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A NOTE ON ENESTRÖM–KAKEYA THEOREM

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Abstract. We develop a technique to find a region containing all the zeros of a polynomial $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree n with real coefficients by adding suitable weights to its coefficients. This technique enables one to use computer programming to convert any polynomial of degree n to Kakeya polynomial and thereby obtain best possible region containing all its zeros.

1. Introduction

The impossibility of solving polynomial equations of degree greater or equal to five by radicals is an important milestone in the history of mathematics, occasioned with the ground breaking discoveries in algebra by N. H. Abel and E. Galois in the first quarter of nineteenth century. This motivated the study of identifying suitable regions in the complex plane containing all the zeros of a given polynomial of degree $n \geqslant 5$. Gauss and Cauchy were the first to contribute in this field. A classical result due to Cauchy [3] in this direction may be stated as:

THEOREM A. *If $P(z) := \sum_{j=0}^{n} a_j z^j$ is a complex polynomial of degree n, then all the zeros of $P(z)$ lie in*

$$|z| < 1 + M,$$

where $M := \max_{0 \leqslant j \leqslant n-1} |a_j/a_n|$.

The study of the distribution of zeros of polynomials due to numerous applications in various fields has been the inspiration for much theoretical research. But algebraic and analytic methods for finding zeros of polynomials or the regions where the zeros lie, in general are quite complicated. Since zeros of a polynomial are continuous functions of its coefficients, therefore, for attaining better and sharp bounds it is desirable to put some restrictions on the coefficients of the polynomials. In this connection the following elegant result betterly known as Eneström-Kakeya Theorem (for refrences see [6], [7]) can be stated as:

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THEOREM B. *If* $P(z) := \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with real coefficients, such that*

$$a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_0 > 0,$$

then all the zeros of $P(z)$ *lie in* $|z| \leqslant 1$.

This is a sharp result and has applications in various fields like Algebra, Algebraic Geometry, Computer Science, Differential Geometry, Physics etc. Keeping in view the importance of this result and the nature of restriction on coefficients, the attention was paid to its generalizations in various ways.

Joyal, Labelle and Rahman [4] while maintaining the condition of monotonicity on the coefficients, dropped the condition of these to be positive and proved the following result.

THEOREM C. *If* $P(z) := \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with real coefficients such that*

$$a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z| \leqslant \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

Aziz and Zargar [2] relaxed the hypothesis of Eneström-Kakeya theorem in a different way and proved the following result.

THEOREM D. *If* $P(z) := \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with real coefficients such that for some* $k \geqslant 1$,

$$ka_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z + k - 1| \leqslant \frac{ka_n - a_0 + |a_0|}{|a_n|}.$$

Liman, Tawheeda and Shah [5] assumed that if the first three coefficients of the polynomial do not satisfy the condition of monotonicity and some weight $k \geqslant 1$ is found such that the coefficients after putting this weight in a suitable way are made to satisfy the Kakeya property, then following result holds.

THEOREM E. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n. *If for some* $k \geqslant 1$

$$k^2 a_n \geqslant ka_{n-1} \geqslant a_{n-2} \geqslant \ldots \geqslant a_1 \geqslant a_0 > 0,$$

then all the zeros of $P(z)$ lie in

$$|z+k-1| \leqslant k+2(k-1)\frac{a_{n-1}}{a_n}.$$

Recently Rather, Dar and Iqbal [8] generalized the above results in a different way and proved:

THEOREM F. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients such that for some $k_j \geqslant 1,\ j = 1,2,\ldots,r,\ 1 \leqslant r \leqslant n$,*

$$k_1 a_n \geqslant k_2 a_{n-1} \geqslant k_3 a_{n-2} \geqslant \ldots \geqslant k_r a_{n-r+1} \geqslant a_{n-r} \geqslant \ldots \geqslant a_1 \geqslant a_0,$$

then all the zeros of $P(z)$ lie in

$$\left|z+(k_1-1)-(k_2-1)\frac{a_{n-1}}{a_n}\right| \leqslant \frac{1}{|a_n|}\Bigg(k_1 a_n-(k_2-1)|a_{n-1}| + 2\sum_{j=2}^{r}(k_j-1)|a_{n-j+1}|-a_0+|a_0|\Bigg).$$

In this paper, we devise a method to transform any given polynomial to a Kakeya-type polynomial by adding suitable weights to its coefficients and prove the following result.

