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FEKETE–SZEGŐ INEQUALITY FOR CLASSES OF ANALYTIC FUNCTIONS CONNECTED WITH THE (p,q)–DERIVATIVE

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Abstract. In this paper, we introduce the new classes $\mathscr{S}^*_{\lambda,p,q}(\eta,\zeta,\varphi)$ and $\mathscr{C}_{\lambda,p,q}(\eta,\zeta,\varphi)$ of analytic functions in the open unit disc, by using the (p,q)-derivative, which are a generalization of the known starlike and convex functions of complex order, respectively. Our aim for these classes is to investigate the Fekete-Szegő inequalities. The various results, which are presented in this paper, would generalize those in related works of several earlier authors.

1. Introduction

Let A denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, by S we shall denote the class of all functions in A which are univalent in $\mathbb{U}$.

For the function f given by (1) and $\zeta \in A$ given by

$$\zeta(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{2}$$

the Hadamard product (or convolution) of f and ζ is defined by

$$(f * \zeta)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (\zeta * f)(z).$$

For $b_n = 1,\ n \geqslant 2$, let $\zeta(z) = I(z)$, then $(f * I)(z) = f(z)$.

The theory of q-calculus plays an important role in many fields of mathematical, physical, and engineering sciences. The first application of the q-calculus was introduced by Jackson in [14, 15]. Recently, there is an extension of q-calculus, denoted by (p,q)-calculus which is obtained by substituting q by q/p in q-calculus. The (p,q)-integer was introduced by Chakrabarti and Jagannathan in [8]. For definitions and properties of the (p,q)-calculus, one may refer to [4, 5, 22].

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For $0<q<p\leqslant 1$, the $(p;q)$-derivative operator for $f*\zeta$ is defined as in [1]:

$$D_{pq}(f*\zeta)(z)=\begin{cases}\frac{(f*\zeta)(pz)-(f*\zeta)(qz)}{(p-q)z}, & \text{if } z\in\mathbb{U}^*:=\mathbb{U}-\{0\};\\ f'(0), & \text{if } z=0.\end{cases} \tag{3}$$

From (3) we deduce that

$$D_{pq}(f*\zeta)(z)=1+\sum_{n=2}^{\infty}[n,p,q]\,a_nb_nz^{n-1} \qquad (z\in\mathbb{U}),$$

where the (p,q)-bracket number is given by

$$\begin{aligned}[n,p,q]&=\frac{p^n-q^n}{p-q}=\sum_{j=0}^{n-1}p^{n-(j+1)}q^j \\ &=p^{n-1}+p^{n-2}q+p^{n-3}q^2+\ldots+q^{n-1} \qquad (0<q<p\leqslant 1),\end{aligned} \tag{4}$$

which is a natural generalization of the q-number. Clearly, we note that $[n,1,q]=[n]_q=\frac{1-q^n}{1-q}$, and $\lim_{q\longrightarrow 1^-}[n,1,q]=n$.

By using (4) the (p,q)-shifted factorial is given by

$$[n,p,q]!=\begin{cases}1, & \text{if } n=0;\\ \prod_{i=1}^{n}[i,p,q], & \text{if } n\in\mathbb{N}:=\{1,2,3,\ldots\},\end{cases}$$

and for any positive number δ, the (p,q)-generalized Pochhammer symbol is defined by

$$[\delta,p,q]_n=\begin{cases}1, & \text{if } n=0;\\ \prod_{i=1}^{n}[\delta+i-1,p,q], & \text{if } n\in\mathbb{N}.\end{cases}$$

For the functions f and ζ are given by (1) and (2), respectively, we define the linear operator $\mathscr{T}_{\zeta}^{\lambda,p,q}:A\to A$ by

$$\mathscr{T}_{\zeta}^{\lambda,p,q}f(z)*\mathscr{M}_{p,q,\lambda+1}=zD_{pq}(f*\zeta)(z) \qquad (\lambda>-1,\ 0<q<p\leqslant 1,\ z\in\mathbb{U}),$$

where the function $\mathscr{M}_{p,q,\lambda+1}$ is given by

$$\mathscr{M}_{p,q,\lambda+1}=z+\sum_{n=2}^{\infty}\frac{[\lambda+1,p,q]_{n-1}}{[n-1,p,q]!}z^n \qquad (\lambda>-1,\ 0<q<p\leqslant 1,\ z\in\mathbb{U}).$$

It is easy to find that

$$\mathscr{T}_{\zeta}^{\lambda,p,q}f(z)=z+\sum_{n=2}^{\infty}\Psi_{n-1}a_nz^n \qquad (\lambda>-1,\ 0<q<p\leqslant 1,\ z\in\mathbb{U}), \tag{5}$$

where

$$\Psi_{n-1} := \frac{[n,p,q]!}{[\lambda+1,p,q]_{n-1}} b_n, \qquad n \geqslant 2. \tag{6}$$

We note that $\mathscr{T}_{\zeta}^{0,1,q} f(z) \longrightarrow z(f*\zeta)'(z)$ as $\lambda = 0,\ p = 1$, and $q \longrightarrow 1^-$, where $(f*\zeta)'$ is the ordinary derivative of the function $f*\zeta$. Also, for $\lambda = b_n = 1$, we have $\mathscr{T}_{I}^{1,p,q} f(z) = f(z)$.

