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LOCATION OF ZEROS OF THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. In this paper, we infer bounds for the moduli of the zeros of the polar derivative of univariate complex polynomial and provide some more general results, which yields some classical bounds as special cases.

1. Introduction

Geometrical relationships between the zeros and the coefficients of a polynomial have been frequently explored since the time of Cauchy and have deeply influenced the development of mathematics throughout the centuries. A classical problem in the *geometry of polynomials* is to locate the zeros of a given polynomial by determining discs and annuli in complex plane in which all its zeros are situated. The problem of locating these regions which contain all or some zeros of a polynomial gains importance in this theory due to non availability of any general method for finding these zeros and over a period large number of results have been obtained in this direction by researchers. Among them the Eneström-Kakeya Theorem ([4], [5]) given below is classical and well known in the theory of location of zeros of polynomials.

THEOREM 1.1. *An nth-order polynomial* $p(z)=\sum\limits_{v=0}^{n} a_v z^v$ *has all its zeros in the disk* $|z|\leqslant 1$ *if its coefficients satisfy the following monotonic condition*

$$a_n \geqslant a_{n-1} \geqslant a_{n-2} \ldots \geqslant a_1 \geqslant a_0 > 0.$$

The Eneström-Kakeya Theorem has been extended and generalized in many ways by researchers see ([1], [2], [3]).

For a complex number α, consider the operator D_α which maps a polynomial $p(z)$ into $D_\alpha p(z)=np(z)+(\alpha-z)p'(z)$. This operator is known as polar derivative of $p(z)$ with respect to α and note that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$. It generalizes the ordinary derivative of $p(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha}=p'(z)$$

uniformly with respect to z for $|z|\leqslant R$, $R>0$.

Concerning the moduli of zeros of the polar derivative of a polynomial P. Ramulu and G. L. Reddy [6] proved the following results.

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THEOREM 1.2. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *and* $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ *be the polar derivative of* $p(z)$ *with respect to real* α *such that*

$$a_{n-1} \geqslant 2a_{n-2} \geqslant 3a_{n-3} \ldots \geqslant (n-2)a_2 \geqslant (n-1)a_1 \geqslant na_0.$$

If $\alpha = 0$ *then all the zeros of* $D_0 p(z)$ *lie in*

$$|z| \leqslant \frac{a_{n-1} + n|a_0| - na_0}{|a_{n-1}|}.$$

THEOREM 1.3. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *and* $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ *be the polar derivative of* $p(z)$ *with respect to real* α *such that*

$$a_{n-1} \leqslant 2a_{n-2} \leqslant 3a_{n-3} \ldots \leqslant (n-2)a_2 \leqslant (n-1)a_1 \leqslant na_0.$$

If $\alpha = 0$ *then all the zeros of* $D_0 p(z)$ *lie in*

$$|z| \leqslant \frac{n|a_0| + na_0 - a_{n-1}}{|a_{n-1}|}.$$

Later on G. L. Reddy et al. [7] generalized the above theorems by proving the followings results.

THEOREM 1.4. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *and* $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ *be the polar derivative of* $p(z)$ *with respect to real* α *such that*

$$(i+2)\alpha a_{i+2} + [n-(i+1)]a_{i+1} \geqslant (i+1)\alpha a_{i+1} + (n-i)a_i \quad \text{for } i = 0,1,2\ldots,n-2.$$

Then all the zeros of $D_\alpha p(z)$ *lie in*

$$|z| \leqslant \frac{n\alpha a_n + a_{n-1} + |\alpha a_1 + na_0| - (\alpha a_1 + na_0)}{|n\alpha a_n + a_{n-1}|}.$$

THEOREM 1.5. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *and* $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ *be the polar derivative of* $p(z)$ *with respect to real* α *such that*

$$(i+2)\alpha a_{i+2} + [n-(i+1)]a_{i+1} \leqslant (i+1)\alpha a_{i+1} + (n-i)a_i \quad \text{for } i = 0,1,2\ldots,n-2.$$

Then all the zeros of $D_\alpha p(z)$ *lie in*

$$|z| \leqslant \frac{(\alpha a_1 + na_0) + |\alpha a_1 + na_0| - (n\alpha a_n + a_{n-1})}{|n\alpha a_n + a_{n-1}|}.$$

2. Main results

In this paper we present generalizations of all the above results and find regions containing all the zeros of the polar derivative of a polynomial. In fact, we prove the following results.

