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NEVANLINNA'S FIVE–VALUE THEOREM FOR DERIVATIVES OF MEROMORPHIC FUNCTIONS IN AN ANGULAR DOMAIN

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Abstract. In this paper, we first obtain the famous Xiong Inequality for meromorphic functions in an angular domain and also generalise Nevanlinna's five-value theorem for derivatives of meromorphic functions by considering weaker assumptions of sharing five values and small functions to partially sharing $k(\geqslant 5)$ values and small functions in an angular domain. As a particular cases of our results, we deduce He Ping result in an angular domain.

1. Introduction

The uniqueness theory of meromorphic functions is an interesting problem in the value distribution theory. We say that f and $\widehat{f}$ share the value a CM (counting multiplicities) if f and $\widehat{f}$ have the same a-points with the same multiplicity and if f and $\widehat{f}$ share the value a IM (ignoring multiplicities) if we do not consider the multiplicities. In 1929, R. Nevanlinna proved that, if f and $\widehat{f}$ be two non-constant meromorphic functions in $\mathbb{C}$ and if they share five distinct values IM, then $f\equiv\widehat{f}$; if they share four distinct values CM, then f is a Mobius transformation of $\widehat{f}$. After this work, many authors proved several results on uniqueness of meromorphic functions concerning shared values in the complex plane. In 2004, J. H. Zheng (see [3]) extended the uniqueness of meromorphic functions dealing with five shared values in an angular domains of $\mathbb{C}$. Also in 2010, He Ping proved some important results on the uniqueness of meromorphic functions sharing values in an angular domain (see [7]) and others have done lots of work in this area (see [3]–[21]). It is interesting to prove some important uniqueness results in the whole of the complex plane to an angular domain. In this paper, we study the famous Xiong Inequality for meromorphic functions in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant \arg z\leqslant\beta\}$ and also generalise Nevanlinna's five-value theorem for derivatives of meromorphic functions by considering weaker assumptions of sharing five values and small functions to partially sharing $k(\geqslant 5)$ values and small functions in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant \arg z\leqslant\beta\}$.

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2. Basic notations and definitions

Nevanlinna theory in an angular domain will play a key role in the proof of theorems. Let $f(z)$ be a meromorphic function on the angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant \arg z\leqslant\beta\}$,

$$A_{\alpha,\beta}(r,f)=\frac{\omega}{\pi}\int_1^r\left(\frac{1}{t^{\omega}}-\frac{t^{\omega}}{r^{2\omega}}\right)\left\{\log^+|f(te^{i\alpha})|+\log^+|f(te^{i\beta})|\right\}\frac{dt}{t},$$

$$B_{\alpha,\beta}(r,f)=\frac{2\omega}{\pi r^{\omega}}\int_{\alpha}^{\beta}\log^+|f(re^{i\theta})|\sin\omega(\theta-a)d\theta,$$

$$C_{\alpha,\beta}(r,f)=\sum_{1<|b_n|<r}\left(\frac{1}{|b_n|^{\omega}}-\frac{|b_n|^{\omega}}{r^{2\omega}}\right)\sin\omega(\theta_n-a)d\theta,$$

where $\omega=\pi/(\beta-\alpha)$ and $b_n=|b_n|e^{i\theta_n}$ are the poles of $f(z)$ on $\overline{\Omega}(\alpha,\beta)$ appearing according to the multiplicities. $C_{\alpha,\beta}$ is called angular counting function of the poles of $f(z)$ on $\overline{\Omega}(\alpha,\beta)$ and Nevanlinna's angular characteristic function is defined as follows

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f).$$

Throughout, we denote by $R_{\alpha,\beta}(r,*)$ a quantity satisfying satisfying

$$R_{\alpha,\beta}(r,*)=O\{\log(rS_{\alpha,\beta}(r,*))\},\ \ r\in E,$$

where E denotes a set of positive real numbers with finite linear measure.

DEFINITION 1. Let $f(z)$ be a meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant\arg z\leqslant\beta\}$. Then function

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f)$$

is called angular Nevanlinna characteristic of $f(z)$.

