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ON IDEAL CONVERGENCE OF NESTED SEQUENCES OF SETS

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Abstract. In this work, we show the equivalence of ideal Wijsman convergence and Wijsman convergence and the equivalence of ideal Hausdorff convergence and Hausdorff convergence for the nested sequences of sets. Then we prove that the ideal Wijsman limit and the ideal Hausdorff limit of the nested sequence of sets are equivalent each other for every admissible ideal.

1. Introduction

Wijsman [13, 14] defined the concept of Wijsman convergence for sequences of sets (see also [8], [11]). Wijsman convergence is defined as the pointwise convergence of the distance function. Hausdorff convergence, which is defined by the Hausdorff distance, corresponds to the uniform convergence of the distance function (see [4], [7], [3], [11]). Therefore, Hausdorff convergence always implies Wijsman convergence. Apreutesei [1] showed that Wijsman convergence and Hausdorff convergence are equivalent for monotone sequences of compact sets.

Nuray and Rhoades [10] introduced the notions of Wijsman statistically convergence and Hausdorff statistically convergence. In 2000, Kostyrko et al. [6] introduced the notion of ideal convergence for sequences defined on the metric spaces, where $\mathscr{I}$ is an ideal on $\mathbb{N}$. In [5] Kişi and Nuray introduced Wijsman $\mathscr{I}$-convergence, and in [12] Talo and Sever introduced Hausdorff $\mathscr{I}$-convergence. The reader can refer to the recent monographs [2] and [9] on the sequence spaces and summability theory, and applications.

We show that $\mathscr{I}$-Wijsman convergence is equivalent to Wijsman convergence for every $\mathscr{I}$ admissible ideal and the nested sequences of nonempty closed sets (see Theorem 1). Similarly, we give the equivalence of $\mathscr{I}$-Hausdorff convergence and Hausdorff convergence (see Theorem 2). Finally, we show that $\mathscr{I}$-Wijsman limit and $\mathscr{I}$-Hausdorff limit of the nested sequence of sets are equivalent each other for every admissible ideal (see Corollary 1).

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2. Preliminaries

Throughout this paper, (X,ρ) denotes a metric space. We denote the family of all nonempty closed subsets and the family of all nonempty compact subsets of X by $\mathrm{Cl}(X)$ and $\mathscr{K}(X)$, respectively. $\mathrm{cl}(A)$ denotes the closure of a set A.

The *distance* $d(x,A)$ from a point $x \in X$ to a set $A \subseteq X$ is defined as

$$d(x,A) = \inf_{y\in A} \rho(x,y)$$

(see [13]).

Hausdorff distance of nonempty sets $A,B \subseteq X$ is defined as

$$H(A,B) = \max\{h(A,B), h(B,A)\}$$

where $h(A,B) = \sup_{a\in A} d(a,B)$, or equivalently

$$H(A,B) = \inf\{\varepsilon > 0 : A \subseteq B^{\varepsilon} \text{ and } B \subseteq A^{\varepsilon}\}$$

where $A^{\varepsilon} = \bigcup_{a\in A}\{x \in X : \rho(a,x) < \varepsilon\} = \{x \in X : d(x,A) < \varepsilon\}$ is the ε-enlargement of A.

A sequence $(A_n)_{n\in\mathbb{N}}$ of nonempty subsets of X is said to be *Hausdorff convergent* to a set $A \subseteq X$ if

$$\lim_{n\to\infty} H(A_n, A) = 0.$$

In this case, we write $H - \lim A_n = A$ or $A_n \xrightarrow{H} A$ ([3], [7], [11]).

We say that a sequence $(A_n)_{n\in\mathbb{N}}$ of nonempty subsets of X is *Wijsman convergent* to a set $A \subseteq X$ if

$$\lim_{n\to\infty} d(x,A_n) = d(x,A) \text{ for each } x \in X.$$

In this case, we write $W - \lim A_n = A$ or $A_n \xrightarrow{W} A$ ([13], [14]).