THEOREM 1. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *and* $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ *be the polar derivative of* $p(z)$ *with respect to real* α *such that, for some* $k \geqslant 1$

$$\begin{aligned} na_0 &\leqslant 2\alpha a_2 + (n-1-\alpha)a_1 \leqslant (n-1)a_1 \leqslant 3\alpha a_3 + (n-2-2\alpha)a_2 \leqslant (n-2)a_2 \\ &\leqslant 4\alpha a_4 + (n-3-3\alpha)a_3 \leqslant \ldots \leqslant 2a_{n-2} \leqslant [k(n\alpha a_n + a_{n-1}) - (n-1)\alpha a_{n-1}]. \end{aligned}$$

Then all the zeros of $D_\alpha p(z)$ *lie in*

$$|z+k-1| \leqslant \frac{k(n\alpha a_n + a_{n-1}) + |\alpha a_1 + na_0| - (\alpha a_1 + na_0)}{|n\alpha a_n + a_{n-1}|}.$$

COROLLARY 1. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *and* $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ *be the polar derivative of* $p(z)$ *with respect to real* α *such that, for some* $k \geqslant 1$

$$\begin{aligned} 0 < na_0 &\leqslant 2\alpha a_2 + (n-1-\alpha)a_1 \leqslant (n-1)a_1 \leqslant 3\alpha a_3 + (n-2-2\alpha)a_2 \leqslant (n-2)a_2 \\ &\leqslant 4\alpha a_4 + (n-3-3\alpha)a_3 \leqslant \ldots \leqslant 2a_{n-2} \leqslant [k(n\alpha a_n + a_{n-1}) - (n-1)\alpha a_{n-1}]. \end{aligned}$$

Then all the zeros of $D_\alpha p(z)$ *lie in*

$$|z+k-1| \leqslant k.$$

REMARK 1. Theorem 1 is a generalization of Theorem 1.4. For $k = 1$ we get Theorem 1.4.

REMARK 2. On setting $k = 1$ and $\alpha = 0$ in Theorem 1 we get Theorem 1.2.

THEOREM 2. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *and* $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ *be the polar derivative of* $p(z)$ *with respect to real* α *such that, for some* $k \geqslant 1$

$$\begin{aligned} na_0 &\geqslant 2\alpha a_2 + (n-1-\alpha)a_1 \geqslant (n-1)a_1 \geqslant 3\alpha a_3 + (n-2-2\alpha)a_2 \geqslant (n-2)a_2 \\ &\geqslant 4\alpha a_4 + (n-3-3\alpha)a_3 \geqslant \ldots \geqslant 2a_{n-2} \geqslant [k(n\alpha a_n + a_{n-1}) - (n-1)\alpha a_{n-1}]. \end{aligned}$$

Then all the zeros of $D_\alpha p(z)$ *lie in*

$$|z+k-1| \leqslant \frac{|\alpha a_1 + na_0| + (\alpha a_1 + na_0) - k(n\alpha a_n + a_{n-1})}{|n\alpha a_n + a_{n-1}|}.$$

REMARK 3. Theorem 2 is a generalization of Theorem 1.5. For $k = 1$ we get Theorem 1.5 also for $\alpha = 0$ and $k = 1$ we get Theorem 1.3.

COROLLARY 2. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *and* $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ *be the polar derivative of* $p(z)$ *with respect to real* α *such that, for some* $k \geqslant 1$

$$na_0 \geqslant 2\alpha a_2 + (n-1-\alpha)a_1 \geqslant (n-1)a_1 \geqslant 3\alpha a_3 + (n-2-2\alpha)a_2 \geqslant (n-2)a_2$$
$$\geqslant 4\alpha a_4 + (n-3-3\alpha)a_3 \geqslant \ldots \leqslant 2a_{n-2} \geqslant [k(n\alpha a_n + a_{n-1}) - (n-1)\alpha a_{n-1}] > 0.$$

Then all the zeros of $D_\alpha p(z)$ *lie in*

$$|z + k - 1| \leqslant \frac{2(\alpha a_1 + na_0) - k(n\alpha a_n + a_{n-1})}{n\alpha a_n + a_{n-1}}.$$

Next we find lower bounds for the moduli of the zeros of $D_\alpha p(z)$ under the same conditions on the coefficients. In fact, we prove the followings theorems.