3. Some lemmas

LEMMA 1. [3] *Let $f(z)$ be a meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant\arg z\leqslant\beta\}$, $a\in\mathbb{C}$*

$$S_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)=S_{\alpha,\beta}(r,f)+O(1).$$

and for an integer $p\geqslant 0$,

$$S_{\alpha,\beta}(r,f^{(p)})\leqslant 2pS_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,f),$$

$$A_{\alpha,\beta}\left(r,\frac{f^{(p)}}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f^{(p)}}{f}\right)=R_{\alpha,\beta}(r,f),$$

and $R_{\alpha,\beta}(r,f^{(p)})=R_{\alpha,\beta}(r,f)$.

LEMMA 2. [3] *Let* $f(z)$ *be a meromorphic function in an angular domain* $\overline{\Omega}(\alpha,\beta) = \{z:\alpha\leqslant\arg z\leqslant\beta\}$. *Then for arbitrary* q *distinct* $a_j\in\overline{\mathbb{C}}$ $(1\leqslant j\leqslant q)$, *we have*

$$(q-2)S_{\alpha,\beta}(r,f)\leqslant\sum_{j=1}^{q}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)+R_{\alpha,\beta}(r,f),$$

where the term $\overline{C}_{\alpha,\beta}(r,1/f-a_j)$ *will be replaced by* $\overline{C}_{\alpha,\beta}(r,f)$ *when some* $a_j=\infty$.

We use $\overline{C}^{k)}_{\alpha,\beta}(r,1/f-a_j)$ to denote the zeros of $f(z)-a$ in $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant\arg z\leqslant\beta\}$ whose multiplicities are no greater than k and are counted only once. Likewise, we use $\overline{C}^{(k+1}_{\alpha,\beta}(r,1/f-a_j)$ to denote the zeros of $f(z)-a$ in $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant\arg z\leqslant\beta\}$ whose multiplicities are greater than k and are counted only once. In 2010, He Ping proved some uniqueness theorems for meromorphic functions in an angular domain.

THEOREM 1. [7] *Let* $f(z)$ *and* $g(z)$ *be two non-constant meromorphic functions,* a_j $(j=1,2,3,4,5)$ *be different complex numbers. If* $\overline{E}(a_j,\Delta_\delta,f)\subseteq\overline{E}(a_j,\Delta_\delta,g)$ $(j=1,2,3,4,5)$ *and*

$$\liminf_{r\to\infty}\frac{\sum_{j=1}^{5}\overline{C}\left(r,\Delta_\delta,\frac{1}{f-a_j}\right)}{\sum_{j=1}^{5}\overline{C}\left(r,\Delta_\delta,\frac{1}{\hat{f}-a_j}\right)}>\frac{1}{2},$$

then $f\equiv g$.

4. Main results

Let $f(z)$ be a meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant\arg z\leqslant\beta\}$ and a be a complex number in the extended complex plane. Write $E_{\alpha,\beta}(a,f) = \{z\in\overline{\Omega}(\alpha,\beta):f(z)-a=0\}$, where each zero with multiplicity m is counted m times. If we ignore the multiplicity, then the set is denoted by $\overline{E}_{\alpha,\beta}(a,f)$. We use $\overline{E}^{k)}_{\alpha,\beta}(a,f)$ to denote the set of zeros of $f-a$ with multiplicities not greater than k, in which each zero is counted only once.

In this paper, we say that two meromorphic functions in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z:\alpha\leqslant\arg z\leqslant\beta\}$ share a function $a(z)$ if we have $f(z)-a(z)=0$ if and only if $\hat{f}-a(z)=0$. Now we consider the case that two meromorphic functions partially share small functions.

DEFINITION 2. Let $f(z)$ be a meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$ and $a(z)$ be a small function of $f(z)$. We define

$$\overline{E}_{\alpha,\beta}(a,f) = \{z | f(z) - a(z) = 0\}$$

in which each zero is counted only once.

We say that a meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$ $f(z)$ partially shares a value a with a meromorphic function $\widehat{f}(z)$ in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$ if

$$\overline{E}_{\alpha,\beta}(a,f) \subseteq \overline{E}_{\alpha,\beta}(a,\widehat{f})$$

To prove our main theorem, we need to get the following Xiong inequality for meromorphic functions in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$.