Let A_n's be nonempty subsets of X. The sequence $(A_n)_{n\in\mathbb{N}}$ is called a nested sequence of sets if it is monotone increasing or monotone decreasing according to the inclusion relation, that is,

$$A_n \subseteq A_{n+1} \ (\forall n \in \mathbb{N}) \text{ or } A_{n+1} \subseteq A_n \ (\forall n \in \mathbb{N}).$$

A family $\mathscr{I}$ of subsets of $\mathbb{N}$ is said to be an *ideal* on $\mathbb{N}$ if the following conditions are satisfied:

(i) $\emptyset \in \mathscr{I}$,

(ii) $A \cup B \in \mathscr{I}$ for each $A,B \in \mathscr{I}$,

(iii) $B \in \mathscr{I}$ for each $A \in \mathscr{I}$ such that $B \subseteq A$.

An ideal is called *proper* if $\mathbb{N} \notin \mathscr{I}$, and proper ideal is called *admissible* if $\{n\} \in \mathscr{I}$ for each $n \in \mathbb{N}$. Obviously, an admissible ideal includes all finite subset of $\mathbb{N}$. If $\mathscr{I}$ is an ideal on $\mathbb{N}$ then the family $\mathscr{F}(\mathscr{I}) = \{\mathbb{N} \setminus A : A \in \mathscr{I}\}$ is a filter on $\mathbb{N}$ ([6]).

Since we will use the distance function and Hausdorff distance in the next definitions of convergence, let's remind the definition of ideal convergence in $\mathbb{R}$: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Let $\mathscr{I}$ be any ideal on $\mathbb{N}$. If for every $\varepsilon > 0$

$$\{n \in \mathbb{N} : |x_n - x_0| \geqslant \varepsilon\} \in \mathscr{I}$$

then (x_n) is said to be $\mathscr{I}$-convergent to x_0. Then we write $\mathscr{I} - \lim x_n = x_0$ ([6]).

The family $\mathscr{I}_f = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$ is the minimum admissible ideal according to the inclusion relation. Then $\mathscr{I}_f$-convergence and classical convergence is equivalent to each other. If $\mathscr{I}$ is a proper ideal with $\mathscr{I} \supseteq \mathscr{I}_f$ then $\mathscr{I}$ is an admissible ideal.

DEFINITION 1. ([12]) We say that the sequence $(A_n)_{n\in\mathbb{N}}$ of nonempty subsets of X is *$\mathscr{I}$-Hausdorff convergent* to the set A if

$$\left\{n \in \mathbb{N} : \sup_{x\in X} |d(x,A_n) - d(x,A)| \geqslant \varepsilon\right\} \in \mathscr{I}$$

for every $\varepsilon > 0$, or if $\mathscr{I} - \lim H(A_n, A) = 0$, i.e., for every $\varepsilon > 0$

$$\{n \in \mathbb{N} : H(A_n, A) \geqslant \varepsilon\} \in \mathscr{I}$$

or equivalently

$$\{n \in \mathbb{N} : h(A_n, A) \geqslant \varepsilon \text{ or } h(A, A_n) \geqslant \varepsilon\} \in \mathscr{I}.$$

In this case, we write $\mathscr{I}\text{-}H - \lim A_n = A$ or $A_n \xrightarrow{\mathscr{I}\text{-}H} A$.

DEFINITION 2. ([5]) We say that a sequence $(A_n)_{n\in\mathbb{N}}$ of nonempty subsets of X is *$\mathscr{I}$-Wijsman convergent* to a set $A \subseteq X$ if

$$\mathscr{I} - \lim d(x, A_n) = d(x, A) \text{ for each } x \in X,$$

i.e., for each $\varepsilon > 0$ and each $x \in X$

$$\{n \in \mathbb{N} : |d(x,A_n) - d(x,A)| \geqslant \varepsilon\} \in \mathscr{I}.$$

In this case, we write $\mathscr{I}\text{-}W - \lim A_n = A$ or $A_n \xrightarrow{\mathscr{I}\text{-}W} A$.