REMARK 1. The linear operator $\mathscr{T}_{\zeta}^{\lambda,p,q}$ is a generalization of many other linear operators considered earlier, we obtain the next special cases:

(i) For $p = 1$, we obtain the operators

$$\mathscr{H}_{\zeta}^{\lambda,q} f(z) := z + \sum_{n=2}^{\infty} \Phi_{n-1} a_n z^n \qquad (\lambda > -1,\ 0 < q < 1,\ z \in \mathbb{U}),$$

where

$$\Phi_{n-1} = \frac{[n,q]!}{[\lambda+1,q]_{n-1}} b_n,$$

and

$$\mathscr{T}_{\zeta}^{\lambda} f(z) := \lim_{q \longrightarrow 1^-} \mathscr{T}_{\zeta}^{\lambda,1,q} f(z) = z + \sum_{n=2}^{\infty} \frac{n!}{(\lambda+1)_{n-1}} a_n b_n z^n \qquad (\lambda > -1,\ z \in \mathbb{U}),$$

where the operators $\mathscr{H}_{\zeta}^{\lambda,q}$ and $\mathscr{T}_{\zeta}^{\lambda}$ were introduced and studied by El-Deeb et al. [11];

(ii) For $p = 1$ and $b_n = \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)}$, $\nu > 0$, $\lambda > -1$, we obtain the operator

$$\mathscr{N}_{\nu,q}^{\lambda} f(z) := z + \sum_{n=2}^{\infty} \frac{[n,q]!}{[\lambda+1,q]_{n-1}} \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)} a_n z^n \qquad (z \in \mathbb{U}),$$

where the operator $\mathscr{N}_{\nu,q}^{\lambda}$ was studied by El-Deeb and Bulboacă [10];

(iii) For $p = 1$ and $b_n = \left(\frac{k+1}{k+n}\right)^{\alpha}$, $\alpha > 0$, $k \geqslant 0$, we obtain the operator

$$\mathscr{M}_{k,q}^{\lambda,\alpha} f(z) := z + \sum_{n=2}^{\infty} \left(\frac{k+1}{k+n}\right)^{\alpha} \frac{[n,q]!}{[\lambda+1,q]_{n-1}} a_n z^n \quad (z \in \mathbb{U}),$$

where the operator $\mathscr{M}_{k,q}^{\lambda,\alpha}$ was studied by El-Deeb and Bulboacă [9];

(iv) For $p = 1$ and $b_n = 1$, we obtain the the operator

$$\mathscr{J}_{q}^{\lambda} f(z) := z + \sum_{n=2}^{\infty} \frac{[n,q]!}{[\lambda+1,q]_{n-1}} a_n z^n \quad (z \in \mathbb{U}),$$

where the operator $\mathscr{J}_{q}^{\lambda}$ was studied by Arif et al. [3];

(v) For $p=1$ and $b_n=\frac{m^{n-1}}{(n-1)!}e^{-m}$, $m>0$, we obtain the q-analogue of Poisson operator:

$$\mathscr{I}_q^{\lambda,m}f(z):=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}\cdot\frac{[n,q]!}{[\lambda+1,q]_{n-1}}a_nz^n \quad (z\in\mathbb{U}),$$

where the operator $\mathscr{I}_q^{\lambda,m}$ was studied by Porwal [18];

(vi) For $p=1$ and $b_n=\left[\frac{1+\ell+\mu(k-1)}{1+\ell}\right]^m$, $m\in\mathbb{Z}$, $\ell>0$, $\mu\geqslant 0$, we obtain the q-analogue of Prajapat operator [19], defined by:

$$\mathscr{J}_{q,\ell,\mu}^{\lambda,m}f(z):=z+\sum_{n=2}^{\infty}\left[\frac{1+\ell+\mu(n-1)}{1+\ell}\right]^m\cdot\frac{[n,q]!}{[\lambda+1,q]_{n-1}}a_nz^n \quad (z\in\mathbb{U}).$$

For two functions f and ζ, which are analytic in $\mathbb{U}$, we say that f is subordinate to ζ, written $f(z)\prec\zeta(z)$ if there exists a Schwarz function s, which (by definition) is analytic in $\mathbb{U}$ with $s(0)=0$ and $|s(z)|<1$ for all $z\in\mathbb{U}$, such that $f(z)=\zeta(s(z))$, $z\in\mathbb{U}$. Furthermore, if the function ζ is univalent in $\mathbb{U}$, then we have the following equivalence, (cf., e.g., [6] , and [17]):

$$f(z)\prec\zeta(z)\Leftrightarrow f(0)=\zeta(0) \text{ and } f(\mathbb{U})\subset\zeta(\mathbb{U}).$$

