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COEFFICIENT PROBLEMS OF A CLASS OF q–STARLIKE FUNCTIONS ASSOCIATED WITH q–ANALOGUE OF AL–OBOUDI–AL–QAHTANI INTEGRAL OPERATOR AND NEPHROID DOMAIN

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Abstract. This investigation is on a set $SN_q^*(n,\tau;\eta)$ of q-starlike functions defined by using a newly defined q-analogue of Al-Oboudi-Al-Qahtani integral operator along with subordination and nephroid domain. This new q-operator generalizes some known integral operators. Results such as coefficient bounds, Fekete-Szegö problem (for real and complex parameters) and bounds of some Hankel determinants are presented. Our results generalize some known and new ones.

1. Introduction and definitions

In this work we let A represent the set of *normalized analytic* functions of the form

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j,\quad f(0)=0,\ f'(0)=1 \text{ and } z\in E:=\{z\in\mathbb{C}:|z|<1\}. \tag{1.1}$$

Also, we let S represent the set of functions *analytic* and *univalent* in E. A function $f\in S$ that satisfies the geometric condition $Re\{zf'(z)/f(z)\}>0,\ z\in E$, is called a starlike function. Let $S^\star$ represent the set of starlike functions in the unit disk E.

The convolution of two functions

$$f_1(z)=z+\sum_{j=2}^{\infty}a_jz^j,\ f_2(z)=z+\sum_{j=2}^{\infty}\alpha_jz^j\in A,\quad z\in E, \tag{1.2}$$

is defined by the function $(f_1\star f_2)(z):=z+\sum_{j=2}^{\infty}a_j\alpha_jz^j=:(f_2\star f_1)(z)\in A$. Likewise, from (1.2), $f_1(z)$ is said to be *subordinate* to $f_2(z)$, notationally represented by $f_1(z)\prec f_2(z)$, $z\in E$, if there exists an analytic function

$$s(z)=s_1z+s_2z^2+s_3z^3+\cdots\quad (s(0)=0,\ |s(z)|\leqq|z|<1,\ z\in E) \tag{1.3}$$

such that

$$f_1(z)=f_2(z)\circ s(z)=f_2(s(z)).$$

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In case $f_2(z)$ is univalent in E, then $f_1(z) \prec f_2(z) \implies f_1(0) = f_2(0)$ and $f_1(E) \subset f_2(E)$.

In the works of Pommerenke [35, 36], the ith-Hankel determinant

$$H_{i,j}(f) := \begin{vmatrix} a_j & a_{j+1} & \cdots & \cdots & a_{j+i-1} \\ a_{j+1} & a_{j+2} & \cdots & \cdots & a_{j+i} \\ a_{j+2} & a_{j+3} & \cdots & \cdots & a_{j+i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j+i-1} & a_{j+i} & \cdots & \cdots & a_{j+2(i-1)} \end{vmatrix}$$

was reported for $i, j \in \mathbb{N}$ where all the $a_j s$ are the coefficients of z^j for $f \in S$ of the form (1.1). Observe that for some varied parameters we obtain

$$|H_{2,1}(f)| = |a_3 - a_2^2|, \tag{1.4}$$

$$|H_{2,2}(f)| = |a_2 a_4 - a_3^2| \tag{1.5}$$

and

$$|H_{3,1}(f)| \leqq |a_3||H_{2,2}(f)| + |a_4||a_2 a_3 - a_4| + |a_5||H_{2,1}(f)|. \tag{1.6}$$

In deed, Pommerenke [35] mentioned the application of Hankel determinants in the study of singularities of complex functions. Cantor [5] used Hankel determinants to solve some problems of power series with integral coefficients. Junod [18], Dilcher and Jiu [8], and Its and Krasovsky [13] reported that some specific types of problems of orthogonal polynomials can be solved by using Hankel determinants. Chu [7] applied Hankel determinants to solve some problems involving factorial fractions. The asymptotic behaviour of Hankel determinants were described in separate works of Noor [32], Charliera and Gharakhloo [6] and Ul-Haq and Noor [48]. Some notations and properties of Hankel matrices and their determinants were extensively discussed by Layman [29]. More properties of Hankel determinants can be found in [3, 4, 26, 34]. Another interesting aspect of (1.4) is its kin association with the well-known Fekete-Szegö functional

$$F_\lambda(f) = \left|a_3 - \lambda a_2^2\right| \tag{1.7}$$

in [10]. It can easily be verified that $|H_{2,1}(f)| = F_1(f)$. So (1.7) is a generalization of (1.4). See the works in [3, 4, 26] for more details.