THEOREM 3. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *and* $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ *be the polar derivative of* $p(z)$ *with respect to real* α *such that, for some* $k \geqslant 1$

$$na_0 \leqslant 2\alpha a_2 + (n-1-\alpha)a_1 \leqslant (n-1)a_1 \leqslant 3\alpha a_3 + (n-2-2\alpha)a_2 \leqslant (n-2)a_2$$
$$\leqslant 4\alpha a_4 + (n-3-3\alpha)a_3 \leqslant \ldots \leqslant 2a_{n-2} \leqslant [k(n\alpha a_n + a_{n-1}) - (n-1)\alpha a_{n-1}].$$

Then all the zeros of $D_\alpha p(z)$ *lie in*

$$|z + k - 1| \geqslant \frac{|\alpha a_1 + na_0| - k(n\alpha a_n + a_{n-1}) + (\alpha a_1 + na_0)}{|n\alpha a_n + a_{n-1}|}.$$

Combining Theorem 1 and Theorem 3, we get the following result.

THEOREM 4. *Let* $p(z)$ *be a polynomial of degree* n *and* $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ *be the polar derivative of* $p(z)$ *with respect to real* α *such that, for some* $k \geqslant 1$

$$na_0 \leqslant 2\alpha a_2 + (n-1-\alpha)a_1 \leqslant (n-1)a_1 \leqslant 3\alpha a_3 + (n-2-2\alpha)a_2 \leqslant (n-2)a_2$$
$$\leqslant 4\alpha a_4 + (n-3-3\alpha)a_3 \leqslant \ldots \leqslant 2a_{n-2} \leqslant [k(n\alpha a_n + a_{n-1}) - (n-1)\alpha a_{n-1}].$$

Then all the zeros of $D_\alpha p(z)$ *lie in*

$$\frac{|\alpha a_1 + na_0| - k(n\alpha a_n + a_{n-1}) + (\alpha a_1 + na_0)}{|n\alpha a_n + a_{n-1}|} \leqslant |z + k - 1|$$
$$\leqslant \frac{k(n\alpha a_n + a_{n-1}) + |\alpha a_1 + na_0| - \alpha a_1 - na_0}{|n\alpha a_n + a_{n-1}|}.$$

Taking $k = 1$ and the coefficients to be positive, we get the following result from Theorem 4.

COROLLARY 3. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *and* $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ *be the polar derivative of* $p(z)$ *with respect to real* α *such that*

$$\begin{aligned} 0 < na_0 &\leqslant 2\alpha a_2 + (n-1-\alpha)a_1 \leqslant (n-1)a_1 \leqslant 3\alpha a_3 + (n-2-2\alpha)a_2 \leqslant (n-2)a_2 \\ &\leqslant 4\alpha a_4 + (n-3-3\alpha)a_3 \leqslant \ldots \leqslant 2a_{n-2} \leqslant [k(n\alpha a_n + a_{n-1}) - (n-1)\alpha a_{n-1}]. \end{aligned}$$

Then all the zeros of $D_\alpha p(z)$ *lie in*

$$\frac{2(\alpha a_1 + na_0) - k(n\alpha a_n + a_{n-1})}{n\alpha a_n + a_{n-1}} \leqslant |z| \leqslant 1.$$

THEOREM 5. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *and* $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ *be the polar derivative of* $p(z)$ *with respect to real* α *such that, for some* $k \geqslant 1$

$$\begin{aligned} na_0 &\geqslant 2\alpha a_2 + (n-1-\alpha)a_1 \geqslant (n-1)a_1 \geqslant 3\alpha a_3 + (n-2-2\alpha)a_2 \geqslant (n-2)a_2 \\ &\geqslant 4\alpha a_4 + (n-3-3\alpha)a_3 \geqslant \ldots \geqslant 2a_{n-2} \geqslant [k(n\alpha a_n + a_{n-1}) - (n-1)\alpha a_{n-1}]. \end{aligned}$$

Then all the zeros of $D_\alpha p(z)$ *lie in*

$$|z + k - 1| \geqslant \frac{|\alpha a_1 + na_0| + k(n\alpha a_n + a_{n-1}) - (\alpha a_1 + na_0)}{|n\alpha a_n + a_{n-1}|}.$$

Combining Theorem 2 and Theorem 5, we obtain the following result.