Ma and Minda [16] unified various subclasses of starlike and convex functions consist of functions $f\in A$, satisfying the subordination $\frac{zf'(z)}{f(z)}\prec\varphi(z)$ and $1+\frac{zf''(z)}{f'(z)}\prec\varphi(z)$ respectively, where the function $\varphi(z)$ is analytic in $\mathbb{U}$ with $Re(\varphi(z))>0$, $\varphi(0)=1$ and $\varphi'(0)>0$. Seoudy and Aouf [23] introduce the classes of q-starlike and q-convex functions of complex order, by using the q-derivative and the principle of subordination. The classes of (p,q)-starlike and (p,q)-convex functions of complex order, by using the (p,q)-derivative in terms of the subordination principle, are defined by Yatkın and Kadıoğlu [27].

Let $\mathscr{P}$ be a subclass of all functions $\varphi\in A$, which are analytic and univalent in $\mathbb{U}$ with $Re(\varphi(z))>0$, $\varphi(0)=1$ and $\varphi'(0)>0$. By using principals of subordination and the (p,q)-derivative, we now introduce the following classes of analytic functions:

DEFINITION 1. A function f given by (1) is said to be in the class $\mathscr{S}^*_{\lambda,p,q}(\eta,\zeta,\varphi)$ $(\lambda>-1,\ 0<q<p\leqslant 1,\ z\in\mathbb{U})$, if it satisfies the following subordination condition:

$$1+\frac{1}{\eta}\left(\frac{zD_{pq}\left(\mathscr{T}_{\zeta}^{\lambda,p,q}f(z)\right)}{\mathscr{T}_{\zeta}^{\lambda,p,q}f(z)}-1\right)\prec\varphi(z), \tag{7}$$

where $\eta\in\mathbb{C}^*:=\mathbb{C}-\{0\}$, $\varphi\in\mathscr{P}$, and ζ is given by (2).

DEFINITION 2. A function f given by (1) is said to be in the class $\mathscr{C}_{\lambda,p,q}(\eta,\zeta,\varphi)$ $(\lambda>-1,\ 0<q<p\leqslant 1,\ z\in\mathbb{U})$, if it satisfies the following subordination condition:

$$1+\frac{1}{\eta}\left(\frac{D_{pq}\left(zD_{pq}\left(\mathscr{T}_{\zeta}^{\lambda,p,q}f(z)\right)\right)}{D_{pq}\left(\mathscr{T}_{\zeta}^{\lambda,p,q}f(z)\right)}-1\right)\prec\varphi(z), \tag{8}$$

where $\eta\in\mathbb{C}^*,\varphi\in\mathscr{P}$, and ζ is given by (2).

We note that:

(i) $\mathscr{S}^*_{1,p,q}(\eta,I,\varphi)=\mathscr{S}^{\eta}_{p,q}(\varphi)$, and $\mathscr{C}_{1,p,q}(\eta,I,\varphi)=\mathscr{C}^{\eta}_{p,q}(\varphi)$ (Yatkın and Kadıoğlu [27]);

(ii) $\mathscr{S}^*_{1,p,q}(1,I,\varphi)=\mathscr{S}^*_{p,q}(\varphi)$, and $\mathscr{C}_{1,p,q}(1,I,\varphi)=\mathscr{C}_{p,q}(\varphi)$ (Srivastava et al. [25]);

(iii) $\mathscr{S}^*_{1,1,q}(\eta,I,\varphi)=\mathscr{S}_{q,\eta}(\varphi)$, and $\mathscr{C}_{1,1,q}(\eta,I,\varphi)=\mathscr{C}_{q,\eta}(\varphi)$ (Seoudy and Aouf [23]);

(iv) $\mathscr{S}^*_{1,1,q}(1,I,\varphi)=\mathscr{S}^*_{q}(\varphi)$, and $\mathscr{C}_{1,1,q}(1,I,\varphi)=\mathscr{C}_{q}(\varphi)$ (Cetinkaya et al. [7]);

(v) $\lim_{q\longrightarrow 1^-}\mathscr{S}^*_{1,1,q}(\eta,I,\varphi)=\mathscr{S}^*_{\eta}(\varphi)$, and $\lim_{q\longrightarrow 1^-}\mathscr{C}_{1,1,q}(\eta,I,\varphi)=\mathscr{C}_{\eta}(\varphi)$ (Ravichandran et al. [20]);

(vi) $\lim_{q\longrightarrow 1^-}\mathscr{S}^*_{1,1,q}(1,I,\varphi)=\mathscr{S}^*(\varphi)$, and $\lim_{q\longrightarrow 1^-}\mathscr{C}_{1,1,q}(1,I,\varphi)=\mathscr{C}(\varphi)$ (Ma and Minda [16]);

(vii) $\lim_{q\longrightarrow 1^-}\mathscr{S}^*_{1,1,q}\left(\eta,I,\frac{1+(1-2\beta)z}{1-z}\right)=\mathscr{S}^*_{\beta}(\eta)$, and $\lim_{q\longrightarrow 1^-}\mathscr{C}_{1,1,q}\left(\eta,I,\frac{1+(1-2\beta)z}{1-z}\right)=\mathscr{C}_{\beta}(\eta)$ $(0\leqslant\beta<1)$ (Frasin [13]);