LEMMA 1. ([1]) *Let $(A_n)_{n\in\mathbb{N}}$ be a nested sequence where $A_n \in \mathscr{K}(X)$ for every $n \in \mathbb{N}$.*

(i) If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence then

$$W - \lim A_n = H - \lim A_n = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right).$$

(ii) If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence then

$$W-\lim A_n = H-\lim A_n = \bigcap_{n\in\mathbb{N}} A_n.$$

In the following, we give an example of a nested sequence where Hausdorff convergence and Wijsman convergence are not equivalent. This example does not contradict Lemma 1 because the elements of the sequence are not compact.

EXAMPLE 1. In the Euclidean space $\mathbb{R}^2$, let's consider the sequence $(A_n)_{n\in\mathbb{N}}$ defined as

$$A_n = \left\{(x,y)\in\mathbb{R}^2 : -\frac{|x|}{n} \leqslant y \leqslant \frac{|x|}{n}\right\} \text{ for each } n\in\mathbb{N}$$

and let

$$A = \left\{(x,y)\in\mathbb{R}^2 : y=0\right\}.$$

(A_n) is a monotone decreasing sequence with $A_{n+1}\subseteq A_n$ for every $n\in\mathbb{N}$. The sets A_n are closed but not compact. We have

$$W-\lim A_n = A,$$

but (A_n) is not *Hausdorff convergent*.

LEMMA 2. *Let $A, A_n \in \mathrm{Cl}(X)$ $(n\in\mathbb{N})$ and $\mathscr{I}$ be an admissible ideal on $\mathbb{N}$.*

(i) If $A_n \xrightarrow{W} A$ then $A_n \xrightarrow{\mathscr{I}\text{-}W} A$.

(ii) If $A_n \xrightarrow{H} A$ then $A_n \xrightarrow{\mathscr{I}\text{-}H} A$.

EXAMPLE 2. Let $\mathscr{I}_1 = \{A\subseteq\mathbb{N} : A\cap\mathbb{N}_k = \emptyset \text{ for some } k\in\mathbb{N}\}$ where $\mathbb{N}_k = \{nk : n\in\mathbb{N}\}$ and $\mathscr{I}_2 = \{A\subseteq\mathbb{N} : A\subseteq 2\mathbb{N}\}$. $\mathscr{I}_1$ is an admissible ideal, but $\mathscr{I}_2$ is not.

In the Euclidean space $\mathbb{R}^2$, consider the sequence $(A_n)_{n\in\mathbb{N}}$ defined as

$$A_n = \begin{cases} \left\{(x,y)\in\mathbb{R}^2 : y = 1/n\right\} & ,n\neq 3k-1 \\ \left\{(x,y)\in\mathbb{R}^2 : |x| + \left|ny-n^2\right| = n^2\right\} & ,n = 3k-1 \end{cases} \quad (k\in\mathbb{N})$$

and $A = \left\{(x,y)\in\mathbb{R}^2 : y=0\right\}$. The sequence $(A_n)_{n\in\mathbb{N}}$ is not nested.

We have $A_n \xrightarrow{W} A$ and so $A_n \xrightarrow{\mathscr{I}_1-W} A$ (because $\mathscr{I}_1$ is an admissible ideal). But (A_n) is not $\mathscr{I}_2$-Wijsman convergent and therefore it is not $\mathscr{I}_2$-Hausdorff convergent. Also we have $A_n \overset{H}{\not\longrightarrow} A$, but $A_n \xrightarrow{\mathscr{I}_1-H} A$.

3. Main results

In this section, we give the equivalence of Wijsman convergence, Hausdorff convergence, ideal Wijsman convergence and ideal Hausdorff convergence for the nested sequences of sets.

THEOREM 1. *Let $(A_n)_{n\in\mathbb{N}}$ be a nested sequence of closed subsets of X and $A \in \mathrm{Cl}(X)$. Let $\mathscr{I}$ be an admissible ideal on $\mathbb{N}$. Then we have:*

$$A_n \xrightarrow{W} A \Longleftrightarrow A_n \xrightarrow{\mathscr{I}-W} A.$$

Proof. $(\Longrightarrow)$: It was given in Lemma 2.