THEOREM 6. *Let* $p(z)$ *be a polynomial of degree* n *and* $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ *be the polar derivative of* $p(z)$ *with respect to real* α *such that, for some* $k \geqslant 1$

$$\begin{aligned} na_0 &\geqslant 2\alpha a_2 + (n-1-\alpha)a_1 \geqslant (n-1)a_1 \geqslant 3\alpha a_3 + (n-2-2\alpha)a_2 \geqslant (n-2)a_2 \\ &\geqslant 4\alpha a_4 + (n-3-3\alpha)a_3 \geqslant \ldots \geqslant 2a_{n-2} \geqslant [k(n\alpha a_n + a_{n-1}) - (n-1)\alpha a_{n-1}]. \end{aligned}$$

Then all the zeros of $D_\alpha p(z)$ *lie in*

$$\begin{aligned} &\frac{|\alpha a_1 + na_0| + k(n\alpha a_n + a_{n-1}) - (\alpha a_1 + na_0)}{|n\alpha a_n + a_{n-1}|} \leqslant |z + k - 1| \\ &\leqslant \frac{|\alpha a_1 + na_0| + \alpha a_1 + na_0 - k(n\alpha a_n + a_{n-1})}{|n\alpha a_n + a_{n-1}|}. \end{aligned}$$

Taking $k = 1$ and imposing positive monotonic conditions on coefficients in Theorem 6, we get the following result.

COROLLARY 4. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *and* $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ *be the polar derivative of* $p(z)$ *with respect to real* α *such that*

$$\begin{aligned}&na_0 \geqslant 2\alpha a_2 + (n-1-\alpha)a_1 \geqslant (n-1)a_1 \geqslant 3\alpha a_3 + (n-2-2\alpha)a_2 \geqslant (n-2)a_2\\ &\geqslant 4\alpha a_4 + (n-3-3\alpha)a_3 \geqslant \ldots \geqslant 2a_{n-2} \geqslant [k(n\alpha a_n + a_{n-1}) - (n-1)\alpha a_{n-1}] > 0.\end{aligned}$$

Then all the zeros of $D_\alpha p(z)$ *lie in*

$$1 \leqslant |z| \leqslant \frac{2(\alpha a_1 + na_0) - (n\alpha a_n + a_{n-1})}{n\alpha a_n + a_{n-1}}.$$

3. Proofs of the Theorems

Proof of Theorem 1. Let $p(z)$ be a polynomial of degree n and $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ be the polar derivative of $p(z)$ with respect to real α of degree at most $n-1$. Then

$$D_\alpha p(z) = \sum_{v=0}^{n-1} [(v+1)\alpha a_{v+1} + (n-v)a_v] z^v.$$

Let us consider the polynomial

$$\begin{aligned}H(z) &= (1-z)D_\alpha p(z)\\ &= -[n\alpha a_n + a_{n-1}]z^n + [(n\alpha a_n + a_{n-1}) - ((n-1)\alpha a_{n-1} + 2a_{n-2})]z^{n-1}\\ &\quad + [((n-1)\alpha a_{n-1} + 2a_{n-2}) - ((n-2)\alpha a_{n-2} + 3a_{n-3})]z^{n-2} + \ldots\\ &\quad + [(3\alpha a_3 + (n-2)a_2) - (2\alpha a_2 + (n-1)a_1)]z^2\\ &\quad + [(2\alpha a_2 + (n-1)a_1) - (\alpha a_1 + na_0)]z + (\alpha a_1 + na_0).\end{aligned}$$

Equivalently,

$$\begin{aligned}H(z) = &-[n\alpha a_n + a_{n-1}]z^n + [n\alpha a_n + a_{n-1}]z^{n-1} - k[n\alpha a_n + a_{n-1}]z^{n-1}\\ &+ [k(n\alpha a_n + a_{n-1}) - ((n-1)\alpha a_{n-1} + 2a_{n-2})]z^{n-1} + \ldots\\ &+ [(3\alpha a_3 + (n-2)a_2) - (2\alpha a_2 + (n-1)a_1)]z^2\\ &+ [(2\alpha a_2 + (n-1)a_1) - (\alpha a_1 + na_0)]z + (\alpha a_1 + na_0).\end{aligned}$$