(viii) $\lim_{q\longrightarrow 1^-}\mathscr{S}^*_{1,1,q}\left(\eta,I,\frac{1+z}{1-z}\right)=\mathscr{S}^*(\eta)$, and $\lim_{q\longrightarrow 1^-}\mathscr{C}_{1,1,q}\left(\eta,I,\frac{1+z}{1-z}\right)=\mathscr{C}(\eta)$ (Wiatrowski [26]);

(ix) $\lim_{q\longrightarrow 1^-}\mathscr{S}^*_{1,1,q}\left(1-\beta,I,\frac{1+z}{1-z}\right)=\mathscr{S}^*(\beta)$, and $\lim_{q\longrightarrow 1^-}\mathscr{C}_{1,1,q}\left(1-\beta,I,\frac{1+z}{1-z}\right)=\mathscr{C}(\beta)$ $(0\leqslant\beta<1)$ (Robertson [21]);

(x) $\lim_{q\longrightarrow 1^-}\mathscr{S}^*_{1,1,q}\left(\eta e^{-i\theta}\cos\theta,I,\frac{1+z}{1-z}\right)=\mathscr{S}^{\theta}(\eta)$, and $\lim_{q\longrightarrow 1^-}\mathscr{C}_{1,1,q}(\eta e^{-i\theta}\cos\theta,I,\frac{1+z}{1-z})=\mathscr{C}^{\theta}(\eta)$ $(|\theta|<\frac{\pi}{2})$ (Al-Oboudi and Haidan [2]).

In order to derive our main results we need to use the following lemmas of Ma and Minda [16]:

LEMMA 1. *If* $p(z)=1+c_1z+c_2z^2+c_3z^3+\ldots$ *is a function with positive real part in* $\mathbb{U}$ *and* $\vartheta\in\mathbb{C}$, *then*

$$\left|c_2-\vartheta c_1^2\right|\leqslant 2\max\{1;|2\vartheta-1|\}.$$

The result is sharp for giving two choices of the function p *as follows:*

$$p(z)=\frac{1+z^2}{1-z^2}\quad \textit{and}\quad p(z)=\frac{1+z}{1-z}.$$

LEMMA 2. *If* $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ *is a function with positive real part in* $\mathbb{U}$ *and* $\vartheta \in \mathbb{C}$*, then*

$$\left|c_2 - \vartheta c_1^2\right| \leqslant \begin{cases} -4\vartheta + 2, & \vartheta \leqslant 0; \\ 2, & 0 \leqslant \vartheta \leqslant 1; \\ 4\vartheta - 2, & \vartheta \geqslant 1. \end{cases}$$

The result is sharp for $\vartheta < 0$ *or* $\vartheta > 1$ *if and only if* $p(z)$ *is* $\frac{1+z}{1-z}$ *or one of its rotations. When* $0 < \vartheta < 1$*, then the equality holds if and only if* $p(z)$ *is* $\frac{1+z^2}{1-z^2}$ *or one of its rotations. Further, the result is sharp for* $\vartheta = 0$*, if and only if* $p(z) = \left(\frac{1+\xi}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\xi}{2}\right)\frac{1-z}{1+z}$ $(0 \leqslant \xi \leqslant 1)$ *or one of its rotations. If* $\vartheta = 1$*, the result is sharp if and only if* $p(z)$ *is the reciprocal of one of the functions such that equality holds in the case of* $\vartheta = 0$.

Also the above upper bound is sharp, and it can be improved as follows when $0 < \vartheta < 1$*:*

$$\left|c_2 - \vartheta c_1^2\right| + \vartheta \left|c_1\right|^2 \leqslant 2 \quad \left(0 < \vartheta \leqslant \frac{1}{2}\right), \tag{9}$$

and

$$\left|c_2 - \vartheta c_1^2\right| + (1-\vartheta)\left|c_1\right|^2 \leqslant 2 \quad \left(\frac{1}{2} \leqslant \vartheta < 1\right). \tag{10}$$

The Feteke-Szegő problem is to find the coefficients estimates for second and third coefficients of functions in any class of analytic function having a specified geometric property [12]. For some history of Feketo-Szegő problem for class of starlike, convex and close-to-convex functions, refer to work produced by by Srivastava et al. [24].

In the present paper, we obtain the Feteke-Szegő inequalities for the classes $\mathscr{S}^*_{\lambda,p,q}(\eta,\zeta,\varphi)$ and $\mathscr{C}_{\lambda,p,q}(\eta,\zeta,\varphi)$. The results presented in this paper would generalize some recent works of [7, 20, 23, 25, 27].

