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UNIQUENESS OF NON–HOMOGENEOUS DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS SHARING A SMALL FUNCTION

DILIP CHANDRA PRAMANIK* AND JAYANTA ROY

Abstract. In this paper, we study the uniqueness of differential polynomials of meromorphic functions that share a small function. Our results improve and generalize results of Lahiri and Pal [12].

1. Introduction and main results

Let $\mathbb{C}$ denote the finite complex plane and let f be a non-constant meromorphic function defined on $\mathbb{C}$. We assume that the reader is familiar with the standard definitions and notations used in the Nevanlinna value distribution theory, such as $T(r,f), m(r,f), N(r,f)$ (see [5, 17, 21]). By $S(r,f)$ we denote any quantity satisfying the condition $S(r,f) = \circ(T(r,f))$ as $r \to \infty$ possibly outside an exceptional set of finite linear measure. A meromorphic function a is called a small function with respect to f if either $a \equiv \infty$ or $T(r,a) = S(r,f)$. We denote by $S(f)$ the collection of all small functions with respect to f. Clearly $\mathbb{C} \cup \{\infty\} \subset S(f)$ and $S(f)$ is a field over the set of complex numbers. For $b \in \mathbb{C} \cup \{\infty\}$ the quantities

$$\delta(b,f) = 1 - \limsup_{r\to\infty} \frac{N(r,b;f)}{T(r,f)}$$

and

$$\Theta(b,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,b;f)}{T(r,f)}$$

are respectively called the deficiency and ramification index of b for the function f.

We write $E(a,f) = \{z \in \mathbb{C} : f(z) - a = 0\}$, where each zero with multiplicity m is counted m times. If we ignore the multiplicity, then the set is denoted by $\overline{E}(a,f)$. For any two non-constant meromorphic functions f and g, and $a \in S(f) \cap S(g)$ we say that f and g share a IM (CM) provided that $\overline{E}(a,f) = \overline{E}(a,g)$ $(E(a,f) = E(a,g))$.

In 1976 Yang [16] posted the following question:

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* Corresponding author.

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Suppose that f and g are two transcendental entire functions such that f and g share the value 0 CM and $f^{(1)}$, $g^{(1)}$ share the value 1 CM. What can be said about the relationship between f and g?

Many authors, including Shibazaki [15], Yi [18, 19], Yang and Yi [20], Hua [6], Mues and Reinders [14], Lahiri [8, 9] studied the question of Yang [16].

The following result of Yi [18] is an answer to the question of Yang [16] when the n-th derivatives of f and g share the value 1 CM.

THEOREM 1. [18] *Let f and g be two non-constant entire functions and let n be a positive integer. If f, g share the value 0 CM, $f^{(n)}$, $g^{(n)}$ share the value 1 CM and $\delta(0;f) > \frac{1}{2}$, then either $f \equiv g$ or $f^{(n)}.g^{(n)} \equiv 1$.*

The following theorems are improvement of Theorem 1.

THEOREM 2. [22] *Let n be a positive integer and f, g be two non-constant meromorphic functions such that $f^{(n)}$, $g^{(n)}$ share the value 1 CM. If*

$$2\delta(0;f) + (n+4)\Theta(\infty;f) > n+5$$

and

$$2\delta(0;g) + (n+4)\Theta(\infty;g) > n+5,$$

then either $f \equiv g$ or $f^{(n)}.g^{(n)} \equiv 1$.

THEOREM 3. [22] *Let n be a positive integer and f, g be two non-constant meromorphic functions such that $f^{(n)}$, $g^{(n)}$ share the value 1 IM. If*

$$5\delta(0;f) + (4n+7)\Theta(\infty;f) > 4n+11$$

and

$$5\delta(0;g) + (4n+7)\Theta(\infty;g) > 4n+11,$$

then either $f \equiv g$ or $f^{(n)}.g^{(n)} \equiv 1$.

Let $n \geqslant 2$ be a positive integer. An expression of the form

$$L(f) = f^{(n)} + a_{k-1}f^{(n-1)} + \ldots + a_0 f, \tag{1}$$

where $a_0, a_1, \ldots, a_{n-1}$ are complex constants, is called a linear differential polynomial of f.

In 2015, Li and Li considered the problem of replacing the nth derivatives in Theorems 1–3 by the respective linear differential polynomials. They proved the following theorems:

THEOREM 4. [13] *Let f and g be two non-constant entire functions. Suppose that f, g share the value 0 CM and $L(f)$, $L(g)$ share the value 1 CM and $\delta(0;f) > \frac{1}{2}$. If $\rho(f) \neq 1$, then either $f \equiv g$ or $L(f).L(g) \equiv 1$.*

THEOREM 5. [13] *Let f and g be two non-constant entire functions. Suppose that f, g share the value 0 CM and $L(f)$, $L(g)$ share the value 1 IM and $\delta(0;f) > \frac{4}{5}$. If $\rho(f) \neq 1$, then either $f \equiv g$ or $L(f).L(g) \equiv 1$.*