Lately, geometric function theorists have been inspired to study different kinds of *natural* image domains such as circular domain [17], domain of lemniscate of Bernoulli [43], conic domain [21], stripe-like domain [25], shell-like domain [9], cardioid domain [42], lune-like domain [11], leaf-like domain [37], petal-like domain [31] and many others, for various subsets of A. In particular, Wani and Swaminathan [50, 51], Swaminathan and Wani [46] and Khan et al. [23] investigated the set

$$SN^*(\eta) := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \eta(z) = 1 + z - \frac{1}{3}z^3,\ z \in E \right\} \tag{1.8}$$

where $\eta(z)$ is a function that *univalently* maps the boundary ∂E of the unit disk E onto a 2-cusped kidney-shaped curve called *nephroid*. It was also demonstrated in [50, 51] that the function $\eta(z)$ maps the unit disk E onto the domain bounded by the nephroid curve

$$\left((u-1)^2+v^2-\left(\frac{2}{3}\right)^2\right)^3=\frac{4}{3}v^2. \tag{1.9}$$

It is important to note that the nephroid curve in (1.9) is symmetric about the real axis and falls entirely on the *open-right-half-plane*. Interested readers can see [23, 46, 50, 51] for some illustrative diagrams and more properties of the nephroid curve.

Recent studies show that the concept of q-calculus has enticed many geometric function theorists. The concept of q-difference and q-integral were introduced in the works of Jackson [14, 15, 16] and since then, many researchers (for instance see [1, 12, 26, 27, 28, 38, 45]) have used it in various ways to define, extend and establish many properties of some subsets of A. In deed, Srivastava [44] and Kac and Cheung [19] extensively discussed some areas of applications of q-analysis in the fields of Pure and Applied Mathematics.

DEFINITION 1. ([14, 15, 19]) For function $f(z)\in A$ of the form (1.1) and for $0<q<1$, the q-derivative operator $D_q : A \longrightarrow A$ of $f(z)$ is defined by

$$\left.\begin{aligned}
&D_qf(z)=\tfrac{f(z)-f(qz)}{z(1-q)}=1+\sum_{j=2}^{\infty}[j]_q a_j z^{j-1} \quad (z\neq 0)\\
&D_qf(0)=f'(0)=1 \quad (z=0) \quad \text{if it exists}\\
&D_q^2f(z)=D_q(D_qf(z))=\sum_{j=2}^{\infty}[j]_q[j-1]_q a_j z^{j-2}\\
&\text{where } [j]_q=\tfrac{1-q^j}{1-q}=1+q+q^2+\cdots+q^{j-1} \quad \text{so that } \lim_{q\to 1^-}[j]_q=j.
\end{aligned}\right\} \tag{1.10}$$

The importance and uses of operators (and q-operators) in geometric function theory is not hidden anymore. In fact, many researchers (for example see [20, 33, 41, 49]) find it convenient to define q-operators by using the principle of convolution. In particular, Aldweby and Darus [1] introduced and studied the class

$$S_q^{\star}(M):=\left\{f\in A:\ \frac{zD_qf(z)}{f(z)}\prec M(z)=1+\sum_{j=1}^{\infty}m_jz^j,\ 0<q<1,\ z\in E\right\} \tag{1.11}$$

of Ma-Minda q-starlike functions. Note that $\lim_{q\to 1^-} S_q^{\star}(M)=S^{\star}(M)$ is the well-known class of Ma-Minda-starlike functions in E. Since $M(z)$ is known to unify several subclasses of analytic-univalent functions, geometric function theorists have studied (1.11) by taking different forms of function $M(z)$. For instance see [24, 38, 40, 45, 49] for some details.

Now using Definition 1 along with the concept of convolution, we define the q-Al-Oboudi-Al-Qahtani operator as follows.