2. Main results

To get our results, we use the similar methods studied by Seoudy and Aouf [23]. Unless otherwise mentioned, we assume throughout this paper that, ζ, $[n,p,q]$ and Ψ_{n-1}, $n \in \{2,3\}$ are given by (2), (4) and (6), respectively, $\varphi \in \mathscr{P}$, $\lambda > -1$, $0 < q < p \leqslant 1$, $\mu \in \mathbb{C}$, $\eta \in \mathbb{C}^*$, and $z \in \mathbb{U}$.

THEOREM 1. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$*, where* $B_1, B_2, \ldots \in \mathbb{R}$ *with* $B_1 \neq 0$*. If the function* f *given by* (1) *belongs to the class* $\mathscr{S}^*_{\lambda,p,q}(\eta,\zeta,\varphi)$*, then*

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{|\eta B_1|}{|[3,p,q]-1|\Psi_2} \times \max\left\{1; \left|\frac{B_2}{B_1} + \frac{\eta B_1}{[2,p,q]-1}\left(1 - \frac{([3,p,q]-1)\Psi_2}{([2,p,q]-1)\Psi_1^2}\mu\right)\right|\right\}. \tag{11}$$

The result is sharp.

Proof. If $f \in \mathscr{S}^{*}_{\lambda,p,q}(\eta,\zeta,\varphi)$, from (7) and the definition of subordination it follows that there exist a Schwarz function u such that

$$1+\frac{1}{\eta}\left(\frac{zD_{pq}\left(\mathscr{T}_{\zeta}^{\lambda,p,q}f(z)\right)}{\mathscr{T}_{\zeta}^{\lambda,p,q}f(z)}-1\right)=\varphi(u(z)). \tag{12}$$

Now, we define the function r with $Re\{r(z)\}>0$ and $r(0)=1$ by:

$$r(z)=\frac{1+u(z)}{1-u(z)}=1+u_1z+u_2z^2+u_3z^3+\ldots. \tag{13}$$

Since u is a Schwarz function.
Therefore,

$$\begin{aligned}\varphi(u(z)) &= \varphi\left(\frac{r(z)-1}{r(z)+1}\right)\\ &= \varphi\left(\frac{u_1}{2}z+\frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)z^2+\ldots\right)\\ &= 1+\frac{B_1u_1}{2}z+\left[\frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)B_1+\frac{1}{4}u_1^2B_2\right]z^2+\ldots.\end{aligned} \tag{14}$$

On the other hand

$$\begin{aligned}&1+\frac{1}{\eta}\left(\frac{zD_{pq}\left(\mathscr{T}_{\zeta}^{\lambda,p,q}f(z)\right)}{\mathscr{T}_{\zeta}^{\lambda,p,q}f(z)}-1\right)\\ &=1+\frac{([2,p,q]-1)\Psi_1}{\eta}a_2z\\ &\quad+\frac{1}{\eta}\left[([3,p,q]-1)\Psi_2a_3-([2,p,q]-1)\Psi_1^2a_2^2\right]z^2+\ldots.\end{aligned} \tag{15}$$

Now, equating the coefficients in (14) and (15), we get

$$\frac{([2,p,q]-1)\Psi_1}{\eta}a_2=\frac{B_1u_1}{2}, \tag{16}$$

$$\frac{1}{\eta}\left[([3,p,q]-1)\Psi_2a_3-([2,p,q]-1)\Psi_1^2a_2^2\right]=\left[\frac{B_1u_2}{2}-\frac{B_1u_1^2}{4}+\frac{B_2u_1^2}{4}\right]. \tag{17}$$

From (16) and (17), we get

$$a_2=\frac{\eta B_1u_1}{2([2,p,q]-1)\Psi_1}, \tag{18}$$

and

$$a_3=\frac{\eta B_1}{2([3,p,q]-1)\Psi_2}\left[u_2-\frac{1}{2}\left(1-\frac{B_2}{B_1}-\frac{\eta B_1}{[2,p,q]-1}\right)u_1^2\right]. \tag{19}$$

Therefore, we have

$$a_3-\mu a_2^2=\frac{\eta B_1}{2\left([3,p,q]-1\right)\Psi_2}\left[u_2-\vartheta u_1^2\right], \tag{20}$$

where

$$\vartheta=\frac{1}{2}\left(1-\frac{B_2}{B_1}-\frac{\eta B_1}{[2,p,q]-1}\left(1-\frac{\left([3,p,q]-1\right)\Psi_2}{\left([2,p,q]-1\right)\Psi_1^2}\mu\right)\right). \tag{21}$$

Hence, by applying Lemma 1, we get the Feteke-Szegő inequality, given by (11), for the class $\mathscr{S}^*_{\lambda,p,q}(\eta,\zeta,\varphi)$.

The result is sharp for the function $r(z)=\frac{1+z}{1-z}=1+2z+2z^2+\ldots$, and (12) gives

$$\begin{aligned}1+\frac{1}{\eta}\left(\frac{zD_{pq}\left(\mathscr{T}_\zeta^{\lambda,p,q}f(z)\right)}{\mathscr{T}_\zeta^{\lambda,p,q}f(z)}-1\right)&=\varphi(z) \\ &=1+B_1z+B_2z^2+\ldots.\end{aligned} \tag{22}$$

By comparing (14) and (22), we have $u_1=2$ and $u_2=2$, and (20) gives the equality sign in the place of the inequality given by (11).