Then for $|z| > 1$, i.e. ($\frac{1}{|z|} < 1$) and by application of triangle inequality, we have,

$$\begin{aligned}|H(z)| = &|(n\alpha a_n + a_{n-1})z^{n-1}[z + k - 1]\\ &+ [k(n\alpha a_n + a_{n-1}) - ((n-1)\alpha a_{n-1} + 2a_{n-2})]z^{n-1} + \ldots\\ &+ [(3\alpha a_3 + (n-2)a_2) - (2\alpha a_2 + (n-1)a_1)]z^2\\ &+ [(2\alpha a_2 + (n-1)a_1) - (\alpha a_1 + na_0)]z + (\alpha a_1 + na_0)|.\end{aligned}$$

This gives,

$$\begin{aligned}|H(z)| \geqslant\ & |n\alpha a_n + a_{n-1}||z|^{n-1}|z+k-1| \\ & - |z|^{n-1}\Big|[k(n\alpha a_n + a_{n-1}) - ((n-1)\alpha a_{n-1} + 2a_{n-2})] \\ & + [((n-1)\alpha a_{n-1} + 2a_{n-2}) - ((n-2)\alpha a_{n-2} + 3a_{n-3})]\frac{1}{z} + \dots \\ & + [(3\alpha a_3 + (n-2)a_2) - (2\alpha a_2 + (n-1)a_1)]\frac{1}{z^{n-3}} \\ & + [(2\alpha a_2 + (n-1)a_1) - (\alpha a_1 + na_0)]\frac{1}{z^{n-2}} + (\alpha a_1 + na_0)\frac{1}{z^{n-1}}\Big|.\end{aligned}$$

Therefore,

$$\begin{aligned}|H(z)| \geqslant\ & |z|^{n-1}\Big[|n\alpha a_n + a_{n-1}||z+k-1| \\ & - \Big(|k(n\alpha a_n + a_{n-1}) - ((n-1)\alpha a_{n-1} + 2a_{n-2})| \\ & + |((n-1)\alpha a_{n-1} + 2a_{n-2}) - ((n-2)\alpha a_{n-2} + 3a_{n-3})|\frac{1}{|z|} + \dots \\ & + |(3\alpha a_3 + (n-2)a_2) - (2\alpha a_2 + (n-1)a_1)|\frac{1}{|z|^{n-3}} \\ & + |(2\alpha a_2 + (n-1)a_1) - (\alpha a_1 + na_0)|\frac{1}{|z|^{n-2}} + |\alpha a_1 + na_0|\frac{1}{|z|^{n-1}}\Big)\Big],\end{aligned}$$

that is, for $\frac{1}{|z|} < 1$,

$$\begin{aligned}|H(z)| \geqslant\ & |z|^{n-1}[|n\alpha a_n + a_{n-1}||z+k-1| \\ & - \{k(n\alpha a_n + a_{n-1}) - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|\}] \\ >\ & 0\end{aligned}$$

if

$$|n\alpha a_n + a_{n-1}||z+k-1| - \{k(n\alpha a_n + a_{n-1}) - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|\} > 0,$$

i.e., if

$$|n\alpha a_n + a_{n-1}||z+k-1| > \{k(n\alpha a_n + a_{n-1}) - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|\}.$$

Hence $|H(z)| > 0$ if

$$|z+k-1| > \frac{1}{|n\alpha a_n + a_{n-1}|}\{k(n\alpha a_n + a_{n-1}) - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|\}.$$

Hence all the zeros of $H(z)$ with $|z| > 1$ lie in the disk

$$|z+k-1| \leqslant \frac{1}{|n\alpha a_n + a_{n-1}|}\{k(n\alpha a_n + a_{n-1}) - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|\}.$$

Those zeros with $|z| \leqslant 1$ trivially satisfy the above inequality. Since the set of zeros of $D_\alpha p(z)$ is subset of the set of zeros of $H(z)$, therefore it follows that all the zeros of $D_\alpha p(z)$ lie in

$$|z+k-1| \leqslant \frac{1}{|n\alpha a_n + a_{n-1}|}\left\{k(n\alpha a_n + a_{n-1}) - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|\right\}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. We have $D_\alpha p(z) = \sum_{v=0}^{n-1}[(v+1)\alpha a_{v+1} + (n-v)a_v]z^v$. Consider the polynomial