DEFINITION 2. Let $n = \{0,1,2,\ldots\}$, $\tau \geqq 0$ and $0 < q < 1$, then

$$L_q^{n,\tau} f(z) = f(z) \star \left(z + \sum_{j=2}^{\infty} \frac{1}{\{1 + ([j]_q - 1)\tau\}^n} z^j \right), \quad z \in E$$

where equivalently we have

$$L_q^{n,\tau} f(z) = z + \sum_{j=2}^{\infty} \frac{1}{\{1 + ([j]_q - 1)\tau\}^n} a_j z^j, \quad z \in E \tag{1.12}$$

or

$$L_q^{n,\tau} f(z) = z + \sum_{j=2}^{\infty} \gamma_j a_j z^j, \quad z \in E \tag{1.13}$$

where

$$\gamma_j = \frac{1}{\{1 + ([j]_q - 1)\tau\}^n}.$$

From (1.12), we observe that

1. $\lim_{q \to 1^-} L_q^{0,\tau} f(z) = \lim_{q \to 1^-} L_q^{n,0} f(z) = \lim_{q \to 1^-} L_q^{0,0} f(z) = f(z) \in A$ in (1.1),
2. $\lim_{q \to 1^-} L_q^{n,1} f(z) = L^n f(z)$ is the Sălăgean integral operator studied in [39] and
3. $\lim_{q \to 1^-} L_q^{n,\tau} f(z) = L^{n,\tau} f(z)$ is the Al-Oboudi-Al-Qahtani integral operator studied in [2].

The new class investigated in this paper is therefore defined as follows.

DEFINITION 3. Let $n = \{0,1,2,\ldots\}$, $\tau \geqq 0$ and $0 < q < 1$. A function $f(z) \in A$ is said to be a member of the set $SN_q^{\star}(n,\tau;\eta)$ if the q-differential subordination

$$\frac{z D_q(L_q^{n,\tau} f(z))}{L_q^{n,\tau} f(z)} \prec \eta(z), \quad z \in E \tag{1.14}$$

holds true where $L_q^{n,\tau}$ is the q-operator defined in (1.12) and $\eta(z)$ is defined in (1.8).

Note that from Definition 3,

$$\lim_{q \to 1^-} SN_q^{\star}(0,\tau;\eta) = \lim_{q \to 1^-} SN_q^{\star}(n,0;\eta) = \lim_{q \to 1^-} SN_q^{\star}(0,0;\eta) = SN^{\star}(\eta)$$

is the class earlier studied in [23, 46, 50, 51].

The purpose of our present investigation is to solve some coefficient problems such as the coefficient bounds, the Fekete-Szegö problem and some bounds of Hankel determinants.

2. Applicable lemmas

Let P be the well-known set of analytic functions of the form

$$c(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad z \in E \tag{2.1}$$

that have positive real parts in E. Then the following lemmas hold for $c(z) \in P$.

LEMMA 1. ([47]) *For* $j \in \mathbb{N}$, $|c_j| \leqq 2$.

LEMMA 2. ([47]) *For* $\kappa \in \mathbb{R}$,

$$|c_2 - \kappa c_1^2| \leqq \begin{cases} 2 - 4\kappa & \text{for } \kappa \leqq 0, \\ 2 & \text{for } 0 \leqq \kappa \leqq 1, \\ 4\kappa - 2 & \text{for } \kappa \geqq 1. \end{cases}$$

LEMMA 3. ([22]) *For* $\varkappa \in \mathbb{C}$, $|c_2 - \varkappa c_1^2| \leqq 2\max\{1, |2\varkappa - 1|\}$.

LEMMA 4. ([23, 30]) *For* $i, j \in \mathbb{N}$; $|c_{i+j} - \lambda c_i c_j| \leqq 2$ *for* $0 \leqq \lambda \leqq 1$.

LEMMA 5. ([22, 23]) *For* $i, j \in \mathbb{N}$; $|c_{i+2j} - \sigma c_i c_j^2| \leqq 2(1 + 2\sigma)$ *where* $\sigma \in \mathbb{R}$.

3. Main results

Henceforth, let $n = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$, $\tau \geqq 0$ and $0 < q < 1$ unless otherwise stated.