Similarly, for $r(z)=\frac{1+z^2}{1-z^2}=1+2z^2+\ldots$, and (12) gives

$$\begin{aligned}1+\frac{1}{\eta}\left(\frac{zD_{pq}\left(\mathscr{T}_\zeta^{\lambda,p,q}f(z)\right)}{\mathscr{T}_\zeta^{\lambda,p,q}f(z)}-1\right)&=\varphi\left(z^2\right) \\ &=1+B_1z^2+\ldots.\end{aligned} \tag{23}$$

By comparing (14) and (23), we have $u_1=0$ and $u_2=2$, then (20) gives the equality sign in the place of the inequality given by (11). This completes the proof of Theorem 1. □

Similarly, we can investigate the Feteke-Szegő inequality for the class $\mathscr{C}_{\lambda,p,q}(\eta,\zeta,\varphi)$ in the following theorem:

THEOREM 2. *Let $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\ldots$, where $B_1,B_2,\ldots\in\mathbb{R}$ with $B_1\neq 0$. If the function f given by (1) belongs to the class $\mathscr{C}_{\lambda,p,q}(\eta,\zeta,\varphi)$, then*

$$\begin{aligned}\left|a_3-\mu a_2^2\right|&\leqslant\frac{|\eta B_1|}{[3,p,q]\,|[3,p,q]-1|\,\Psi_2} \\ &\quad\times\max\left\{1;\left|\frac{B_2}{B_1}+\frac{\eta B_1}{[2,p,q]-1}\left(1-\frac{[3,p,q]\left([3,p,q]-1\right)\Psi_2}{[2,p,q]^2\left([2,p,q]-1\right)\Psi_1^2}\mu\right)\right|\right\}.\end{aligned} \tag{24}$$

The result is sharp.

Taking $\lambda=b_n=1$ in Theorem 1, we get the following corollary which obtained by Yatkın and Kadıoğlu [[27], Theorem 4]:

COROLLARY 1. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots,$ *where* $B_1, B_2, \ldots \in \mathbb{R}$ *with* $B_1 \neq 0$. *If the function* f *given by* (1) *belongs to the class* $\mathscr{S}_{p,q}^{\eta}(\varphi)$, *then*

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{|\eta B_1|}{|[3,p,q]-1|} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{\eta B_1}{[2,p,q]-1}\left(1 - \frac{[3,p,q]-1}{[2,p,q]-1}\mu\right)\right|\right\}.$$

The result is sharp.

Taking $p = 1$, and $q \longrightarrow 1^-$ in Corollary 1, we obtain the following result which improves the result of Ravichandran et al. [[20], Theorem 4.1]:

COROLLARY 2. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots,$ *where* $B_1, B_2, \ldots \in \mathbb{R}$ *with* $B_1 \neq 0$. *If the function* f *given by* (1) *belongs to the class* $\mathscr{S}_{\eta}^{*}(\varphi)$, *then*

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{|\eta B_1|}{2} \max\left\{1; \left|\frac{B_2}{B_1} + (1 - 2\mu)\eta B_1\right|\right\}.$$

The result is sharp.

Taking $\lambda = b_n = 1$ in Theorem 2, we get the following corollary which obtained by Yatkın and Kadıoğlu [[27], Theorem 5]:

COROLLARY 3. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots,$*where* $B_1, B_2, \ldots \in \mathbb{R}$ *with* $B_1 \neq 0$. *If the function* f *given by* (1) *belongs to the class* $\mathscr{C}_{p,q}^{\eta}(\varphi)$, *then*

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| \leqslant &\frac{|\eta B_1|}{[3,p,q]\,|[3,p,q]-1|} \\ &\times \max\left\{1; \left|\frac{B_2}{B_1} + \frac{\eta B_1}{[2,p,q]-1}\left(1 - \frac{[3,p,q]([3,p,q]-1)}{[2,p,q]^2([2,p,q]-1)}\mu\right)\right|\right\}.\end{aligned}$$

The result is sharp.

REMARK 2. (i) Taking $\eta = 1$ in Corollary 1, we obtain the result obtained by Srivastava et al. [[25], Theorem 2.1];

(ii) Taking $p = 1$ in Corollary 1, we obtain the result obtained by Seoudy and Aouf [[23], Theorem 1];

(iii) Taking $\eta = 1$ in Corollary 3, we obtain the result obtained by Srivastava et al. [[25], Theorem 2.2];

(iv) Taking $p = 1$ in Corollary 3, we obtain the result obtained by Seoudy and Aouf [[23], Theorem 2];

(v) Taking $p = \eta = 1$ in Corollary 1, we obtain the result obtained by Cetinkaya et al. [[7], Theorem 3];

(vi) Taking $p=\eta=1$ in Corollary 3, we obtain the result obtained by Cetinkaya et al. [[7], Theorem 6].