DEFINITION 1. Let $n_{0j}, n_{1j}, n_{2j}, \ldots, n_{kj}$ be non-negative integers. The expression

$$M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}}(f^{(2)})^{n_{2j}} \ldots (f^{(k)})^{n_{kj}}$$

is called a differential monomial generated by f of degree $d(M_j) = \sum\limits_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum\limits_{i=0}^{k} (i+1)n_{ij}$. Let $a_j \in S(f)$ and $a_j \not\equiv 0$ $(j = 1, 2, \ldots, t)$. The sum

$$P[f] = \sum_{j=1}^{t} a_j M_j[f] \tag{2}$$

is called a differential polynomial generated by f of degree $\overline{d}(P)$, lower degree $\underline{d}$(P), where

$$\overline{d}(P) = \max\{d(M_j) : 1 \leqslant j \leqslant t\},$$

$$\underline{d}(P) = \min\{d(M_j) : 1 \leqslant j \leqslant t\}.$$

The quantity Q is defined by

$$Q = \max\{\Gamma_{M_j} - d(M_j) : 1 \leqslant j \leqslant t\} = \max\left\{\sum_{i=0}^{k} i.n_{ij} : 1 \leqslant j \leqslant t\right\}.$$

Further, $P[f]$ is said to be homogeneous differential polynomial of degree d if $\overline{d}(P) = \underline{d}(P) = d$.

When we consider $P[f]$ and $P[g]$ are non-constant differential polynomials of non-constant meromorphic functions f and g respectively, then we understand that the coefficients $a_j \in S(f) \cap S(g)$.

Recently Lahiri and Pal [12] extended the results of Li and Li [13] to homogeneous differential polynomials and proved the following theorem.

THEOREM 6. *Let f and g be two non-constant meromorphic functions, $a = a(z)$ $(\not\equiv 0, \infty) \in S(f) \cap S(g)$.*

Suppose $P[f] = \sum_{j=1}^{t} a_j \prod_{i=0}^{k} (f^{(i)})^{n_{ij}}$ and $P[g] = \sum_{j=1}^{t} a_j \prod_{i=0}^{k} (g^{(i)})^{n_{ij}}$ are non-constant homogeneous differential polynomials of f and g respectively. If $P[f]$ and $P[g]$ share a IM, and

$$\min\left\{5\delta(0,f) + \frac{4Q+7}{d}\Theta(\infty,f), 5\delta(0,g) + \frac{4Q+7}{d}\Theta(\infty,g)\right\} > \frac{4Q+4d+7}{d}, \tag{3}$$

then either $P[f] \equiv P[g]$ or $P[f].P[g] \equiv a^2$.

REMARK 1. If $P[f]$ and $P[g]$ share a CM, then the condition (3) of Theorem 6 can be replaced by the condition

$$\min\left\{2\delta(0,f)+\frac{Q+4}{d}\Theta(\infty,f),2\delta(0,g)+\frac{Q+4}{d}\Theta(\infty,g)\right\}>\frac{Q+d+4}{d}.$$

One can ask the following question:

QUESTION 1. *Can we get the same conclusion as in Theorem 6 when non-homogeneous differential polynomials $P[f]$ and $P[g]$ share a small function?*

To present the main results we recall the following definition of weighted sharing which is between CM and IM sharing.

DEFINITION 2. [10, 11] Let l be a non-negative integer or infinity and $a\in S(f)$. We denote by $E_l(a,f)$ the set of all zeros of $f-a$, where a zero of multiplicity m is counted m times if $m\leqslant l$ and $l+1$ times if $m>l$. If $E_l(a,f)=E_l(a,g)$, we say that f, g share the function a with weight l. We write f and g share (a,l) to mean that f and g share the function a with weight l. Since $E_l(a,f)=E_l(a,g)$ implies that $E_s(a,f)=E_s(a,g)$ for any integer $s\,(0\leqslant s<l)$, if f, g share (a,l), then f, g share (a,s), $(0\leqslant s<l)$. Moreover, we note that f and g share the function a IM or CM if and only if f and g share $(a,0)$ or (a,∞) respectively.

Suppose f and g share 1 IM and let z_0 be a zero of $f-1$ of multiplicity p and a zero of $g-1$ of multiplicity q. Throughout this paper we denote by $\overline{N}_L(r,\frac{1}{f-1})$ the reduced counting function of those 1-points of f and g where $p>q\geqslant 1$; $\overline{N}_L(r,\frac{1}{g-1})$ is defined similarly. By $N_E^{1)}(r,\frac{1}{f-1})$ the counting function of those 1-points of f and g where $p=q=1$ and denote by $\overline{N}_E^{(2}(r,\frac{1}{f-1})$ the counting function of those 1-points of f and g where $p=q\geqslant 2$, where each such zero is counted only once.

DEFINITION 3. Let p be a positive integer and $a\in S(f)$.

(i) $\overline{N}_{p)}(r,\frac{1}{f-a})$ denotes the counting function of those a-points of f whose multiplicities are not greater than p, where each a-point is counted only once.

(ii) $\overline{N}_{(p}(r,\frac{1}{f-a})$ denotes the counting function of those a-points of f whose multiplicities are not less than p, where each a-point is counted only once.

In this paper we take up Question 1 and prove the uniqueness of non-homogeneous differential polynomials $P[f]$ and $P[g]$.