THEOREM 2. *Let $f(z)$ be a meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$ and b_j $(j = 1,2,\ldots,q)$ be distinct finite non zero complex numbers. Then for any positive integer n, we have*

$$\begin{aligned} qS_{\alpha,\beta}(r,f) < \overline{C}_{\alpha,\beta}(r,f) + qC_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \sum_{j=1}^{q} C_{\alpha,\beta}\left(r,\frac{1}{f^{(n)}-b_j}\right) \\ -\left[(q-1)C_{\alpha,\beta}\left(r,\frac{1}{f^{(n)}}\right) + C_{\alpha,\beta}\left(r,\frac{1}{f^{(n+1)}}\right)\right] + R_{\alpha,\beta}(r,f). \end{aligned} \tag{1}$$

Proof. We have

$$\begin{aligned} S_{\alpha,\beta}(r,f') &= S_{\alpha,\beta}\left(r,f\frac{f'}{f}\right) \leqslant S_{\alpha,\beta}(r,f) + S_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + O(1) \\ &\leqslant S_{\alpha,\beta}(r,f) + A_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + B_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + C_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + O(1) \\ &= S_{\alpha,\beta}(r,f) + \overline{C}_{\alpha,\beta}(r,f) + R_{\alpha,\beta}(r,f) \\ &= 2S_{\alpha,\beta}(r,f) + R_{\alpha,\beta}(r,f). \end{aligned} \tag{2}$$

Hence, by Lemma 1 and (2), we have

$$R_{\alpha,\beta}(r,f^{(k)}) = O(\log rS_{\alpha,\beta}(r,f^{(k)})) = O(\log rS_{\alpha,\beta}(r,f)) = R_{\alpha,\beta}(r,f). \tag{3}$$

$$A_{\alpha,\beta}\left(r,\frac{f^{(k)}}{f-a_i}\right) + B_{\alpha,\beta}\left(r,\frac{f^{(k)}}{f-a_i}\right) = R_{\alpha,\beta}(r,f). \tag{4}$$

From Lemma 1, (3) and (4), we have

$$A_{\alpha,\beta}\left(r,\frac{f^{(k)}}{\prod_{i=1}^{p}(f-a_i)}\right) + B_{\alpha,\beta}\left(r,\frac{f^{(k)}}{\prod_{i=1}^{p}(f-a_i)}\right) = R_{\alpha,\beta}(r,f^{(k)}),$$

$$A_{\alpha,\beta}\left(r,\frac{f^{(k+1)}}{f^{(k)}\prod\limits_{j=1}^{q}(f^{(k)}-b_j)}\right)+B_{\alpha,\beta}\left(r,\frac{f^{(k+1)}}{f^{(k)}\prod\limits_{j=1}^{q}(f^{(k)}-b_j)}\right)=R_{\alpha,\beta}(r,f^{(k)})$$

and

$$\frac{1}{\prod\limits_{i=1}^{p}(f-a_i)^n}=\left\{\frac{f^{(k)}}{\prod\limits_{i=1}^{p}(f-a_i)}\right\}^n\frac{f^{(k+1)}}{f^{(k)}\prod\limits_{j=1}^{q}(f^{(k)}-b_j)}\frac{\prod\limits_{j=1}^{q}(f^{(k)}-b_j)}{(f^k)^{n-1}f^{(k+1)}}$$

Then

$$\begin{aligned}&nA_{\alpha,\beta}\left(r,\frac{1}{\prod_{i=1}^{p}(f-a_i)}\right)+nB_{\alpha,\beta}\left(r,\frac{1}{\prod_{i=1}^{p}(f-a_i)}\right)\\ \leqslant\, &A_{\alpha,\beta}\left(r,\frac{\prod_{j=1}^{q}(f^{(k)}-b_j)}{(f^k)^{n-1}f^{(k+1)}}\right)+B_{\alpha,\beta}\left(r,\frac{\prod_{j=1}^{q}(f^{(k)}-b_j)}{(f^k)^{n-1}f^{(k+1)}}\right)+R_{\alpha,\beta}(r,f^{(k)}).\end{aligned}\tag{5}$$