2. Statements of the results

THEOREM 1. (Main) *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients such that for some $k_j \geqslant 0,\ j = 1,2,\ldots,r,\ 1 \leqslant r \leqslant n$,*

$$k_n + a_n \geqslant k_{n-1} + a_{n-1} \geqslant \ldots \geqslant k_r + a_r \geqslant a_{r-1} \geqslant \ldots \geqslant a_1 \geqslant a_0,$$

then all the zeros of $P(z)$ lie in

$$\left|z+\frac{k_n-k_{n-1}}{a_n}\right| \leqslant \frac{1}{|a_n|}\left[(k_n-k_{n-1})+a_n-a_0+|a_0|+2\sum_{j=1}^{n-r}k_{n-j}\right]. \tag{1}$$

Theorem 1 is applicable to all those polynomials whose first r coefficients are not monotonic. In this case we can find best possible weights $k_j, j = 1,2,\ldots,r,\ 1 \leqslant r \leqslant n$, so that these coefficients after adding these weights can be made monotonic.

REMARK 1. The essence of Theorem 1 is that we can make use of computer programming to convert any polynomial of degree n to Kakeya-type polynomial and there by locate the region containing all its zeros.

If we choose the transformation $k_j = (\lambda_j - 1)a_j,\ a_j > 0,$ so that for $k_j \geqslant 0,$ we have $\lambda_j \geqslant 1,$ for all $j = 1,2,\ldots,r,\ 1 \leqslant r \leqslant n,$ then we obtain the following result which is due to Rather, Dar and Iqbal [8].

COROLLARY 1. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with real coefficients such that for some* $\lambda_j \geqslant 1,\ j = 1,2,\dots,r,\ 1 \leqslant r \leqslant n,$

$$\lambda_n a_n \geqslant \lambda_{n-1} a_{n-1} \geqslant \dots \geqslant \lambda_r a_r \geqslant a_{r-1} \geqslant \dots \geqslant a_1 \geqslant a_0$$

then all the zeros of $P(z)$ *lie in*

$$\left| z + (\lambda_n - 1) - (\lambda_{n-1} - 1)\frac{a_{n-1}}{a_n} \right| \leqslant \frac{1}{|a_n|}\Big[\lambda_n a_n - (\lambda_{n-1} - 1)|a_{n-1}| - a_0 + |a_0| + 2\sum_{j=1}^{n-r}(\lambda_{n-j} - 1)|a_{n-j}|\Big].$$

Choosing $k_j = k$, for all $j = 1,\dots,r,\ 1 \leqslant r \leqslant n$, in Theorem 1 we get the following result.

COROLLARY 2. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with real coefficients such that for some* $k \geqslant 0$,

$$k + a_n \geqslant k + a_{n-1} \geqslant \dots \geqslant k + a_r \geqslant a_{r-1} \geqslant \dots \geqslant a_1 \geqslant a_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z| \leqslant \frac{1}{|a_n|}\Big[a_n - a_0 + |a_0| + 2(n-r)k\Big].$$

Corollary 2 is very interesting and has practical application to a polynomial with coefficients $a_j,\ j = 1,2,\dots,n$, such that for some $r,\ 1 \leqslant r \leqslant n$,

$$a_n \geqslant a_{n-1} \geqslant \dots \geqslant a_r \leqslant a_{r-1} \geqslant a_{r-2} \geqslant \dots \geqslant a_0.$$

In this case we can easily find a real number $k > 0$, such that

$$k + a_n \geqslant k + a_{n-1} \geqslant \dots \geqslant k + a_r \geqslant a_{r-1} \geqslant a_{r-2} \geqslant \dots \geqslant a_0$$

and the radius of the circle containing all the zeros of $P(z)$ depends on this k, which can be as small as possible.