THEOREM 7. *Let f and g be two non-constant meromorphic functions, $a=a(z)$ $(\not\equiv 0,\infty)\in S(f)\cap S(g)$. Suppose that $P[f]$ and $P[g]$, as defined by (2), are non-constant non-homogeneous differential polynomials of f and g respectively of the same degree, lower degree and quantity Q. If $P[f]$ and $P[g]$ share (a,l) with one of the following conditions:*

(i) $l \geqslant 2$ *and*

$$\min\left\{2\overline{d}(P)\delta(0,f)+(Q+4)\Theta(\infty,f),2\overline{d}(P)\delta(0,g)+(Q+4)\Theta(\infty,g)\right\} > Q+4+\overline{d}(P), \tag{4}$$

(ii) $l=1$ *and*

$$\min\{5\underline{d}(P)\delta(0,f)+(3Q+9)\Theta(\infty,f),5\underline{d}(P)\delta(0,g)+(3Q+9)\Theta(\infty,g)\} > 3Q+3\overline{d}(P)+9, \tag{5}$$

(iii) $l=0$ *and*

$$\min\{5\underline{d}(P)\delta(0,f)+(4Q+7)\Theta(\infty,f),5\underline{d}(P)\delta(0,g)+(4Q+7)\Theta(\infty,g)\} > 4Q+4\overline{d}(P)+7, \tag{6}$$

then either $P[f]\equiv P[g]$ *or* $P[f].P[g]\equiv a^2$.

THEOREM 8. *Let* f *and* g *be two non-constant meromorphic functions,* $a = a(z)$ $(\not\equiv 0,\infty) \in S(f)\cap S(g)$. *Let* f *and* g *share the values* 0 *CM and* ∞ *IM. Let* $P[f]$ *and* $P[g]$ *be same as in Theorem 7. If* $P[f]$ *and* $P[g]$ *share* (a,l) *with one of the following conditions:*

(i) $l \geqslant 2$ *and*

$$2\overline{d}(P)\delta(0,f)+(Q+4)\Theta(\infty,f) > Q+4+\overline{d}(P),$$

(ii) $l=1$ *and*

$$5\underline{d}(P)\delta(0,f)+(3Q+9)\Theta(\infty,f) > 3Q+3\overline{d}(P)+9,$$

(iii) $l=0$ *and*

$$5\underline{d}(P)\delta(0,f)+(4Q+7)\Theta(\infty,f) > 4Q+4\overline{d}(P)+7,$$

then either $P[f]\equiv P[g]$ *or* $P[f].P[g]\equiv a^2$.

THEOREM 9. *Let* f *and* g *be two non-constant entire functions,* $a = a(z)$ $(\not\equiv 0,\infty) \in S(f)\cap S(g)$. *Let* f *and* g *share the values* 0 *CM. Let* $P[f]$ *and* $P[g]$ *be same as in Theorem 7. If* $P[f]$ *and* $P[g]$ *share* (a,l) *with one of the following conditions:*

(i) $l \geqslant 2$ *and*

$$\delta(0,f) > \frac{1}{2},$$

(ii) $l=1$ *and*

$$\delta(0,f) > \frac{3}{5},$$

(iii) $l=0$ *and*

$$\delta(0,f) > \frac{4}{5},$$

then either $P[f]\equiv P[g]$ *or* $P[f].P[g]\equiv a^2$.

COROLLARY 1. *Let f and g be two non-constant entire functions such that f and g share the value 0 CM. Suppose that $L(f)$ and $L(g)$ are non-constant linear differential polynomials of f and g respectively as defined by (1). If $L(f)$ and $L(g)$ share $(1,l)$ with one of the following conditions:*

(i) $l \geqslant 2$ and

$$\delta(0,f) > \frac{1}{2},$$

(ii) $l = 1$ and

$$\delta(0,f) > \frac{3}{5},$$

(iii) $l = 0$ and

$$\delta(0,f) > \frac{4}{5},$$

then either $f \equiv g$ or $L(f).L(g) \equiv 1$ under any one of the following conditions:

(i) $\rho(f) \neq 1$,

(ii) $\rho(f) = 1$ and

(a) f has at most a finite number of zeros, or

(b) f has infinitely many zeros and f is of minimal type.

2. Lemmas

In this section we present some lemmas which will be needed in proofing the main theorems.

LEMMA 1. [4] *Let f be a non-constant meromorphic function and $P[f]$ be defined by (2), then*

$$(i)\ N\left(r,\frac{1}{P[f]}\right) \leqslant T\left(r,P[f]\right) - \overline{d}(P)T(r,f) + \left(\overline{d}(P) - \underline{d}(P)\right)m\left(r,\frac{1}{f}\right)$$
$$+\overline{d}(P)N\left(r,\frac{1}{f}\right) + S(r,f) + S(r,g),$$
$$(ii)\ N\left(r,\frac{1}{P[f]}\right) \leqslant Q\overline{N}(r,f) + \left(\overline{d}(P) - \underline{d}(P)\right)m\left(r,\frac{1}{f}\right)$$
$$+\overline{d}(P)N\left(r,\frac{1}{f}\right) + S(r,f) + S(r,g).$$