From (3) and Lemma 1, we have

$$\begin{aligned}&A_{\alpha,\beta}\left(r,\frac{\prod\limits_{j=1}^{q}(f^{(k)}-b_j)}{(f^{(k)})^{n-1}f^{(k+1)}}\right)+B_{\alpha,\beta}\left(r,\frac{\prod\limits_{j=1}^{q}(f^{(k)}-b_j)}{(f^{(k)})^{n-1}f^{(k+1)}}\right)\\ &=C_{\alpha,\beta}\left(r,\frac{(f^{(k)})^{n-1}f^{(k+1)}}{\prod\limits_{j=1}^{q}(f^{(k)}-b_j)}\right)-C_{\alpha,\beta}\left(r,\frac{\prod\limits_{j=1}^{q}(f^{(k)}-b_j)}{(f^{(k)})^{n-1}f^{(k+1)}}\right)+R_{\alpha,\beta}(r,f^{(k)})\\ &=\overline{C}_{\alpha,\beta}(r,f)-(q-n)C_0(r,f^{(k)})+\sum_{j=1}^{q}C_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}-b_j}\right)\\ &\quad-(n-1)C_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}}\right)-C_{\alpha,\beta}\left(r,\frac{1}{f^{(k+1)}}\right)+R_{\alpha,\beta}(r,f^{(k)}).\end{aligned}\tag{6}$$

From (2), (3), (6) and Lemma 1, we obtain

$$\begin{aligned}&nA_{\alpha,\beta}\left(r,\frac{1}{\prod\limits_{i=1}^{p}(f-a_i)}\right)+nB_{\alpha,\beta}\left(r,\frac{1}{\prod\limits_{i=1}^{p}(f-a_i)}\right)\\ &=nS_{\alpha,\beta}\left(r,\prod_{i=1}^{p}(f-a_i)\right)-nC_{\alpha,\beta}\left(r,\frac{1}{\prod\limits_{i=1}^{p}(f-a_i)}\right)+O(1)\\ &=npS_{\alpha,\beta}(r,f)-n\sum_{i=1}pC_{\alpha,\beta}\left(r,\frac{1}{(f-a_i)}\right)+R_{\alpha,\beta}(r,f^{(k)}).\end{aligned}\tag{7}$$

Put (6) and (7) into (5), then we have

$$npS_{\alpha,\beta}(r,f) \leqslant \overline{C}_{\alpha,\beta}(r,f)+n\sum_{i=1}^{p}C_{\alpha,\beta}\left(r,\frac{1}{(f-a_i)}\right)+\sum_{j=1}^{q}C_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}-b_j}\right)$$
$$-(q-n)C_{\alpha,\beta}(r,f^{(k)})-(n-1)C_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}}\right)-C_{\alpha,\beta}\left(r,\frac{1}{f^{(k+1)}}\right)+R_{\alpha,\beta}(r,f).$$

Let $n=q$, $p=1$, we can get the inequality (1). The proof of Theorem 4.1 is completed. □

Next, we prove our main result of the paper as follows.

THEOREM 3. *Let $f(z)$ and $\widehat{f}(z)$ be two meromorphic functions in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant \arg z\leqslant\beta\}$ and a_j $(j=1,2,\ldots,k)$ be k distinct small functions, where $k\geqslant 5$ and for a non negative integers n. Let $f(z)$ and $\widehat{f}(z)$ be two meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant\arg z\leqslant\beta\}$ and*

$$E_{\alpha,\beta}(a_j,f^{(n)})\subseteq E_{\alpha,\beta}(a_j,\widehat{f}^{(n)}),\quad \textit{for all } 1\leqslant j\leqslant k, \tag{8}$$

$$E_{\alpha,\beta}(0,f)\subseteq E_{\alpha,\beta}(0,f^{(n)}) \textit{ and } E_{\alpha,\beta}(0,\widehat{f})\subseteq E_{\alpha,\beta}(0,\widehat{f}^{(n)}), \tag{9}$$

and

$$\frac{\liminf\limits_{r\to\infty}\sum_{j=1}^{k}C_{\alpha,\beta}\left(r,\frac{1}{f^{(n)}-a_j}\right)}{\liminf\limits_{r\to\infty}\sum_{j=1}^{k}C_{\alpha,\beta}\left(r,\frac{1}{\widehat{f}^{(n)}-a_j}\right)}>\frac{n+1}{k-(n+3)}, \tag{10}$$

then $f^{(n)}(z)\equiv\widehat{f}^{(n)}(z)$.