$$\begin{aligned}
S(z) &= (1-z)D_\alpha p(z) \\
&= -[n\alpha a_n + a_{n-1}]z^n + \sum_{j=0}^{n-1}\left\{(j+1)\alpha a_{j+1} + [(n-j)-\alpha j]a_j - (n-j+1)a_{j-1}\right\}z^j, \\
&\qquad a_{-1} = 0 \\
&= -[n\alpha a_n + a_{n-1}]z^n + [n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}]z^{n-1} \\
&\quad + \sum_{j=0}^{n-2}\left\{(j+1)\alpha a_{j+1} + [(n-j)-\alpha j]a_j - (n-j+1)a_{j-1}\right\}z^j \\
&= -[n\alpha a_n + a_{n-1}]z^n + [n\alpha a_n + a_{n-1}]z^{n-1} - k[n\alpha a_n + a_{n-1}]z^{n-1} \\
&\quad + [k(n\alpha a_n + a_{n-1}) + (\alpha - n\alpha)a_{n-1} - 2a_{n-2})]z^{n-1} \\
&\quad + \sum_{j=0}^{n-2}\left\{(j+1)\alpha a_{j+1} + [(n-j)-\alpha j]a_j - (n-j+1)a_{j-1}\right\}z^j,
\end{aligned}$$

i.e.,

$$\begin{aligned}
S(z) = &- (n\alpha a_n + a_{n-1})z^{n-1}[z+k-1] + \{[k(n\alpha a_n + a_{n-1}) + (\alpha - n\alpha)a_{n-1} - 2a_{n-2}]z^{n-1} \\
&+ \sum_{j=0}^{n-2}\{(j+1)\alpha a_{j+1} + [(n-j)-\alpha j]a_j - (n-j+1)a_{j-1}\}z^j\}.
\end{aligned}$$

Then for $|z| > 1$, i.e. ($\frac{1}{|z|} < 1$), we have

$$\begin{aligned}
|S(z)| \geqslant |z|^{n-1}\Big\{&|n\alpha a_n + a_{n-1}||z+k-1| - \Big[|k(n\alpha a_n + a_{n-1}) + (\alpha - n\alpha)a_{n-1} - 2a_{n-2}| \\
&+ \sum_{j=0}^{n-2}|(j+1)\alpha a_{j+1} + [(n-j)-\alpha j]a_j - (n-j+1)a_{j-1}|\frac{1}{|z|^{n-j-1}}\Big]\Big\} \\
\geqslant |z|^{n-1}\{&|n\alpha a_n + a_{n-1}||z+k-1| - [(\alpha a_1 + na_0) + |\alpha a_1 + na_0| - k(n\alpha a_n + a_{n-1})]\}.
\end{aligned}$$

Hence, $|S(z)| > 0$ if

$$|z+k-1| > \frac{1}{|n\alpha a_n + a_{n-1}|}\{(\alpha a_1 + na_0) + |\alpha a_1 + na_0| - k(n\alpha a_n + a_{n-1})\}.$$

That is, all the zeros of $S(z)$ with $|z|>1$ lie in the disk

$$|z+k-1|\leqslant \frac{1}{|n\alpha a_n+a_{n-1}|}\{(\alpha a_1+na_0)+|\alpha a_1+na_0|-k(n\alpha a_n+a_{n-1})\}.$$

Since all the zeros of $S(z)$ with $|z|\leqslant 1$ already lie in

$$|z+k-1|\leqslant \frac{1}{|n\alpha a_n+a_{n-1}|}\{(\alpha a_1+na_0)+|\alpha a_1+na_0|-k(n\alpha a_n+a_{n-1})\},$$

therefore, it follows that all the zeros of $S(z)$ lie in

$$|z+k-1|\leqslant \frac{1}{|n\alpha a_n+a_{n-1}|}\{(\alpha a_1+na_0)+|\alpha a_1+na_0|-k(n\alpha a_n+a_{n-1})\}.$$

Thus all the zeros of $D_\alpha p(z)$ lie in

$$|z+k-1|\leqslant \frac{1}{|n\alpha a_n+a_{n-1}|}\{(\alpha a_1+na_0)+|\alpha a_1+na_0|-k(n\alpha a_n+a_{n-1})\}. \quad \square$$