LEMMA 2. [3] *Let f be a non-constant meromorphic function and $P[f]$ be defined by (2), then*

$$T(r,P[f]) \leqslant Q\overline{N}(r,f) + \overline{d}(P)T(r,f) + S(r,f).$$

LEMMA 3. [1] *If F, G be non-constant meromorphic functions such that F and G share $(1,1)$, then $2\overline{N}_L(r,\frac{1}{F-1}) + 2\overline{N}_L(r,\frac{1}{G-1}) + \overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right) - \overline{N}_{F>2}(r,\frac{1}{G-1}) \leqslant N(r,\frac{1}{G-1}) - \overline{N}(r,\frac{1}{G-1}) + S(r,F) + S(r,G)$.*

LEMMA 4. [1] *If* F, G *be non-constant meromorphic functions such that* F *and* G *share* $(1,1)$, *then* $\overline{N}_{F>2}(r,\frac{1}{G-1}) < \frac{1}{2}\overline{N}(r,F) + \frac{1}{2}\overline{N}(r,\frac{1}{F}) - \frac{1}{2}N_0(r,\frac{1}{F^{(1)}})$.

LEMMA 5. [2] *If* F, G *be non-constant meromorphic functions such that* F *and* G *share* $(1,0)$, *then* $\overline{N}_L(r,\frac{1}{F-1}) < \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + S(r,F)$.

LEMMA 6. [7] *Let* f *be a transcendental meromorphic function and* $P[f]$ *as defined by (2) be non-constant with* $\underline{d}(P) \geqslant 1$. *Then*

$$\underline{d}(P)T(r,f) \leqslant \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{P[f]-1}\right) + \underline{d}(P)N\left(r,\frac{1}{f}\right) - N_0\left(r,\frac{1}{(P(f))^{(1)}}\right) + S(r,f),$$

where $N_0\big(r,\frac{1}{(P[f])^{(1)}}\big)$ *denotes the counting function corresponding to the zeros of* $(P[f])^{(1)}$ *which are not the zeros of* $P[f]$ *and* $P[f]-1$.

3. Proof of main theorems

Proof of Theorem 7. Let

$$F = \frac{P[f]}{a},\ G = \frac{P[g]}{a}.$$

Since $P[f]$ and $P[g]$ share (a,l), it follows that F, G share $(1,l)$ except at the zeros and poles of a. Also note that $\overline{N}(r,F) = \overline{N}(r,f) + S(r,f)$ and $\overline{N}(r,G) = \overline{N}(r,g) + S(r,g)$.

Define

$$H \equiv \left(\frac{F^{(2)}}{F^{(1)}} - 2\frac{F^{(1)}}{F-1}\right) - \left(\frac{G^{(2)}}{G^{(1)}} - 2\frac{G^{(1)}}{G-1}\right). \tag{7}$$

We shall show that $H \equiv 0$. Suppose on the contrary that $H \not\equiv 0$. Then from (7) we have $m(r,H) = S(r,f) + S(r,g)$.

By second fundamental theorem of Nevanlinna we have

$$\begin{aligned} T(r,F) + T(r,G) \leqslant\ & \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}(r,G) \\ & + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) - N_0\left(r,\frac{1}{F^{(1)}}\right) \\ & - N_0\left(r,\frac{1}{G^{(1)}}\right) + S(r,F) + S(r,G), \end{aligned} \tag{8}$$

where $N_0(r,\frac{1}{F^{(1)}})$ denotes the counting function corresponding to the zeros of $F^{(1)}$ which are not the zeros of F and $F-1$. $N_0(r,\frac{1}{G^{(1)}})$ is defined similarly.

We consider the following cases:

Case 1: $l \geqslant 1$. Then from (7) we have

$$
\begin{aligned}
N_E^{1)}\left(r,\frac{1}{F-1}\right) &\leqslant N\left(r,\frac{1}{H}\right) \leqslant T(r,H)+O(1)\\
&\leqslant N(r,H)+S(r,F)+S(r,G).\\
&\leqslant \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}(r,G)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\
&\quad +\overline{N}_L\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F^{(1)}}\right)+N_0\left(r,\frac{1}{G^{(1)}}\right)+S(r,F)+S(r,G),
\end{aligned}
$$

and so

$$
\begin{aligned}
\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right) &= N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)\\
&\quad +\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\
&\leqslant \overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\
&\quad +2\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)\\
&\quad +\overline{N}\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F^{(1)}}\right)+N_0\left(r,\frac{1}{G^{(1)}}\right)\\
&\quad +S(r,F)+S(r,G). \qquad (9)
\end{aligned}
$$

Subcase 1.1: $l=1$. Using Lemmas 3 and 4 we have

$$
\begin{aligned}
&2\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\
&\leqslant N\left(r,\frac{1}{G-1}\right)+\overline{N}_{F>2}\left(r,\frac{1}{G-1}\right)\\
&\leqslant N\left(r,\frac{1}{G-1}\right)+\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)-\frac{1}{2}N_0\left(r,\frac{1}{F^{(1)}}\right)\\
&\quad +S(r,F)+S(r,G). \qquad (10)
\end{aligned}
$$

Thus from (9) and (10) we have

$$
\begin{aligned}
\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right) &\leqslant \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}(r,G)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\
&\quad +N\left(r,\frac{1}{G-1}\right)+\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)\\
&\quad +\frac{1}{2}N_0\left(r,\frac{1}{F^{(1)}}\right)+N_0\left(r,\frac{1}{G^{(1)}}\right)\\
&\quad +S(r,F)+S(r,G). \qquad (11)
\end{aligned}
$$