REMARK 2. For $k = 0$, Corollary 2 reduces to a result due to Joyalle, Labelle and Rahman [4].

If we choose the transformation $k = (\lambda - 1)a_j,\ a_j > 0,\ \lambda \geqslant 1$, for all $j = 1,2,\dots,r$, $1 \leqslant r \leqslant n$, in Corollary 2, we obtain the following result.

COROLLARY 3. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with real coefficients such that for some* $\lambda \geqslant 1,\ a_j > 0,\ j = 1,2,\dots,r,\ 1 \leqslant r \leqslant n,$

$$\lambda a_n \geqslant \lambda a_{n-1} \geqslant \dots \geqslant \lambda a_r \geqslant a_{r-1} \geqslant \dots \geqslant a_1 \geqslant a_0,$$

then all the zeros of $P(z)$ lie in

$$|z| \leqslant \frac{1}{|a_n|}\left[a_n - a_0 + |a_0| + 2(\lambda - 1)\sum_{j=1}^{n-r} a_{n-j}\right].$$

The following result can also be obtained by choosing the coefficients to be positive.

COROLLARY 4. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients such that for some $k_j \geqslant 0,\ j = 1,2,\dots,r,\ 1 \leqslant r \leqslant n,$*

$$k_n + a_n \geqslant k_{n-1} + a_{n-1} \geqslant \dots \geqslant k_r + a_r \geqslant a_{r-1} \geqslant \dots \geqslant a_1 \geqslant a_0 > 0$$

then all the zeros of $P(z)$ lie in

$$\left|z + \frac{k_n - k_{n-1}}{a_n}\right| \leqslant 1 + \frac{1}{a_n}\left[(k_n - k_{n-1}) + 2\sum_{j=1}^{n-r} k_{n-j}\right].$$

3. Computation and analysis

In this section we give some examples of polynomials to illustrate that Theorem 1 gives comparatively better bound for the region containing the zeros of a polynomial than Theorem A. It is important to mention here that all the existing Eneström-Kakeya type results are not applicable to the type of polynomials considered. Except for polynomial in Example 1, where Theorem F is applicable but gives the same region as given by Theorem 1.

EXAMPLE 1. Let $P(z) = 2.5z^4 + 2.7z^3 + 3z^2 + z + 1$.

Results	Radius of Circle	Area of Circle
Theorem A	2.2	15.205
Theorem F	1.32	5.4739
Theorem 1	1.32	5.4739
Actual Bound	0.96349	2.91638

Table 1: *Example 1*

Here the monotone hypothesis is violated. Therefore, we choose $k_4 = 0.5$ and $k_3 = 0.3, k_2 = 0, k_1 = 0,$ in Theorem 1 and it is evident that the above table gives better bound with 43.18% than Theorem A.

While choosing $k_1 = 1.2$ and $k_2 = 1.1111111111$, in Theorem F, for this polynomial, the radius of the circle containing all the zeros of $P(z)$ is same as that of Theorem 1.

EXAMPLE 2. Let $P(z) = 11z^3 + 12.5z^2 + 1$.

Results	Raduis of Circle	Area of Circle
Theorem A	2.136	14.266
Theorem F	not applicable	
Theorem 1	1.318	5.457
Actual Bound	1.1995	4.520

Table 2: *Example 2*

Here $a_1 = 0$ and the monotone hypothesis is violated. Therefore, we choose $k_3 = 1.5$ and $k_2 = 0, k_1 = 1$ in Theorem 1 and it is evident that the above table gives better bound, with 61.7% than Theorem A.