Proof. Given $\in>0$ and from Theorem 1, we have

$$(k-2-\in)S_{\alpha,\beta}(r,f)\leqslant\overline{C}_{\alpha,\beta}(r,f)+(k-2)C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+\sum_{j=1}^{k-2}C_{\alpha,\beta}\left(r,\frac{1}{f^{(n)}-a_j}\right)$$
$$-(k-3)C_{\alpha,\beta}\left(r,\frac{1}{f^{(n)}}\right)+R_{\alpha,\beta}(r,f) \tag{11}$$

and

$$(k-2-\in)S_{\alpha,\beta}(r,\widehat{f})\leqslant\overline{C}_{\alpha,\beta}(r,\widehat{f})+(k-2)C_0\left(r,\frac{1}{\widehat{f}}\right)+\sum_{j=1}^{k-2}C_{\alpha,\beta}\left(r,\frac{1}{\widehat{f}^{(n)}-a_j}\right)$$
$$-(k-3)C_{\alpha,\beta}\left(r,\frac{1}{\widehat{f}^{(n)}}\right)+R_{\alpha,\beta}(r,\widehat{f}). \tag{12}$$

Using (9), (11) and (12) reduces to

$$(k-2-\in)S_{\alpha,\beta}(r,f)\leqslant\overline{C}_{\alpha,\beta}(r,f)+C_0\left(r,\frac{1}{f^{(n)}}\right)$$
$$+\sum_{j=1}^{k-2}C_{\alpha,\beta}\left(r,\frac{1}{f^{(n)}-a_j}\right)+R_{\alpha,\beta}(r,f) \tag{13}$$

and

$$(k-2-\in)S_{\alpha,\beta}(r,\widehat{f}) \leqslant \overline{C}_{\alpha,\beta}(r,\widehat{f})+C_{\alpha,\beta}\left(r,\frac{1}{\widehat{f}^{(n)}}\right) + \sum_{j=1}^{k-2} C_{\alpha,\beta}\left(r,\frac{1}{\widehat{f}^{(n)}-a_j}\right)+R_{\alpha,\beta}(r,\widehat{f}). \qquad (14)$$

Without loss of generality, we may assume $a_k=\infty$ and $a_{k-1}=0$. First we may assume that all a_j $(1\leqslant j\leqslant k)$ in (8) are finite. Then by (13) and (14), we have

$$(k-3-\in)S_{\alpha,\beta}(r,f) \leqslant \sum_{j=1}^{k-1} C_{\alpha,\beta}\left(r,\frac{1}{f^{(n)}-a_j}\right)+R_{\alpha,\beta}(r,f) \qquad (15)$$

and

$$(k-3-\in)S_{\alpha,\beta}(r,\widehat{f}) \leqslant \sum_{j=1}^{k-1} C_{\alpha,\beta}\left(r,\frac{1}{\widehat{f}^{(n)}-a_j}\right)+R_{\alpha,\beta}(r,\widehat{f}). \qquad (16)$$

From (15), (16) and by Remark 3.2, we have

$$\begin{aligned}&(q-3-\in)[S_{\alpha,\beta}(r,f)+S_{\alpha,\beta}(r,\widehat{f})]\\ \leqslant& \sum_{j=1}^{k-1}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f^{(n)}-a_j}\right)+\sum_{j=1}^{q}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\widehat{f}^{(n)}-a_j}\right)\\ &+R_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,\widehat{f}),\end{aligned} \qquad (17)$$

Assume that $f^{(n)}(z)\not\equiv\widehat{f}^{(n)}(z)$. Then from (8), we have

$$\begin{aligned}\sum_{j=1}^{k-1}\overline{C}_{\alpha,beta}\left(r,\frac{1}{f^{(n)}-a_j}\right) &\leqslant \sum_{j=1}^{k-1}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f^{(n)}-\widehat{f}^{(n)}}\right)\\ &\leqslant (n+1)[S_{\alpha,\beta}(r,f)+S_{\alpha,\beta}(r,\widehat{f})]+O(1).\end{aligned} \qquad (18)$$

From (15), (16) and (18), we have

$$\begin{aligned}&\sum_{j=1}^{k-1}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f^{(n)}-a_j}\right)\\ \leqslant& \left(\frac{n+1}{k-3)}+O(1)\right)\left[\sum_{j=1}^{k-1}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f^{(n)}-a_j}\right)+\sum_{j=1}^{k-1}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\widehat{f}^{(n)}-a_j}\right)\right]\end{aligned}$$

for $r\notin E$, which implies

$$\begin{aligned}&\left(\frac{k-(n+4)-\in}{(k-3)-\in}+O(1)\right)\sum_{j=1}^{k-1}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f^{(n)}-a_j}\right)\\ \leqslant& \left(\frac{n+1}{k-3-\in}+O(1)\right)\sum_{j=1}^{k-1}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\widehat{f}^{(n)}-a_j}\right)\end{aligned}$$

for $r \notin E$.