Using Lemma 1 and (11), we obtain from (8) that

$$\begin{aligned}
T(r,F) \leqslant\ & 2\overline{N}(r,F)+2\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right) \\
& +\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,F)+S(r,G). \\
\leqslant\ & \frac{5}{2}\overline{N}(r,F)+2\overline{N}(r,G)+N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right) \\
& +\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,F)+S(r,G). \\
\leqslant\ & \frac{5}{2}\overline{N}(r,f)+2\overline{N}(r,g)+T(r,F)-\overline{d}(P)T(r,f) \\
& +\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)+\overline{d}(P)N\left(r,\frac{1}{f}\right)+Q\overline{N}(r,g) \\
& +\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{g}\right)+\overline{d}(P)N\left(r,\frac{1}{g}\right)+\frac{Q}{2}\overline{N}(r,f) \\
& +\frac{1}{2}\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)+\frac{\overline{d}(P)}{2}N\left(r,\frac{1}{f}\right)+S(r,f)+S(r,g).
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \overline{d}(P)T(r,f) \leqslant\ & \frac{Q+5}{2}\overline{N}(r,f)+(Q+2)\overline{N}(r,g)+\frac{3\overline{d}(P)}{2}N\left(r,\frac{1}{f}\right) \\
& +\overline{d}(P)N\left(r,\frac{1}{g}\right)+\frac{3}{2}\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right) \\
& +\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g). \\
\leqslant\ & \frac{Q+5}{2}\overline{N}(r,f)+(Q+2)\overline{N}(r,g)+\frac{3\underline{d}(P)}{2}N\left(r,\frac{1}{f}\right) \\
& +\underline{d}(P)N\left(r,\frac{1}{g}\right)+\frac{3}{2}\left(\overline{d}(P)-\underline{d}(P)\right)T(r,f) \\
& +\left(\overline{d}(P)-\underline{d}(P)\right)T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \left(\frac{3\underline{d}(P)-\overline{d}(P)}{2}\right)T(r,f)\ +\ & \left(\underline{d}(P)-\overline{d}(P)\right)T(r,g) \\
& \leqslant \frac{Q+5}{2}\overline{N}(r,f)+\frac{3\underline{d}(P)}{2}N\left(r,\frac{1}{f}\right) \\
& \quad +(Q+2)\overline{N}(r,g)+\underline{d}(P)N\left(r,\frac{1}{g}\right) \\
& \quad +S(r,f)+S(r,g). \qquad (12)
\end{aligned}$$

Similarly,

$$\Rightarrow \left(\frac{3\underline{d}(P)-\overline{d}(P)}{2}\right)T(r,g)+\left(\underline{d}(P)-\overline{d}(P)\right)T(r,f)$$

$$\leqslant \frac{Q+5}{2}\overline{N}(r,g)+\frac{3\underline{d}(P)}{2}N\left(r,\frac{1}{g}\right)+(Q+2)\overline{N}(r,f)$$
$$+\underline{d}(P)N\left(r,\frac{1}{f}\right)+S(r,f)+S(r,g). \tag{13}$$

Adding (12) and (13) we get

$$\left(\frac{5\underline{d}(P)-3\overline{d}(P)}{2}\right)\{T(r,f)+T(r,g)\}$$
$$\leqslant \frac{3Q+9}{2}\overline{N}(r,f)+\frac{5\underline{d}(P)}{2}N\left(r,\frac{1}{f}\right)+\frac{3Q+9}{2}\overline{N}(r,g)+\frac{5\underline{d}(P)}{2}N\left(r,\frac{1}{g}\right)$$
$$+S(r,f)+S(r,g),$$
$$\Rightarrow \left\{5\underline{d}(P)\delta(0,f)+(3Q+9)\Theta(\infty,f)-(3Q+3\overline{d}(P)+9)\right\}T(r,f)$$
$$+\left\{5\underline{d}(P)\delta(0,g)+(3Q+9)\Theta(\infty,g)-(3Q+3\overline{d}(P)+9)\right\}T(r,g)$$
$$\leqslant S(r,f)+S(r,g),$$

which contradicts (5).

Subcase 1.2: $l\geqslant 2$. For this case we have

$$2\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)$$
$$\leqslant N(r,\frac{1}{G-1})+S(r,F)+S(r,G) \tag{14}$$

From Lemma 1, (8), (9) and (14) we obtain

$$T(r,F)\leqslant 2\overline{N}(r,F)+2\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)$$
$$+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G).$$
$$\leqslant 2\overline{N}(r,f)+2\overline{N}(r,g)+N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G).$$

$$\Rightarrow \overline{d}(P)T(r,f)\leqslant 2\overline{N}(r,f)+2\overline{N}(r,g)+\overline{d}(P)N\left(r,\frac{1}{f}\right)+\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)$$
$$+Q\overline{N}(r,g)+\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{g}\right)+\overline{d}(P)N\left(r,\frac{1}{g}\right)$$
$$+S(r,f)+S(r,g).$$
$$\leqslant 2\overline{N}(r,f)+(Q+2)\overline{N}(r,g)+\underline{d}(P)N\left(r,\frac{1}{f}\right)+\underline{d}(P)N\left(r,\frac{1}{g}\right)$$
$$+\left(\overline{d}(P)-\underline{d}(P)\right)T(r,f)+\left(\overline{d}(P)-\underline{d}(P)\right)T(r,g)+S(r,f)+S(r,g).$$

$$\Rightarrow \underline{d}(P)T(r,f)+\left(\underline{d}(P)-\overline{d}(P)\right)T(r,g)\leqslant 2\overline{N}(r,f)+\underline{d}(P)N\left(r,\frac{1}{f}\right)+(Q+2)\overline{N}(r,g)$$
$$+\underline{d}(P)N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g). \tag{15}$$