THEOREM 1. *Let* $f(z) \in SN_q^*(n, \tau; \eta)$, *then*

$$|a_2| \leqq \frac{1}{([2]_q - 1)\gamma_2} \tag{3.1}$$

$$|a_3| \leqq \frac{1}{([2]_q - 1)([3]_q - 1)\gamma_3} \tag{3.2}$$

$$|a_4| \leqq 2(L + 2J) \tag{3.3}$$

$$|a_5| \leqq 2(E + 4B). \tag{3.4}$$

where

$$\left.\begin{aligned}
J &= \tfrac{1}{([4]_q-1)\gamma_4}\left\{\tfrac{3-[2]_q(4-[2]_q)+[3]_q([2]_q-1)}{8([2]_q-1)^2([3]_q-1)}-\tfrac{1}{12}\right\},\\
K &= \tfrac{1}{([4]_q-1)\gamma_4}\left\{\tfrac{1}{2}-\tfrac{[3]_q+[2]_q-2}{4([2]_q-1)([3]_q-1)}\right\},\\
L &= \tfrac{1}{2([4]_q-1)\gamma_4}\\
A &= \tfrac{1}{([5]_q-1)\gamma_5}\left[\tfrac{1}{16([2]_q-1)^2}\left\{\tfrac{(2-[2]_q)^2}{([3]_q-1)}+\tfrac{1}{([2]_q-1)}+\tfrac{(2-[2]_q)(2[2]_q+[3]_q-3)}{([2]_q-1)([3]_q-1)}\right\}\right.\\
&\qquad\left.-\tfrac{([4]_q+[2]_q-2)}{8([2]_q-1)([4]_q-1)}\left\{\tfrac{3-[2]_q(4-[2]_q)+[3]_q([2]_q-1)}{2([2]_q-1)^2([3]_q-1)}-\tfrac{1}{3}\right\}\right]\\
B &= \tfrac{1}{([5]_q-1)\gamma_5}\left[\tfrac{([4]_q+[2]_q-2)}{4([2]_q-1)([4]_q-1)}\left\{1-\tfrac{[3]_q+[2]_q-2}{2([2]_q-1)([3]_q-1)}\right\}\right.\\
&\qquad\left.+\tfrac{1}{4([2]_q-1)([3]_q-1)}\left\{\tfrac{(2[2]_q+[3]_q-3)}{2([2]_q-1)}-(2-[2]_q)\right\}-\tfrac{1}{4}\right]\\
C &= \tfrac{1}{([5]_q-1)\gamma_5}\left[\tfrac{1}{2}-\tfrac{([4]_q+[2]_q-2)}{4([2]_q-1)([4]_q-1)}\right]\\
D &= \tfrac{1}{([5]_q-1)\gamma_5}\left[\tfrac{1}{4}-\tfrac{1}{4([3]_q-1)}\right]\\
E &= \tfrac{1}{2([5]_q-1)\gamma_5}.
\end{aligned}\right\} \tag{3.5}$$

The results are non-sharp.

Proof. By the definition of subordination, (1.14) can be expressed as

$$\frac{zD_q(L_q^{n,\tau}f(z))}{L_q^{n,\tau}f(z)}=\eta(s(z))=1+s(z)-\frac{1}{3}(s(z))^3,\quad z\in E \tag{3.6}$$

where $s(z)$ is given in (1.3).

Simple computation shows that by using (1.13), the LHS of (3.6) can be binomially expanded as

$$\begin{aligned}
\frac{zD_q(L_q^{n,\tau}f(z))}{L_q^{n,\tau}f(z)} &= 1+\{[2]_q-1\}\gamma_2a_2z\\
&\quad+\{([3]_q-1)\gamma_3a_3-([2]_q-1)\gamma_2^2a_2^2\}z^2\\
&\quad+\{(2-[2]_q-[3]_q)\gamma_2\gamma_3a_2a_3+([2]_q-1)\gamma_2^3a_2^3+([4]_q-1)\gamma_4a_4\}z^3\\
&\quad+\{(2-[2]_q-[4]_q)\gamma_2\gamma_4a_2a_4+(1-[3]_q)\gamma_3^2a_3^2+(1-[4]_q)\gamma_2^4a_2^4\\
&\qquad+([5]_q-1)\gamma_5a_5+(2[2]_q+[3]_q-3)\gamma_2^2\gamma_3a_2^2a_3\}z^4\\
&\quad+\cdots.
\end{aligned} \tag{3.7}$$