Now, we estimate the coefficient bounds for the coefficients of z and z^2 of the functions belonging to the class $\mathscr{S}^*_{\lambda,p,q}(\eta,\zeta,\varphi)$:

THEOREM 3. *Let* $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\ldots$ *with* $B_1>0$ *and* $B_2\geqslant 0$. *Let*

$$\sigma_1=\frac{([2,p,q]-1)\Psi_1^2\left[\eta B_1^2+([2,p,q]-1)(B_2-B_1)\right]}{([3,p,q]-1)\Psi_2\eta B_1^2}, \tag{25}$$

$$\sigma_2=\frac{([2,p,q]-1)\Psi_1^2\left[\eta B_1^2+([2,p,q]-1)(B_2+B_1)\right]}{([3,p,q]-1)\Psi_2\eta B_1^2}, \tag{26}$$

$$\sigma_3=\frac{([2,p,q]-1)\Psi_1^2\left[\eta B_1^2+([2,p,q]-1)B_2\right]}{([3,p,q]-1)\Psi_2\eta B_1^2}. \tag{27}$$

If the function f *given by* (1) *belongs to the class* $\mathscr{S}^*_{\lambda,p,q}(\eta,\zeta,\varphi)$ *with* $\eta>0$, *then*

$$\left|a_3-\mu a_2^2\right|\leqslant\begin{cases}\frac{\eta B_2}{([3,p,q]-1)\Psi_2}+\frac{\eta^2B_1^2}{([2,p,q]-1)([3,p,q]-1)\Psi_2}\left(1-\frac{([3,p,q]-1)\Psi_2}{([2,p,q]-1)\Psi_1^2}\mu\right) & \text{if} \quad \mu\leqslant\sigma_1;\\ \frac{\eta B_1}{([3,p,q]-1)\Psi_2} & \text{if} \quad \sigma_1\leqslant\mu\leqslant\sigma_2;\\ -\frac{\eta B_2}{([3,p,q]-1)\Psi_2}-\frac{\eta^2B_1^2}{([2,p,q]-1)([3,p,q]-1)\Psi_2}\left(1-\frac{([3,p,q]-1)\Psi_2}{([2,p,q]-1)\Psi_1^2}\mu\right) & \text{if} \quad \mu\geqslant\sigma_2.\end{cases} \tag{28}$$

Further, if $\sigma_1<\mu\leqslant\sigma_3$, *then*

$$\left|a_3-\mu a_2^2\right|+\frac{([2,p,q]-1)^2\Psi_1^2}{([3,p,q]-1)\Psi_2\eta B_1^2}\left[\frac{B_1-B_2}{-\frac{\eta B_1^2}{[2,p,q]-1}\left(1-\frac{([3,p,q]-1)\Psi_2}{([2,p,q]-1)\Psi_1^2}\mu\right)}\right]|a_2|^2 \leqslant\frac{\eta B_1}{([3,p,q]-1)\Psi_2}, \tag{29}$$

and if $\sigma_3\leqslant\mu<\sigma_2$, *then*

$$\left|a_3-\mu a_2^2\right|+\frac{([2,p,q]-1)^2\Psi_1^2}{([3,p,q]-1)\Psi_2\eta B_1^2}\left[\frac{B_2+B_1}{+\frac{\eta B_1^2}{([2,p,q]-1)}\left(1-\frac{([3,p,q]-1)\Psi_2}{([2,p,q]-1)\Psi_1^2}\mu\right)}\right]|a_2|^2 \leqslant\frac{\eta B_1}{([3,p,q]-1)\Psi_2}. \tag{30}$$

The result is sharp.

Proof. If $\vartheta\leqslant 0$, then (21), gives

$$\mu\leqslant\frac{([2,p,q]-1)\Psi_1^2\left[\eta B_1^2+([2,p,q]-1)(B_2-B_1)\right]}{([3,p,q]-1)\Psi_2\eta B_1^2}.$$

Let $\sigma_1 = \frac{([2,p,q]-1)\Psi_1^2\left[\eta B_1^2+([2,p,q]-1)(B_2-B_1)\right]}{([3,p,q]-1)\Psi_2\eta B_1^2}$, then from the above relation, we have $\mu \leqslant \sigma_1$.

Applying Lemma 2 to (20) and (21), we obtain

$$\left|a_3-\mu a_2^2\right| \leqslant \frac{\eta B_1}{([3,p,q]-1)\Psi_2}\left[\frac{B_2}{B_1}+\frac{\eta B_1}{([2,p,q]-1)}\left(1-\frac{([3,p,q]-1)\Psi_2}{([2,p,q]-1)\Psi_1^2}\mu\right)\right], \tag{31}$$

where $\mu \leqslant \sigma_1$.

Simplifying the Inequality (31), we get the first inequality of Assertion (28).