Similarly,

$$\begin{aligned}
&\underline{d}(P)T(r,g)+\left(\underline{d}(P)-\overline{d}(P)\right)T(r,f)\\
\leqslant\ &2\overline{N}(r,g)+\underline{d}(P)N\left(r,\frac{1}{g}\right)+(Q+2)\overline{N}(r,f)\\
&+\underline{d}(P)N\left(r,\frac{1}{f}\right)+S(r,f)+S(r,g). \qquad (16)
\end{aligned}$$

Adding (15) and (16) we get

$$\begin{aligned}
&(2\underline{d}(P)-\overline{d}(P))\{T(r,f)+T(r,g)\}\\
\leqslant\ &(Q+4)\overline{N}(r,f)+2\underline{d}(P)N\left(r,\frac{1}{f}\right)+(Q+4)\overline{N}(r,g)\\
&+2\underline{d}(P)N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g)\\
\Rightarrow\ &\left\{2\underline{d}(P)\delta(0,f)+(Q+4)\Theta(\infty,f)-(Q+\overline{d}(P)+4)\right\}T(r,f)\\
&+\left\{2\underline{d}(P)\delta(0,g)+(Q+4)\Theta(\infty,g)-(Q+\overline{d}(P)+4)\right\}T(r,g)\\
\leqslant\ &S(r,f)+S(r,g),
\end{aligned}$$

which is a contradiction to the hypothesis (4).

Case 2: $l=0$. Then we have

$$\begin{aligned}
N_E^{1)}\left(r,\frac{1}{F-1}\right)&=N_E^{1)}\left(r,\frac{1}{G-1}\right)+S(r,G)\\
\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)&=\overline{N}_E^{(2}\left(r,\frac{1}{G-1}\right)+S(r,G).
\end{aligned}$$

From (7) we have

$$\begin{aligned}
\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)&=N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\
&\quad+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\
&\leqslant N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+N\left(r,\frac{1}{G-1}\right)\\
&\leqslant \overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\
&\quad+2\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)\\
&\quad+N\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F^{(1)}}\right)+N_0\left(r,\frac{1}{G^{(1)}}\right)\\
&\quad+S(r,F)+S(r,G). \qquad (17)
\end{aligned}$$

Now using (17), Lemma 1 and Lemma 5 we obtain from (8) that

$$
\begin{aligned}
T(r,F) \leqslant\ & 2\overline{N}(r,F)+2\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)\\
&+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+2\overline{N}(r,F)+2\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)\\
&+S(r,F)+S(r,G).\\
\leqslant\ & 4\overline{N}(r,F)+3\overline{N}(r,G)+N\left(r,\frac{1}{F}\right)+2N\left(r,\frac{1}{G}\right)+2N\left(r,\frac{1}{F}\right)+S(r,F)+S(r,G).\\
\leqslant\ & 4\overline{N}(r,f)+3\overline{N}(r,g)+T(r,F)-\overline{d}(P)T(r,f)+\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)\\
&+\overline{d}(P)N\left(r,\frac{1}{f}\right)+2Q\overline{N}(r,g)+2\overline{d}(P)N\left(r,\frac{1}{g}\right)\\
&+2\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{g}\right)+2Q\overline{N}(r,f)+2\overline{d}(P)N\left(r,\frac{1}{f}\right)\\
&+2\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)+S(r,f)+S(r,g).
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \overline{d}(P)T(r,f) \leqslant\ & (2Q+4)\overline{N}(r,f)+(2Q+3)\overline{N}(r,g)+3\underline{d}(P)N\left(r,\frac{1}{f}\right)\\
&+2\underline{d}(P)N\left(r,\frac{1}{g}\right)+2\left(\overline{d}(P)-\underline{d}(P)\right)T(r,g)\\
&+3\left(\overline{d}(P)-\underline{d}(P)\right)T(r,f)+S(r,f)+S(r,g).
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow &\left(3\underline{d}(P)\ -\ 2\overline{d}(P)\right)T(r,f)+2\left(\underline{d}(P)-\overline{d}(P)\right)T(r,g)\\
&\leqslant (2Q+4)\overline{N}(r,f)+(2Q+3)\overline{N}(r,g)+3\underline{d}(P)N\left(r,\frac{1}{f}\right)\\
&\quad+2\underline{d}(P)N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g). \qquad (18)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left(3\underline{d}(P)-2\overline{d}(P)\right)T(r,g)+2\left(\underline{d}(P)-\overline{d}(P)\right)T(r,f)\\
&\leqslant (2Q+4)\overline{N}(r,g)+(2Q+3)\overline{N}(r,f)+3\underline{d}(P)N\left(r,\frac{1}{g}\right)\\
&\quad+2\underline{d}(P)N\left(r,\frac{1}{f}\right)+S(r,f)+S(r,g). \qquad (19)
\end{aligned}
$$