$(\Longleftarrow)$: We will prove the sufficient condition in two stages, depending on whether the sequence $(A_n)_{n\in\mathbb{N}}$ is increasing or decreasing.

(1) Let $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence such that $A_n \subseteq A_{n+1}$ for every $n \in \mathbb{N}$. Let's assume that $A_n \xrightarrow{\mathscr{I}-W} A$.

Firstly, we show that $A_n \subseteq A$ for every $n \in \mathbb{N}$. Let's fix $n \in \mathbb{N}$ and let $u \in A_n$. Since (A_n) is increasing, for every $m \geqslant n$ we have $u \in A_m$ and so $d(u,A_m) = 0$. From $A_n \xrightarrow{\mathscr{I}-W} A$, we have

$$K(u,\varepsilon) := \{m \in \mathbb{N} : |d(u,A_m) - d(u,A)| < \varepsilon\} \in \mathscr{F}(\mathscr{I})$$

for each $\varepsilon > 0$. For each $\varepsilon > 0$ there is an $m_\varepsilon \in \mathbb{N}$ which is $m_\varepsilon \in K(u,\varepsilon)$ and $m_\varepsilon \geqslant n$. Hence we get

$$d(u,A) = |d(u,A_{m_\varepsilon}) - d(u,A)| < \varepsilon \tag{1}$$

for each $\varepsilon > 0$. Then we obtain $u \in A$ from the closeness of A.

Then we can write $d(x,A) \leqslant d(x,A_n)$ and so

$$d(x,A_n) - d(x,A) \geqslant 0 \tag{2}$$

for each $x \in X$ and each $n \in \mathbb{N}$.

Now, let $x \in X$ and $\varepsilon > 0$. From $A_n \xrightarrow{\mathscr{I}-W} A$, we have

$$L(x,\varepsilon) := \{n \in \mathbb{N} : |d(x,A_n) - d(x,A)| < \varepsilon\} \in \mathscr{F}(\mathscr{I}). \tag{3}$$

Let $n_0 = n_0(x,\varepsilon) := \min L(x,\varepsilon)$. Since (A_n) is increasing, we have

$$d(x,A_n) \leqslant d\left(x,A_{n_0}\right) \tag{4}$$

for every $n \geqslant n_0$. From (3) and (4), we get

$$d(x,A_n) - d(x,A) \leqslant d\left(x,A_{n_0}\right) - d(x,A) < \varepsilon \tag{5}$$

for every $n \geqslant n_0$.

From (2) and (5), we get

$$|d(x,A_n) - d(x,A)| < \varepsilon \tag{6}$$

for every $n \geqslant n_0$.

Since $x \in X$ is arbitrary, we obtain

$$\lim_{n\to\infty} d(x,A_n) = d(x,A) \tag{7}$$

for every $x \in X$. Consequently, $A_n \xrightarrow{W} A$.

(2) Let $(A_n)_{n\in\mathbb{N}}$ be a decreasing sequence such that $A_n \supseteq A_{n+1}$ for every $n \in \mathbb{N}$. Let's assume that $A_n \xrightarrow{\mathscr{I}-W} A$.

Firstly, we show that $A \subseteq A_n$ for every $n \in \mathbb{N}$. Let's fix $n \in \mathbb{N}$ and let $u \in A$. From $A_n \xrightarrow{\mathscr{I}-W} A$, we have

$$K(u,\varepsilon) := \{m \in \mathbb{N} : |d(u,A_m) - d(u,A)| < \varepsilon\} \in \mathscr{F}(\mathscr{I})$$

for each $\varepsilon > 0$. For each $\varepsilon > 0$ there is an $m_\varepsilon \in \mathbb{N}$ which is $m_\varepsilon \in K(u,\varepsilon)$ and $m_\varepsilon \geqslant n$. Since (A_n) is decreasing, we have $A_{m_\varepsilon} \subseteq A_n$ and $d(u,A_n) \leqslant d(u,A_{m_\varepsilon})$. Since $d(u,A) = 0$ we get

$$d(u,A_n) \leqslant d(u,A_{m_\varepsilon}) = |d(u,A_{m_\varepsilon}) - d(u,A)| < \varepsilon \tag{8}$$

for each $\varepsilon > 0$. From the closeness of A_n, we obtain $u \in A_n$.