Now we expand the RHS of (3.6). Firstly, it is well-known that the relationship between $s(z)$ in (1.3) and $c(z)$ in (2.1) can be demonstrated as

$$s(z) = \frac{c(z)-1}{c(z)+1} = \frac{(1+c_1z+c_2z^2+c_3z^3+\cdots)-1}{(1+c_1z+c_2z^2+c_3z^3+\cdots)+1}$$
$$= \frac{1}{2}c_1z + \frac{1}{2}\left(-\frac{1}{2}c_1^2+c_2\right)z^2 + \frac{1}{2}\left(\frac{1}{2^2}c_1^3 - c_1c_2 + c_3\right)z^3 + \cdots$$

so that

$$\eta(s(z)) = 1 + \frac{1}{2}c_1z + \frac{1}{2}\left(-\frac{1}{2}c_1^2+c_2\right)z^2 + \frac{1}{2}\left(\frac{1}{6}c_1^3 - c_1c_2 + c_3\right)z^3$$
$$+ \frac{1}{2}\left(\frac{1}{2}c_1^2c_2 - c_1c_3 - \frac{1}{2}c_2^2 + c_4\right)z^4 + \cdots. \quad (3.8)$$

Now comparing of coefficients in (3.7) and (3.8) shows that

$$([2]_q - 1)\gamma_2 a_2 = \frac{1}{2}c_1, \quad (3.9)$$

$$([3]_q - 1)\gamma_3 a_3 - ([2]_q - 1)\gamma_2^2 a_2^2 = \frac{1}{2}\left(-\frac{1}{2}c_1^2 + c_2\right), \quad (3.10)$$

$$(2 - [2]_q - [3]_q)\gamma_2\gamma_3 a_2 a_3 + ([2]_q - 1)\gamma_2^3 a_2^3 + ([4]_q - 1)\gamma_4 a_4 = \frac{1}{2}\left(\frac{1}{6}c_1^3 - c_1c_2 + c_3\right) \quad (3.11)$$

and

$$(2 - [2]_q - [4]_q)\gamma_2\gamma_4 a_2 a_4 + (1 - [3]_q)\gamma_3^2 a_3^2 + (1 - [2]_q)\gamma_2^4 a_2^4 + ([5]_q - 1)\gamma_5 a_5$$
$$+ (2[2]_q + [3]_q - 3)\gamma_2^2\gamma_3 a_2^2 a_3 = \frac{1}{2}\left(\frac{1}{2}c_1^2c_2 - c_1c_3 - \frac{1}{2}c_2^2 + c_4\right). \quad (3.12)$$

By simple calculation, (3.9) simplifies to

$$a_2 = \frac{c_1}{2([2]_q - 1)\gamma_2} \quad (3.13)$$

so that by applying triangle inequality and Lemma 1 we obtain (3.1). Using (3.13) in (3.10) leads to

$$a_3 = \frac{c_1^2(2 - [2]_q) + 2c_2([2]_q - 1)}{4([2]_q - 1)([3]_q - 1)\gamma_3} \quad (3.14)$$

so that by applying triangle inequality and Lemma 1 we obtain (3.2). Using (3.13) and (3.14) in (3.11) leads to

$$a_4 = -\frac{c_1^3}{([4]_q - 1)\gamma_4}\left\{\frac{3 - [2]_q(4 - [2]_q) + [3]_q([2]_q - 1)}{8([2]_q - 1)^2([3]_q - 1)} - \frac{1}{12}\right\}$$
$$- \frac{c_1c_2}{([4]_q - 1)\gamma_4}\left\{\frac{1}{2} - \frac{[3]_q + [2]_q - 2}{4([2]_q - 1)([3]_q - 1)}\right\} + \frac{c_3}{2([4]_q - 1)\gamma_4} \quad (3.15)$$