Adding (18) and (19) we obtain

$$
\begin{aligned}
&\left(5\underline{d}(P)-4\overline{d}(P)\right)\left\{T(r,f)+T(r,g)\right\}\\
&\leqslant (4Q+7)\overline{N}(r,f)+5\underline{d}(P)N\left(r,\frac{1}{f}\right)+(4Q+7)\overline{N}(r,g)\\
&\quad+5\underline{d}(P)N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g),
\end{aligned}
$$

$$\Rightarrow \left\{5\underline{d}(P)\delta(0,f)+(4Q+7)\Theta(\infty,f)-(4Q+4\overline{d}(P)+7)\right\}T(r,f)$$
$$+\left\{5\underline{d}(P)\delta(0,g)+(4Q+7)\Theta(\infty,g)-(4Q+4\overline{d}(P)+7)\right\}T(r,g)$$
$$\leqslant S(r,f)+S(r,g),$$

which contradicts (6).

Therefore $H\equiv 0$. So integrating twice we get

$$\frac{1}{G-1}=\frac{A}{F-1}+B,$$

where $A\ (\neq 0)$ and B are constants.

Thus

$$G=\frac{(B+1)F+(A-B-1)}{BF+(A-B)}, \tag{20}$$

and

$$F=\frac{(B-A)G+(A-B-1)}{BG-(B+1)}. \tag{21}$$

Next we consider the following three subcases:

Subcase 2.1: $B\neq 0,-1$. Then from (21) we have

$$\overline{N}\left(r,\frac{1}{G-\frac{B+1}{B}}\right)=\overline{N}(r,F).$$

By Nevanlinna second fundamental theorem and (ii) of Lemma 1 we get

$$\begin{aligned}T(r,G)&\leqslant \overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}\left(r,\frac{1}{G-\frac{B+1}{B}}\right)+S(r,G)\\&\leqslant \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+S(r,G)\\&\leqslant \overline{N}(r,G)+T(r,G)-\overline{d}(P)T(r,g)+\overline{d}(P)N\left(r,\frac{1}{g}\right)\\&\quad+\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{g}\right)+\overline{N}(r,F)+S(r,F)+S(r,G).\end{aligned}$$

$$\begin{aligned}\Rightarrow \overline{d}(P)T(r,g)&\leqslant \overline{N}(r,f)+\overline{N}(r,g)+(\overline{d}(P)-\underline{d}(P))T(r,g)\\&\quad+\underline{d}(P)N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g).\end{aligned} \tag{22}$$

If $A-B-1\neq 0$, then it follows from (20) that

$$N\left(r,\frac{1}{F-\frac{-A+B+1}{B+1}}\right)=N\left(r,\frac{1}{G}\right).$$

Again by Nevanlinna second fundamental theorem and Lemma 1 we have

$$\begin{aligned} T(r,F) &\leqslant \overline{N}(r,F)+\overline{N}\Big(r,\frac{1}{F}\Big)+N\left(r,\frac{1}{F-\frac{-A+B+1}{B+1}}\right)+S(r,F) \\ &\leqslant \overline{N}(r,F)+T(r,F)-\overline{d}(P)T(r,f)+\overline{d}(P)N\Big(r,\frac{1}{f}\Big) \\ &\quad +\big(\overline{d}(P)-\underline{d}(P)\big)m\Big(r,\frac{1}{f}\Big)+Q\overline{N}(r,g)+\overline{d}(P)N\Big(r,\frac{1}{g}\Big) \\ &\quad +\big(\overline{d}(P)-\underline{d}(P)\big)m\Big(r,\frac{1}{g}\Big)+S(r,f)+S(r,g). \end{aligned}$$

$$\begin{aligned} \Rightarrow \overline{d}(P)T(r,f) &\leqslant \overline{N}(r,f)+\big(\overline{d}(P)-\underline{d}(P)\big)T(r,f)+\underline{d}(P)N\Big(r,\frac{1}{f}\Big) \\ &\quad +Q\overline{N}(r,g)+\underline{d}(P)N\Big(r,\frac{1}{g}\Big)+\big(\overline{d}(P)-\underline{d}(P)\big)T(r,g) \\ &\quad +S(r,f)+S(r,g). \end{aligned} \tag{23}$$

Combining (22) and (23)

$$\begin{aligned} &\underline{d}(P)T(r,f)+\big(2\underline{d}(P)-\overline{d}(P)\big)T(r,g) \\ \leqslant\ & 2\overline{N}(r,f)+\underline{d}(P)N\Big(r,\frac{1}{f}\Big)+(Q+1)\overline{N}(r,g) \\ &+2\underline{d}(P)N\Big(r,\frac{1}{g}\Big)+S(r,f)+S(r,g). \\ \Rightarrow\ & \big\{\underline{d}(P)\delta(0,f)+2\Theta(\infty,f)-2\big\}T(r,f) \\ &+\big\{2\underline{d}(P)\delta(0,g)+(Q+1)\Theta(\infty,g)-(Q+\overline{d}(P)+1)\big\}T(r,g) \\ \leqslant\ & S(r,f)+S(r,g), \end{aligned}$$

which contradicts assumptions (4)–(6).