Therefore, we obtain

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k-1} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right)}{\sum_{j=1}^{k-1} \overline{C}_0\left(r, \frac{1}{\widehat{f}-a_j}\right)} \leqslant \frac{n+1}{k-(n+4)-\in}$$

which is true for all $\in > 0$ and replace $k-1$ by k. Hence

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k-1} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f^{(n)}-a_j}\right)}{\sum_{j=1}^{k-1} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{\widehat{f}^{(n)}-a_j}\right)} \leqslant \frac{n+1}{k-(n+4)}. \tag{19}$$

Where a_k is finite (since all a_j $(1 \leqslant j \leqslant k)$ are finite).

From (4.21) contradicts to (10) and hence $f^{(n)}(z) \equiv \widehat{f}^{(n)}(z)$. Now assume that one of the a_j $(1 \leqslant j \leqslant k)$ in (8) is infinity say $a_k = \infty$. Taking any finite value a such that $a \neq a_j$ $(1 \leqslant j \leqslant k-1)$. Set

$$F^{(n)}(z) = \frac{1}{f^{(n)}-a}, \; G^{(n)}(z) = \frac{1}{\widehat{f}^{(n)}-a}.$$

Put $b_j = \frac{1}{a_j-a}$ $(1 \leqslant j \leqslant k-1)$ and $b_k = 0$.

Since $F^{(n)}(z)$ and $G^{(n)}(z)$ partially share finite values b_j $(1 \leqslant j \leqslant k-1)$ IM. Thus by the above case $F^{(n)}(z) \equiv G^{(n)}(z)$. Which completes the proof of theorem. □

If $n=0$ in Theorem 2, then the conditions $E_{\alpha,\beta}(0,f) \subseteq E_{\alpha,\beta}(0,f^{(n)})$ and $E_{\alpha,\beta}(0,\widehat{f}) \subseteq E_{\alpha,\beta}(0,\widehat{f}^{(n)})$ are obvious and hence in this case, Theorem 2 reduces as follows

THEOREM 4. *Let $f(z)$ and $\widehat{f}(z)$ be two meromorphic functions in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$ and a_j $(j = 1,2,\ldots,k)$ be k distinct small functions, where $k \geqslant 5$. If $\overline{E}_{\alpha,\beta}(a_j,f) \subseteq \overline{E}_{\alpha,\beta}(a_j,\widehat{f})$ for all $1 \leqslant j \leqslant k$. If*

$$\frac{\liminf_{r\to\infty} \sum_{j=1}^{k} C_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right)}{\liminf_{r\to\infty} \sum_{j=1}^{k} C_{\alpha,\beta}\left(r, \frac{1}{\widehat{f}-a_j}\right)} > \frac{1}{k-3},$$

then $f(z) \equiv \widehat{f}(z)$.

If $n=0$ and $k=5$ in Theorem 2, then from Theorem 2 we deduce Theorem 1 as follows

COROLLARY 1. *Let $f(z)$ and $\widehat{f}(z)$ be two meromorphic functions in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant\arg z\leqslant\beta\}$ and a_j $(j=1,2,\ldots,5)$ be $4\nu+1$ distinct small functions. If $\overline{E}_{\alpha,\beta}(a_j,f)\subseteq\overline{E}_{\alpha,\beta}(a_j,\widehat{f})$ for all $1\leqslant j\leqslant 5$, and*

$$\liminf_{r\to\infty}\frac{\sum_{j=1}^{5}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)}{\sum_{j=1}^{5}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\widehat{f}-a_j}\right)}>\frac{1}{2},$$

then $f(z)\equiv\widehat{f}(z)$.