EXAMPLE 3. Let $P(z) = 17z^5 + 13z^4 + 11z^3 + 7z^2 + 1.5$.

Results	Raduis of Circle	Area of Circle
Theorem A	1.764	9.7756
Theorem F	not applicable	
Theorem 1	1.176	4.344
Actual Bound	0.8055	2.038

Table 3: *Example 3*

Here $a_1 = 0$ and the monotone hypothesis is violated. Therefore, we choose $k_1 = 1.5$ and $k_2 = k_3 = k_4 = k_5 = 0$, in Theorem 1 and it is evident that the above table gives better bound, with 55.5% improvement in the area over Theorem A.

EXAMPLE 4. Let $P(z) = 20z^6 + 21z^5 + 21.5z^4 + 2z^3 + z^2 + 1$.

Results	Raduis of Disk	Area of Disk
Theorem A	2.075	13.52
Theorem F	not applicable	
Theorem 1	1.2	4.52
Actual Bound	0.95567	2.869

Table 4: *Example 4*

Here $a_1 = 0$ and the monotone hypothesis is violated. Therefore, we choose $k_5 = 0.5$ $k_6 = 1.5, k_1 = 1$ and $k_2 = k_3 = k_4 = 0$, in Theorem 1 and it is evident that the above table gives better bound, with 66% than Theorem A.

4. Proof of the Theorem 1

Proof. Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z)\\
&= (1-z)(a_nz^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0)\\
&= -a_nz^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \ldots + (a_{r+1} - a_r)z^{r+1}\\
&\quad + (a_r - a_{r-1})z^r + \ldots + (a_1 - a_0)z + a_0\\
&= -a_nz^{n+1} + \Big[(k_n + a_n) - (k_{n-1} + a_{n-1})\Big]z^n\\
&\quad + \Big[(k_{n-1} + a_{n-1}) - (k_{n-2} + a_{n-2})\Big]z^{n-1} + \ldots\\
&\quad + \Big[(k_{r+1} + a_{r+1}) - (k_r + a_r)\Big]z^{r+1} + \Big[(k_r + a_r) - a_{r-1}\Big]z^r + \ldots\\
&\quad + \Big[a_1 - a_0\Big]z + a_0 - \sum_{j=r+1}^{n}(k_j - k_{j-1})z^j - k_rz^r,
\end{aligned}$$

where $k_j \geqslant 0,\ j = 1,2,\ldots,r,\ 1 \leqslant r \leqslant n$.

So that we have

$$\begin{aligned}
|F(z)| \geqslant |z|^n \Bigg[\Big| -a_nz - (k_n - k_{n-1})\Big| - \Bigg\{ &|(k_n + a_n) - (k_{n-1} + a_{n-1})|\\
&+ \frac{|(k_{n-1} + a_{n-1}) - (k_{n-2} + a_{n-2})|}{|z|} + \ldots + \frac{|(k_{r+1} + a_{r+1}) - (k_r + a_r)|}{|z|^{n-r-1}}\\
&+ \frac{|(k_r + a_r) - a_{r-1}|}{|z|^{n-r}} + \ldots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} + \sum_{j=r+1}^{n-1}\frac{(|k_j| + |k_{j-1}|)}{|z|^{n-j}} + \frac{|k_r|}{|z|^{n-r}}\Bigg\}\Bigg].
\end{aligned}$$