and for simplicity we have

$$a_4 = -Jc_1^3 - Kc_1c_2 + Lc_3 \tag{3.16}$$

where J, K and L are defined in (3.5). To obtain bound for a_4, from (3.16),

$$\begin{aligned} |a_4| &= |-Jc_1^3 - Kc_1c_2 + Lc_3| \\ &= \left| \frac{L}{2}\left(c_3 - \frac{2K}{L}c_1c_2\right) + \frac{L}{2}\left(c_3 - \frac{2J}{L}c_1^3\right)\right| \end{aligned}$$

so that by applying triangle inequality and Lemmas 4 and 5 we obtain (3.3). Lastly, using (3.13), (3.14) and (3.16) in (3.12) leads to

$$\begin{aligned} a_5 =& \frac{c_1^4}{([5]_q-1)\gamma_5}\left[\frac{1}{16([2]_q-1)^2}\left\{\frac{(2-[2]_q)^2}{([3]_q-1)} + \frac{1}{([2]_q-1)} + \frac{(2-[2]_q)(2[2]_q+[3]_q-3)}{([2]_q-1)([3]_q-1)}\right\}\right. \\ &\left. - \frac{([4]_q+[2]_q-2)}{8([2]_q-1)([4]_q-1)}\left\{\frac{3-[2]_q(4-[2]_q)+[3]_q([2]_q-1)}{2([2]_q-1)^2([3]_q-1)} - \frac{1}{3}\right\}\right] \\ &- \frac{c_1^2c_2}{([5]_q-1)\gamma_5}\left[\frac{([4]_q+[2]_q-2)}{4([2]_q-1)([4]_q-1)}\left\{1 - \frac{[3]_q+[2]_q-2}{2([2]_q-1)([3]_q-1)}\right\}\right. \\ &\left. + \frac{1}{4([2]_q-1)([3]_q-1)}\left\{\frac{(2[2]_q+[3]_q-3)}{2([2]_q-1)} - (2-[2]_q)\right\} - \frac{1}{4}\right] \\ &- \frac{c_1c_3}{([5]_q-1)\gamma_5}\left[\frac{1}{2} - \frac{([4]_q+[2]_q-2)}{4([2]_q-1)([4]_q-1)}\right] \\ &- \frac{c_2^2}{([5]_q-1)\gamma_5}\left[\frac{1}{4} - \frac{1}{4([3]_q-1)}\right] \\ &+ \frac{c_4}{2([5]_q-1)\gamma_5}. \end{aligned}$$

and for simplicity we have

$$a_5 = Ac_1^4 - Bc_1^2c_2 - Cc_1c_3 - Dc_2^2 + Ec_4 \tag{3.17}$$

where A, B, C, D and E are defined in (3.5). To obtain bound for a_5, from (3.17),

$$\begin{aligned} |a_5| &= \left|Ac_1^4 - Bc_1^2c_2 - Cc_1c_3 - Dc_2^2 + Ec_4\right| \\ &= \left|\frac{E}{2}\left(c_4 - \frac{2C}{E}c_1c_3\right) + \frac{E}{2}\left(c_4 - \frac{2D}{E}c_2^2\right) - Bc_1^2\left(c_2 - \frac{A}{B}c_1^2\right)\right| \end{aligned}$$

so that by applying triangle inequality and Lemmas 1 and 4 we obtain (3.4). □

REMARK 1. By setting $n = 0$ (or $\tau = 0$) and letting $q \longrightarrow 1^-$ make (3.1), (3.2), (3.4), (3.13), (3.14), (3.16) and (3.17) to become the results of Khan et al. [23]. Moreover, $|a_4| = \frac{5}{18}$ in (3.3) gives a better estimate when compared with the estimate given by Khan et al. [23].

THEOREM 2. *Let $f(z) \in SN_q^*(n,\tau;\eta)$. Then for $u \in \mathbb{R}$,*

$$\left|a_3 - ua_2^2\right| \leqq \begin{cases} \frac{1}{([3]_q-1)\gamma_3}\left\{\frac{(2-[2]_q)([2]_q-1)\gamma_2^2-u([3]_q-1)\gamma_3}{([2]_q-1)^2\gamma_2^2}+1\right\} & \text{for } u \leqq U_1 \\ \frac{1}{([3]_q-1)\gamma_3} & \text{for } U_1 \leqq u \leqq U_2 \\ \frac{1}{([3]_q-1)\gamma_3}\left\{\frac{u([3]_q-1)\gamma_3-(2-[2]_q)([2]_q-1)\gamma_2^2}{([2]_q-1)^2\gamma_2^2}-1\right\} & \text{for } u \geqq U_2 \end{cases}$$

where

$$U_1 = \frac{(2-[2]_q-1)([2]_q-1)\gamma_2^2}{([3]_q-1)\gamma_3} \quad \text{and} \quad U_2 = \frac{([2]_q-1)\gamma_2^2\{2([2]_q-1)+(2-[2]_q)\}}{([3]_q-1)\gamma_3}.$$

