\frac{7}{10}\right)\right) - \frac{\pi}{5}\zeta(3),$$

where the golden ration $\phi = \left(1+\sqrt{5}\right)/2$. Similarly we have the result

$$\int_0^1 \frac{\mathrm{Li}_3(-x^{10})}{1+x^2} = \frac{\pi}{20}\left(1+\frac{1}{\phi}\right)\left(\psi^{(2)}\left(\frac{1}{10}\right)+\psi^{(2)}\left(\frac{9}{10}\right)\right) - \frac{\pi}{20\phi}\left(\psi^{(2)}\left(\frac{3}{10}\right)+\psi^{(2)}\left(\frac{7}{10}\right)\right) - \frac{\pi}{10}\zeta(3) + 5\zeta(2)G + 500\beta(4)$$

$$= \sum_{n\geqslant 1} (-1)^n H_n^{(3)} \left(\frac{1}{10n+1} - \frac{1}{10n+3} + \frac{1}{10n+5} - \frac{1}{10n+7} + \frac{1}{10n+9}\right).$$

Concluding Remarks. We have carried out a systematic study of a family of integrals containing log-polylog functions in terms of Euler sums. We believe most of our results are new in the literature and have given many examples some of which are not amenable to a mathematical computer package.

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