Let $|z| > 1$, so that $\dfrac{1}{|z|^{n-j}} < 1,\ j = 0,1,\ldots,n-1$, we have

$$\begin{aligned}
|F(z)| > |z|^n|a_n|\Bigg[\bigg|z + \frac{k_n - k_{n-1}}{a_n}\bigg| - \frac{1}{|a_n|}\Bigg\{ &|(k_n + a_n) - (k_{n-1} + a_{n-1})|\\
&+ |(k_{n-1} + a_{n-1}) - (k_{n-2} + a_{n-2})| + \ldots + |(k_{r+1} + a_{r+1}) - (k_r + a_r)|\\
&+ |(k_r + a_r) - a_{r-1}| + \ldots + |a_1 - a_0| + |a_0| + \sum_{j=r+1}^{n-1}(|k_j| + |k_{j-1}|) + |k_r|\Bigg\}\Bigg]\\
= |z|^n|a_n|\Bigg[\bigg|z + \frac{k_n - k_{n-1}}{a_n}\bigg| - \frac{1}{|a_n|}\Bigg\{ &|(k_n + a_n) - (k_{n-1} + a_{n-1})|\\
&+ |(k_{n-1} + a_{n-1}) - (k_{n-2} + a_{n-2})| + \ldots + |(k_{r+1} + a_{r+1}) - (k_r + a_r)|\\
&+ |(k_r + a_r) - a_{r-1}| + \ldots + |a_1 - a_0| + |a_0| - |k_{n-1}| + 2\sum_{j=r}^{n-1}|k_j|\Bigg\}\Bigg].
\end{aligned}$$

Using the monotone assumption satisfied by the coefficients and the fact that $k_j \geqslant 0$, for all $j = 1, 2, \ldots, r,\ 1 \leqslant r \leqslant n$, we get

$$\begin{aligned}
|F(z)| > |z|^n |a_n| \Bigg[\left| z + \frac{k_n - k_{n-1}}{a_n} \right| - \frac{1}{|a_n|} \Big\{ & (k_n + a_n) - (k_{n-1} + a_{n-1}) \\
& + (k_{n-1} + a_{n-1}) - (k_{n-2} + a_{n-2}) + \ldots + (k_{r+1} + a_{r+1}) - (k_r + a_r) \\
& + (k_r + a_r) - a_{r-1} + \ldots + a_1 - a_0 + |a_0| - k_{n-1} + 2 \sum_{j=1}^{n-r} k_{n-j} \Big\} \Bigg] \\
= |z|^n |a_n| \Bigg[\left| z + \frac{k_n - k_{n-1}}{a_n} \right| - \frac{1}{|a_n|} \Big\{ & k_n + a_n - a_0 + |a_0| - k_{n-1} + 2 \sum_{j=1}^{n-r} k_{n-j} \Big\} \Bigg] \\
= |z|^n |a_n| \Bigg[\left| z + \frac{k_n - k_{n-1}}{a_n} \right| - \frac{1}{|a_n|} \Big\{ & k_n - k_{n-1} + a_n - a_0 + |a_0| + 2 \sum_{j=1}^{n-r} k_{n-j} \Big\} \Bigg] \\
> 0, &
\end{aligned}$$

if

$$\left| z + \frac{k_n - k_{n-1}}{a_n} \right| > \frac{1}{|a_n|} \left\{ k_n - k_{n-1} + a_n - a_0 + |a_0| + 2 \sum_{j=1}^{n-r} k_{n-j} \right\}.$$

This implies that all the zeros of $F(z)$ of modulus greater or equal to 1 lie in

$$\left| z + \frac{k_n - k_{n-1}}{a_n} \right| \leqslant \frac{1}{|a_n|} \left\{ k_n - k_{n-1} + a_n - a_0 + |a_0| + 2 \sum_{j=1}^{n-r} k_{n-j} \right\}.$$

Since all the zeros of $F(z)$ with modulus less or equal to 1 already lie in this region, we conclude that all the zeros of $F(z)$ and consequently those of $p(z)$ lie in

$$\left| z + \frac{k_n - k_{n-1}}{a_n} \right| \leqslant \frac{1}{|a_n|} \left\{ k_n - k_{n-1} + a_n - a_0 + |a_0| + 2 \sum_{j=1}^{n-r} k_{n-j} \right\}.$$

This completes the proof of Theorem 1. □

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