Proof. For $u \in \mathbb{R}$ and using (3.13) and (3.14) in (1.7) leads to

$$\left|a_3 - ua_2^2\right| = \frac{1}{2([3]_q-1)\gamma_3}\left|c_2 - \left\{\frac{u([3]_q-1)\gamma_3-(2-[2]_q)([2]_q-1)\gamma_2^2}{2([2]_q-1)^2\gamma_2^2}\right\}c_1^2\right|$$

which equivalently implies that

$$\left|a_3 - ua_2^2\right| = \frac{1}{2([3]_q-1)\gamma_3}\left|c_2 - \kappa c_1^2\right| \tag{3.18}$$

where

$$\kappa = \frac{u([3]_q-1)\gamma_3-(2-[2]_q)([2]_q-1)\gamma_2^2}{2([2]_q-1)^2\gamma_2^2}. \tag{3.19}$$

Now applying Lemma 2 implies that

$$\left|a_3 - ua_2^2\right| \leqq \frac{1}{([3]_q-1)\gamma_3}\left\{\frac{(2-[2]_q)([2]_q-1)\gamma_2^2-u([3]_q-1)\gamma_3}{([2]_q-1)^2\gamma_2^2}+1\right\}$$

for $u \leqq \frac{(2-[2]_q-1)([2]_q-1)\gamma_2^2}{([3]_q-1)\gamma_3}$,

$$\left|a_3 - ua_2^2\right| \leqq \frac{1}{([3]_q-1)\gamma_3}$$

for $0 \leqq u \leqq \frac{([2]_q-1)\gamma_2^2\{2([2]_q-1)+(2-[2]_q)\}}{([3]_q-1)\gamma_3}$ and

$$\left|a_3 - ua_2^2\right| \leqq \frac{1}{([3]_q-1)\gamma_3}\left\{\frac{u([3]_q-1)\gamma_3-(2-[2]_q)([2]_q-1)\gamma_2^2}{([2]_q-1)^2\gamma_2^2}-1\right\}$$

for $u \geqq \frac{([2]_q-1)\gamma_2^2\{2([2]_q-1)+(2-[2]_q)\}}{([3]_q-1)\gamma_3}$ and the proof completes. □

THEOREM 3. *Let $f(z) \in SN_q^*(n,\tau;\eta)$. Then for $v \in \mathbb{C}$,*

$$\left|a_3 - va_2^2\right| \leqq \frac{1}{([3]_q-1)\gamma_3}\max\left\{1, \left|\frac{v([3]_q-1)\gamma_3-(2-[2]_q)([2]_q-1)\gamma_2^2}{([2]_q-1)^2\gamma_2^2}-1\right|\right\}. \tag{3.20}$$

Proof. For $v \in \mathbb{C}$, using (3.18) and (3.19) in (1.7) and applying Lemma 3 implies that

$$\left|a_3 - va_2^2\right| \leqq \frac{1}{2([3]_q-1)\gamma_3} \times 2\max\left\{1, \left|2\left(\frac{v([3]_q-1)\gamma_3-(2-[2]_q)([2]_q-1)\gamma_2^2}{2([2]_q-1)^2\gamma_2^2}\right)-1\right|\right\}$$

and simple computation gives inequality (3.20). □

REMARK 2. Setting $n = 0$ (or $\tau = 0$) and letting $q \longrightarrow 1^-$ make Theorems 2 and 3 to become the results of Khan et al. [23].