DEFINITION 3. Let $f(z)$ be a meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)$ $=\{z:\alpha\leqslant\arg z\leqslant\beta\}$ and $a(z)$ be a values of $f(z)$. We define

$$\overline{E}_{\alpha,\beta}(a,f)=\{z|f(z)-a(z)=0\}$$

in which each zero is counted only once.

We consider, two meromorphic functions partially share five or more values in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant\arg z\leqslant\beta\}$. Precisely speaking, if $f(z)$ and $\widehat{f}(z)$ be two meromorphic functions in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant\arg z\leqslant\beta\}$ and k be distinct values $a_1,a_2,\ldots,a_k$, $k\geqslant 5$ such that $\overline{E}_{\alpha,\beta}(a_j,f)\subseteq\overline{E}_{\alpha,\beta}(a_j,\widehat{f})$, for all $1\leqslant j\leqslant k$.

Now we can state and prove our theorem as follows

THEOREM 5. *Let $f(z)$ and $\widehat{f}(z)$ be two meromorphic functions in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant\arg z\leqslant\beta\}$ and a_j $(j=1,2,\ldots,k)$ be k distinct values, where $k\geqslant 5$ and for a non negative integers n. If*

$$E_{\alpha,\beta}(a_j,f^{(n)})\subseteq E_{\alpha,\beta}(a_j,\widehat{f}^{(n)}),\quad forall\ \ 1\leqslant j\leqslant k,$$

$$E_{\alpha,\beta}(0,f)\subseteq E_{\alpha,\beta}(0,f^{(n)})\ \ and\ \ E_{\alpha,\beta}(0,\widehat{f})\subseteq E_{\alpha,\beta}(0,\widehat{f}^{(n)}),$$

and

$$\frac{\liminf_{r\to\infty}\sum_{j=1}^{k}C_{\alpha,\beta}\left(r,\frac{1}{f^{(n)}-a_j}\right)}{\liminf_{r\to\infty}\sum_{j=1}^{k}C_{\alpha,\beta}\left(r,\frac{1}{\widehat{f}^{(n)}-a_j}\right)}>\frac{n+1}{k-(n+3)},$$

then $f^{(n)}(z)\equiv\widehat{f}^{(n)}(z)$.

Proof. Using a similar argument as Theorem 2, we can prove it. □

If $n=0$ in Theorem 4, then the conditions $E_{\alpha,\beta}(0,f)\subseteq E_{\alpha,\beta}(0,f^{(n)})$ and $E_{\alpha,\beta}(0,\widehat{f})$ $\subseteq E_{\alpha,\beta}(0,\widehat{f}^{(n)})$ are obvious and hence in this case, Theorem 4 reduces as follows

THEOREM 6. *Let $f(z)$ and $\widehat{f}(z)$ be two meromorphic functions in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant\arg z\leqslant\beta\}$ and a_j $(j=1,2,\ldots,k)$ be k distinct values, where $k\geqslant 5$. If $\overline{E}_{\alpha,\beta}(a_j,f)\subseteq\overline{E}_{\alpha,\beta}(a_j,\widehat{f})$ for all $1\leqslant j\leqslant k$. If*

$$\frac{\liminf\limits_{r\to\infty}\sum\limits_{j=1}^{k}C_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)}{\liminf\limits_{r\to\infty}\sum\limits_{j=1}^{k}C_{\alpha,\beta}\left(r,\frac{1}{\widehat{f}-a_j}\right)}>\frac{1}{k-3},$$

then $f(z)\equiv\widehat{f}(z)$.

If $n=0$ and $k=5$ in Theorem 4, then Theorem 4 reduces as follows

COROLLARY 2. *Let $f(z)$ and $\widehat{f}(z)$ be two meromorphic functions in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant\arg z\leqslant\beta\}$ and a_j $(j=1,2,\ldots,5)$ be 5 distinct values. If $\overline{E}_{\alpha,\beta}(a_j,f)\subseteq\overline{E}_{\alpha,\beta}(a_j,\widehat{f})$ for all $1\leqslant j\leqslant 5$, and*

$$\liminf_{r\to\infty}\frac{\sum\limits_{j=1}^{5}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)}{\sum\limits_{j=1}^{5}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\widehat{f}-a_j}\right)}>\frac{1}{2},$$

then $f(z)\equiv\widehat{f}(z)$.

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