Proof of Theorem 3. Let $p(z)=\sum\limits_{v=0}^{n} a_v z^v$ be a polynomial of degree n and $D_\alpha p(z)=np(z)+(\alpha-z)p'(z)$ be the polar derivative of $p(z)$ with respect to real α of degree at most $n-1$. Then

$$D_\alpha p(z)=\sum_{v=0}^{n-1}[(v+1)\alpha a_{v+1}+(n-v)a_v]z^v.$$

Now, consider the polynomial

$$\begin{aligned}
F(z)&=(1-z)D_\alpha p(z)\\
&=-[n\alpha a_n+a_{n-1}]z^n+[(n\alpha a_n+a_{n-1})-((n-1)\alpha a_{n-1}+2a_{n-2})]z^{n-1}+\dots\\
&\quad+[(3\alpha a_3+(n-2)a_2)-(2\alpha a_2+(n-1)a_1)]z^2+[(2\alpha a_2\\
&\quad+(n-1)a_1)-(\alpha a_1+na_0)]z+(\alpha a_1+na_0)\\
&=-[n\alpha a_n+a_{n-1}]z^n+[n\alpha a_n+a_{n-1}]z^{n-1}-k[n\alpha a_n+a_{n-1}]z^{n-1}\\
&\quad+[k(n\alpha a_n+a_{n-1})-((n-1)\alpha a_{n-1}+2a_{n-2})]z^{n-1}+\dots\\
&\quad+[(3\alpha a_3+(n-2)a_2)-(2\alpha a_2+(n-1)a_1)]z^2\\
&\quad+[(2\alpha a_2+(n-1)a_1)-(\alpha a_1+na_0)]z+(\alpha a_1+na_0)\\
&=(\alpha a_1+na_0)+U(z),
\end{aligned}$$

where

$$\begin{aligned}
U(z)=&-[n\alpha a_n+a_{n-1}]z^n+[n\alpha a_n+a_{n-1}]z^{n-1}-k[n\alpha a_n+a_{n-1}]z^{n-1}\\
&+[k(n\alpha a_n+a_{n-1})-((n-1)\alpha a_{n-1}+2a_{n-2})]z^{n-1}+\dots\\
&+[(3\alpha a_3+(n-2)a_2)-(2\alpha a_2+(n-1)a_1)]z^2\\
&+[(2\alpha a_2+(n-1)a_1)-(\alpha a_1+na_0)]z
\end{aligned}$$

$$
\begin{aligned}
&= -(n\alpha a_n + a_{n-1})z^{n-1}[z+k-1] \\
&\quad + [k(n\alpha a_n + a_{n-1}) - ((n-1)\alpha a_{n-1} + 2a_{n-2})]z^{n-1} + \dots \\
&\quad + [(3\alpha a_3 + (n-2)a_2) - (2\alpha a_2 + (n-1)a_1)]z^2 \\
&\quad + [(2\alpha a_2 + (n-1)a_1) - (\alpha a_1 + na_0)]z.
\end{aligned}
$$

For $|z| \leqslant 1$, we have by using hypothesis

$$
\begin{aligned}
|U(z)| &\leqslant |n\alpha a_n + a_{n-1}||z+k-1| + |k(n\alpha a_n + a_{n-1}) - ((n-1)\alpha a_{n-1} + 2a_{n-2})| + \dots \\
&\quad + |(3\alpha a_3 + (n-2)a_2) - (2\alpha a_2 + (n-1)a_1)| \\
&\quad + |(2\alpha a_2 + (n-1)a_1) - (\alpha a_1 + na_0)| \\
&= |n\alpha a_n + a_{n-1}||z+k-1| + k(n\alpha a_n + a_{n-1}) - (\alpha a_1 + na_0).
\end{aligned}
$$

Hence, for $|z| \leqslant 1$,

$$
\begin{aligned}
|F(z)| &= |(\alpha a_1 + na_0) + U(z)| \\
&\geqslant |\alpha a_1 + na_0| - |U(z)| \\
&\geqslant |\alpha a_1 + na_0| - [|n\alpha a_n + a_{n-1}||z+k-1| + k(n\alpha a_n + a_{n-1}) - (\alpha a_1 + na_0)] \\
&> 0
\end{aligned}
$$

if

$$
|z+k-1| < \frac{|\alpha a_1 + na_0| - k(n\alpha a_n + a_{n-1}) + \alpha a_1 + na_0}{|n\alpha a_n + a_{n-1}|}.
$$

This shows that all the zeros of $F(z)$ lie in

$$
|z+k-1| \geqslant \frac{|\alpha a_1 + na_0| - k(n\alpha a_n + a_{n-1}) + \alpha a_1 + na_0}{|n\alpha a_n + a_{n-1}|}.
$$