THEOREM 4. *Let* $f(z) \in SN_q^*(n,\tau;\eta)$. *Then*

$$|H_{2,2}(f)| \leqq 4(Q+P+2M) \tag{3.21}$$

where

$$\left.\begin{aligned} M &= \frac{J}{2([2]_q-1)\gamma_2} + \frac{(2-[2]_q)^2}{16([2]_q-1)^2([3]_q-1)^2\gamma_3^2} \\ N &= \frac{K}{2([2]_q-1)\gamma_2} + \frac{(2-[2]_q)}{4([3]_q-1)^2\gamma_3^2} \\ Q &= \frac{L}{2([2]_q-1)\gamma_2} \\ P &= \frac{1}{4([3]_q-1)^2\gamma_3^2} \end{aligned}\right\} \tag{3.22}$$

and J, K *and* L *are defined in* (3.5).

Proof. By using (3.13), (3.14) and (3.16) in (1.5) we have

$$\begin{aligned} |a_2a_4 - a_3^2| = \Bigg| &-c_1^4\left\{\frac{J}{2([2]_q-1)\gamma_2} + \frac{(2-[2]_q)^2}{16([2]_q-1)^2([3]_q-1)^2\gamma_3^2}\right\} \\ &- c_1^2c_2\left\{\frac{K}{2([2]_q-1)\gamma_2} + \frac{(2-[2]_q)}{4([3]_q-1)^2\gamma_3^2}\right\} \\ &+ c_1c_3\left\{\frac{L}{2([2]_q-1)\gamma_2}\right\} - c_2^2\left\{\frac{1}{4([3]_q-1)^2\gamma_3^2}\right\}\Bigg|. \end{aligned}$$

By equivalence we have

$$\begin{aligned} |a_2a_4 - a_3^2| &= \left|-Mc_1^4 - Nc_1^2c_2 + Qc_1c_3 - Pc_2^2\right| \\ &= \left|\frac{Qc_1}{2}\left(c_3 - \frac{2M}{Q}c_1^3\right) + \frac{Qc_1}{2}\left(c_3 - \frac{2N}{Q}c_1c_2\right) - Pc_2^2\right| \end{aligned}$$

where M, N, Q and P are defined in (3.22). Now applying Lemmas 1, 4 and 5 leads to (3.21). □

THEOREM 5. *Let* $f(z) \in SN_q^*(n,\tau;\eta)$. *Then*

$$|a_2a_3 - a_4| \leqq 2(L+2R) \tag{3.23}$$

where

$$\left.\begin{aligned} R &= \frac{(2-[2]_q)}{8([2]_q-1)^2([3]_q-1)\gamma_2\gamma_3} + J \\ S &= \frac{1}{4([2]_q-1)([3]_q-1)\gamma_2\gamma_3} + K \end{aligned}\right\} \tag{3.24}$$

and J, K *and* L *are defined in* (3.5).

Proof. Recall by using (3.13), (3.14) and (3.16) that

$$|a_2a_3 - a_4| = \left| c_1^3 \left\{ \frac{(2-[2]_q)}{8([2]_q-1)^2([3]_q-1)\gamma_2\gamma_3} + J \right\} + c_1c_2 \left\{ \frac{1}{4([2]_q-1)([3]_q-1)\gamma_2\gamma_3} + K \right\} - Lc_3 \right|.$$

By equivalence we have

$$|a_2a_3 - a_4| = |Rc_1^3 + Sc_1c_2 - Lc_3| = \left| - \left\{ \frac{L}{2}\left(c_3 - \frac{2S}{L}c_1c_2\right) + \frac{L}{2}\left(c_3 - \frac{2R}{L}c_1^3\right) \right\} \right|$$

where R, S and L are defined in (3.24) and (3.5). Now applying Lemmas 1, 4 and 5 leads to (3.23). □

THEOREM 6. *Let* $f(z) \in SN_q^*(n,\tau;\eta)$. *Then*

$$|H_{3,1}(f)| \leqq \frac{4(Q+P+2M)}{([2]_q-1)([3]_q-1)\gamma_3} + \frac{2(E+4B)}{([3]_q-1)\gamma_3} + 4(L+2J)(L+2R). \tag{3.25}$$

Proof. Putting (3.2), (3.3), (3.4), (3.20), (3.21) and (3.23) into (1.6) and simplifying leads to (3.25). □

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