Then we can write $d(x,A_n) \leqslant d(x,A)$ and so

$$d(x,A_n) - d(x,A) \leqslant 0 \tag{9}$$

for each $x \in X$ and each $n \in \mathbb{N}$.

Now, let $x \in X$ and $\varepsilon > 0$. Again from $A_n \xrightarrow{\mathscr{I}-W} A$, we have

$$L(x,\varepsilon) := \{n \in \mathbb{N} : |d(x,A_n) - d(x,A)| < \varepsilon\} \in \mathscr{F}(\mathscr{I}). \tag{10}$$

Let $n_0 = n_0(x,\varepsilon) := \min L(x,\varepsilon)$. Since (A_n) is decreasing, we have

$$d\left(x,A_{n_0}\right) \leqslant d(x,A_n) \tag{11}$$

for every $n \geqslant n_0$. From (10) and (11), we get

$$-\varepsilon < d\left(x,A_{n_0}\right) - d(x,A) \leqslant d(x,A_n) - d(x,A) \tag{12}$$

for every $n \geqslant n_0$.

From (9) and (12), we get

$$|d(x,A_n) - d(x,A)| < \varepsilon \tag{13}$$

for every $n \geqslant n_0$.

Since $x \in X$ is arbitrary, we obtain

$$\lim_{n\to\infty} d(x,A_n) = d(x,A) \tag{14}$$

for every $x \in X$. Consequently, we get $A_n \xrightarrow{W} A$. □

THEOREM 2. *Let* $(A_n)_{n\in\mathbb{N}}$ *be a nested sequence of closed subsets of* X *and* $A \in \mathrm{Cl}(X)$. *Let* $\mathscr{I}$ *be an admissible ideal on* $\mathbb{N}$. *Then we have:*

$$A_n \xrightarrow{H} A \Longleftrightarrow A_n \xrightarrow{\mathscr{I}-H} A.$$

Proof. $(\Longrightarrow)$: It was given in Lemma 2.

$(\Longleftarrow)$: We will prove the sufficient condition in two stages, depending on whether the sequence $(A_n)_{n\in\mathbb{N}}$ is increasing or decreasing.

(1) Let $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence such that $A_n \subseteq A_{n+1}$ for every $n \in \mathbb{N}$. Let's assume that $A_n \xrightarrow{\mathscr{I}-H} A$.

Firstly, we show that $A_n \subseteq A$ for every $n \in \mathbb{N}$. Let's fix $n \in \mathbb{N}$ and let $u \in A_n$. Since (A_n) is increasing, we have $u \in A_m$ for every $m \geqslant n$. From $A_n \xrightarrow{\mathscr{I}-H} A$, we have

$$K(\varepsilon) := \{m \in \mathbb{N} : h(A_m, A) < \varepsilon \text{ and } h(A, A_m) < \varepsilon\} \in \mathscr{F}(\mathscr{I})$$

for each $\varepsilon > 0$. For each $\varepsilon > 0$ there is an $m_\varepsilon \in \mathbb{N}$ which is $m_\varepsilon \in K(\varepsilon)$ and $m_\varepsilon \geqslant n$. Hence we get

$$d(u, A) \leqslant h(A_{m_\varepsilon}, A) < \varepsilon \tag{15}$$

for each $\varepsilon > 0$. Then we obtain $u \in A$ from the closeness of A.