Since all the zeros of $D_\alpha p(z)$ are also the zeros of $F(z)$. The proof of Theorem 3 is complete. □

Proof of Theorem 5. We have $D_\alpha p(z) = \sum_{v=0}^{n-1}[(v+1)\alpha a_{v+1} + (n-v)a_v]z^v$. Consider the polynomial

$$
\begin{aligned}
S(z) &= (1-z)D_\alpha p(z) \\
&= -[n\alpha a_n + a_{n-1}]z^n + \sum_{j=0}^{n-1}\left\{(j+1)\alpha a_{j+1} + [(n-j) - \alpha j]a_j - (n-j+1)a_{j-1}\right\}z^j, \\
&\qquad a_{-1} = 0 \\
&= -[n\alpha a_n + a_{n-1}]z^n + [n\alpha a_n + (1+\alpha - n\alpha)a_{n-1} - 2a_{n-2}]z^{n-1} \\
&\quad + \sum_{j=1}^{n-2}\left\{(j+1)\alpha a_{j+1} + [(n-j) - \alpha j]a_j - (n-j+1)a_{j-1}\right\}z^j + (\alpha a_1 + na_0)
\end{aligned}
$$

$$\begin{aligned}
&= -[n\alpha a_n + a_{n-1}]z^n + [n\alpha a_n + a_{n-1}]z^{n-1} - k[n\alpha a_n + a_{n-1}]z^{n-1} \\
&\quad + [k(n\alpha a_n + a_{n-1}) + (\alpha - n\alpha)a_{n-1} - 2a_{n-2})]z^{n-1} + \\
&\sum_{j=1}^{n-2} \left\{ (j+1)\alpha a_{j+1} + [(n-j) - \alpha j]a_j - (n-j+1)a_{j-1} \right\} z^j + (\alpha a_1 + na_0) \\
&= (\alpha a_1 + na_0) + G(z),
\end{aligned}$$

where

$$\begin{aligned}
G(z) = &-(n\alpha a_n + a_{n-1})z^{n-1}[z + k - 1] + [k(n\alpha a_n + a_{n-1}) + (\alpha - n\alpha)a_{n-1} - 2a_{n-2}]z^{n-1} \\
&+ \sum_{j=1}^{n-2} \{(j+1)\alpha a_{j+1} + [(n-j) - \alpha j]a_j - (n-j+1)a_{j-1}\} z^j.
\end{aligned}$$

For $|z| \leqslant 1$, we have by using hypothesis

$$\begin{aligned}
|G(z)| \leqslant &|n\alpha a_n + a_{n-1}||z + k - 1| + |k(n\alpha a_n + a_{n-1}) + (\alpha - n\alpha)a_{n-1} - 2a_{n-2}| \\
&+ \sum_{j=1}^{n-2} |(j+1)\alpha a_{j+1} + [(n-j) - \alpha j]a_j - (n-j+1)a_{j-1}| \\
&= |n\alpha a_n + a_{n-1}||z + k - 1| - k(n\alpha a_n + a_{n-1}) + (\alpha a_1 + na_0).
\end{aligned}$$

Hence, for $|z| \leqslant 1$,

$$\begin{aligned}
|S(z)| &= |\alpha a_1 + na_0 + G(z)| \\
&\geqslant |\alpha a_1 + na_0| - |G(z)| \\
&\geqslant |\alpha a_1 + na_0| - [|n\alpha a_n + a_{n-1}||z + k - 1| - k(n\alpha a_n + a_{n-1}) + (\alpha a_1 + na_0)] \\
&> 0
\end{aligned}$$

if

$$|z + k - 1| < \frac{|\alpha a_1 + na_0| + k(n\alpha a_n + a_{n-1}) - (\alpha a_1 + na_0)}{|n\alpha a_n + a_{n-1}|}.$$

This shows that all the zeros of $S(z)$ lie in

$$|z + k - 1| \geqslant \frac{|\alpha a_1 + na_0| + k(n\alpha a_n + a_{n-1}) - (\alpha a_1 + na_0)}{|n\alpha a_n + a_{n-1}|}.$$

Since all the zeros of $D_\alpha p(z)$ are also the zeros of $S(z)$. The proof of Theorem 5 is complete. □

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