Therefore $A-B-1=0$. Then by (20)

$$\overline{N}\left(r,\frac{1}{F+\frac{1}{B}}\right)=\overline{N}(r,G).$$

By Nevanlinna second fundamental theorem and Lemma 1 we get

$$\begin{aligned} T(r,F) &< \overline{N}(r,F)+\overline{N}\Big(r,\frac{1}{F}\Big)+\overline{N}\left(r,\frac{1}{F+\frac{1}{B}}\right)+S(r,F), \\ &\leqslant \overline{N}(r,f)+T(r,F)-\overline{d}(P)T(r,f)+\overline{d}(P)N\Big(r,\frac{1}{f}\Big) \\ &\quad +\big(\overline{d}(P)-\underline{d}(P)\big)m\Big(r,\frac{1}{f}\Big)+\overline{N}(r,g)+S(r,f)+S(r,g), \end{aligned}$$

$$\Rightarrow \overline{d}(P)T(r,f) \leqslant \overline{N}(r,f) + \underline{d}(P)N\Big(r,\frac{1}{f}\Big) + \big(\overline{d}(P) - \underline{d}(P)\big)T(r,f)$$
$$+\overline{N}(r,g) + S(r,f) + S(r,g), \tag{24}$$

Combining (22) and (24)

$$\underline{d}(P)\big\{T(r,f)+T(r,g)\big\} \leqslant 2\overline{N}(r,f) + \underline{d}(P)N\Big(r,\frac{1}{f}\Big) + 2\overline{N}(r,g)$$
$$+\underline{d}(P)N\Big(r,\frac{1}{g}\Big) + S(r,f) + S(r,g).$$

which violates assumptions (4)–(6).

Subcase 2.2: $B=-1$. Then

$$G = \frac{A}{A+1-F},$$

and

$$F = \frac{(1+A)G - A}{G}.$$

If $A+1 \neq 0$,

$$\overline{N}\left(r,\frac{1}{F-(A+1)}\right) = \overline{N}(r,G),$$

$$\overline{N}\left(r,\frac{1}{G-\frac{A}{A+1}}\right) = \overline{N}(r,\frac{1}{F}).$$

By similar argument as Subcase 2.1 we get a contradiction.

Therefore $A+1=0$ then $FG=1 \Rightarrow P[f].P[g] \equiv a^2$.

Subcase 2.3: $B=0$. Then (20) and (21) gives $G=\frac{F+A-1}{A}$ and $F = AG+1-A$.

If $A-1\neq 0$, $\overline{N}\left(r,\frac{1}{A-1+F}\right) = \overline{N}\left(r,\frac{1}{G}\right)$ and $\overline{N}\left(r,\frac{1}{G-\frac{A-1}{A}}\right) = \overline{N}(r,\frac{1}{F})$. Proceeding similarly as in Subcase 2.1 we get a contradiction.

Therefore, $A-1=0$ then $F\equiv G$ i.e., $P[f]\equiv P[g]$. This complete the proof. □

Proof of Theorem 8. Let

$$F = \frac{P[f]}{a},\ G = \frac{P[g]}{a}.$$

Since $P[f]$ and $P[g]$ share (a,l), it follows that F, G share $(1,l)$ except at the zeros and poles of a. By Lemma 2 and Lemma 6, we get

$$\underline{d}(P)T(r,f) \leqslant \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{F-1}\right) + \underline{d}(P)N\left(r,\frac{1}{f}\right) + S(r,f),$$
$$= \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{G-1}\right) + \underline{d}(P)N\left(r,\frac{1}{g}\right) + S(r,f),$$
$$\leqslant \big(1+Q+\overline{d}(P)+\underline{d}(P)\big)\,T(r,g) + S(r,f) + S(r,g). \tag{25}$$

Similarly,

$$\underline{d}(P)T(r,g) \leqslant \left(1+Q+\overline{d}(P)+\underline{d}(P)\right)T(r,f)+S(r,f)+S(r,g). \tag{26}$$

From (25) and (26) we get $S(r,f)=S(r,g)$. The rest of the proof is similar to that of Theorem 7. □

Proof of Theorem 9. Proof follows from the proof of Theorem 8 immediately. □

Proof of the Corollary 1. By Theorem 9 we get either $L(f)\equiv L(g)$ or $L(f).L(g)\equiv 1$. Let $L(f)\equiv L(g)$ so that $L(f-g)\equiv 0$. Proceeding similarly as in the proof of Corollary 1.2 of Lahiri and Pal [12], we can prove that $f\equiv g$. □

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Dilip Chandra Pramanik
Department of Mathematics
University of North Bengal
Raja Rammohunpur, Darjeeling-734013, West Bengal, India
e-mail: dcpramanik.nbu2012@gmail.com

Jayanta Roy
Department of Mathematics
Directorate of Distance Education
University of North Bengal
Raja Rammohunpur, Darjeeling-734013, West Bengal, India
e-mail: jayanta2017math@nbu.ac.in