Since $A_n \subseteq A$ for every $n \in \mathbb{N}$, we get

$$h(A_n, A) = 0 \text{ for each } n \in \mathbb{N}. \tag{16}$$

Now, let's fix $\varepsilon > 0$. Let $n_0 = n_0(\varepsilon) := \min K(\varepsilon)$. Since (A_n) is increasing, we have

$$d(x, A_n) \leqslant d\left(x, A_{n_0}\right) \tag{17}$$

for every $n \geqslant n_0$ and every $x \in X$. From $n_0 \in K(\varepsilon)$ we have

$$d\left(x, A_{n_0}\right) \leqslant h\left(A, A_{n_0}\right) < \varepsilon \tag{18}$$

for every $x \in A$. From (17) and (18), we get

$$d(x, A_n) < \varepsilon \tag{19}$$

for every $n \geqslant n_0$ and every $x \in A$. Then we obtain

$$h(A, A_n) = \sup_{x\in A} d(x, A_n) < \varepsilon \tag{20}$$

for every $n \geqslant n_0$. From (16) and (20) we get

$$H(A_n, A) = \max\{h(A_n, A), h(A, A_n)\} = h(A, A_n) < \varepsilon$$

for every $n \geqslant n_0$. Consequently, we get $A_n \xrightarrow{H} A$.

(2) Let $(A_n)_{n\in\mathbb{N}}$ be a decreasing sequence such that $A_n \supseteq A_{n+1}$ for every $n \in \mathbb{N}$. Let's assume that $A_n \xrightarrow{\mathscr{I}-H} A$. Then we have

$$K(\varepsilon) := \{m \in \mathbb{N} : h(A_m, A) < \varepsilon \text{ and } h(A, A_m) < \varepsilon\} \in \mathscr{F}(\mathscr{I})$$

for each $\varepsilon > 0$.

Firstly, we show that $A \subseteq A_n$ for every $n \in \mathbb{N}$. Let's fix $n \in \mathbb{N}$ and let $u \in A$. For each $\varepsilon > 0$ there is an $m_\varepsilon \in \mathbb{N}$ which is $m_\varepsilon \in K(\varepsilon)$ and $m_\varepsilon \geqslant n$. Since (A_n) is decreasing, we have $A_{m_\varepsilon} \subseteq A_n$ and

$$d(u, A_n) \leqslant d(u, A_{m_\varepsilon}) \leqslant h(A, A_{m_\varepsilon}) < \varepsilon \tag{21}$$

for each $\varepsilon > 0$. From the closeness of A_n, we obtain $u \in A_n$.

Therefore we get

$$d(x, A_n) = 0$$

for each $n \in \mathbb{N}$ and each $x \in A$, and so

$$h(A, A_n) = \sup_{x\in A} d(x, A_n) = 0 \tag{22}$$

for each $x \in X$ and each $n \in \mathbb{N}$.

Now, let's fix $\varepsilon > 0$. Let $n_0 = n_0(\varepsilon) := \min K(\varepsilon)$. Since (A_n) is decreasing, we have

$$d(x, A) \leqslant h\left(A_{n_0}, A\right) < \varepsilon \tag{23}$$

for every $n \geqslant n_0$ and every $x \in A_n \subseteq A_{n_0}$. Hence we get

$$h(A_n, A) = \sup_{x\in A_n} d(x, A) < \varepsilon \tag{24}$$

for every $n \geqslant n_0$. From (22) and (24), we get

$$H(A_n, A) = \max\{h(A_n, A), h(A, A_n)\} < \varepsilon \tag{25}$$

for every $n \geqslant n_0$. Consequently, we obtain $A_n \xrightarrow{H} A$. □

Now, from Theorem 1, Theorem 2 and Lemma 1, we can give the following corollary.

COROLLARY 1. *Let $(A_n)_{n\in\mathbb{N}}$ be a nested sequence where $A_n \in \mathscr{K}(X)$ for every $n \in \mathbb{N}$.*

(i) If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence then

$$\mathscr{I}\text{-}W - \lim A_n = \mathscr{I}\text{-}H - \lim A_n = \operatorname{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right)$$

for every admissible ideal $\mathscr{I}$ on $\mathbb{N}$.

(ii) If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence then

$$\mathscr{I}\text{-}W - \lim A_n = \mathscr{I}\text{-}H - \lim A_n = \bigcap_{n\in\mathbb{N}} A_n$$

for every admissible ideal $\mathscr{I}$ on $\mathbb{N}$.

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