Again, if we take $0 \leqslant \vartheta \leqslant 1$, then (21), gives

$$\sigma_1 \leqslant \mu \leqslant \frac{([2,p,q]-1)\Psi_1^2\left[\eta B_1^2+([2,p,q]-1)(B_2+B_1)\right]}{([3,p,q]-1)\Psi_2\eta B_1^2},$$

where σ_1 is given by (25).

Let $\sigma_2 = \frac{([2,p,q]-1)\Psi_1^2\left[\eta B_1^2+([2,p,q]-1)(B_2+B_1)\right]}{([3,p,q]-1)\Psi_2\eta B_1^2}$, then from the above relation, we have $\sigma_1 \leqslant \mu \leqslant \sigma_2$.

Now, using Lemma 2 for $0 \leqslant \vartheta \leqslant 1$ in (20), we obtain the following inequality, which gives the second inequality of Assertion (28):

$$\left|a_3-\mu a_2^2\right| \leqslant \frac{\eta B_1}{([3,p,q]-1)\Psi_2}, \tag{32}$$

where $\sigma_1 \leqslant \mu \leqslant \sigma_2$.

Next, if we take $\vartheta \geqslant 1$, then (21), gives that $\mu \geqslant \sigma_2$, where σ_2 is given by (26).

Now, applying Lemma 2 to (20) and (21), we obtain

$$\left|a_3-\mu a_2^2\right| \leqslant \frac{\eta B_1}{([3,p,q]-1)\Psi_2}\left[-\frac{B_2}{B_1}-\frac{\eta B_1}{([2,p,q]-1)}\left(1-\frac{([3,p,q]-1)\Psi_2}{([2,p,q]-1)\Psi_1^2}\mu\right)\right]. \tag{33}$$

Simplifying the above inequality, we obtain the third inequality of Assertion (28).

Further, if $0 < \vartheta \leqslant \frac{1}{2}$, then using (21), we have

$$\sigma_1 < \mu \leqslant \frac{([2,p,q]-1)\Psi_1^2\left[\eta B_1^2+([2,p,q]-1)B_2\right]}{([3,p,q]-1)\Psi_2\eta B_1^2},$$

where σ_1 is given by (25).

Let $\sigma_3 = \frac{([2,p,q]-1)\Psi_1^2\left[\eta B_1^2+([2,p,q]-1)B_2\right]}{([3,p,q]-1)\Psi_2\eta B_1^2}$, then from the above relation, we have $\sigma_1 < \mu \leqslant \sigma_3$.

Now, using (18), (20) and (25), we get

$$\left|a_3-\mu a_2^2\right|+(\mu-\sigma_1)\left|a_2\right|^2 \tag{34}$$

$$=\frac{\eta B_1}{2([3,p,q]-1)\Psi_2}\left[\begin{array}{c}\left[u_2-\vartheta u_1^2\right]\\ +\frac{1}{2}\left(\mu-\frac{([2,p,q]-1)\Psi_1^2\left[\eta B_1^2-([2,p,q]-1)(B_2-B_1)\right]}{([3,p,q]-1)\Psi_2\eta B_1^2}\right)\frac{([3,p,q]-1)\Psi_2\eta B_1}{([2,p,q]-1)^2\Psi_1^2}u_1^2\end{array}\right]$$

$$=\frac{\eta B_1}{2([3,p,q]-1)\Psi_2}\left[\begin{array}{c}\left[u_2-\vartheta u_1^2\right]\\ +\frac{1}{2}\left(1-\frac{B_2}{B_1}-\frac{\eta B_1}{([2,p,q]-1)}\left(1-\mu\frac{([3,p,q]-1)\Psi_2}{([2,p,q]-1)\Psi_1^2}\right)\right)u_1^2\end{array}\right]$$

$$=\frac{\eta B_1}{2([3,p,q]-1)\Psi_2}\left[\left[u_2-\vartheta u_1^2\right]+\vartheta u_1^2\right].$$

Since $0<\vartheta\leqslant\frac{1}{2}$, therefore using Inequality (9) of Lemma 2 in (34), we get

$$\left|a_3-\mu a_2^2\right|+(\mu-\sigma_1)\left|a_2\right|^2\leqslant\frac{\eta B_1}{([3,p,q]-1)\Psi_2}. \tag{35}$$

Using (25) in (35) and then simplifying, we obtain the following inequality, which gives Assertion (29):

$$\begin{aligned}&\left|a_3-\mu a_2^2\right|+\frac{([2,p,q]-1)^2\Psi_1^2}{([3,p,q]-1)\Psi_2\eta B_1^2}\\ &\times\left[B_1-B_2-\frac{\eta B_1^2}{[2,p,q]-1}\left(1-\frac{([3,p,q]-1)\Psi_2}{([2,p,q]-1)\Psi_1^2}\mu\right)\right]\left|a_2\right|^2\\ \leqslant&\frac{\eta B_1}{([3,p,q]-1)\Psi_2}.\end{aligned}$$

Similarly, if we take $\frac{1}{2}\leqslant\vartheta<1$, then (21), gives that $\sigma_3\leqslant\mu<\sigma_2$, where σ_2 and σ_3 are given by (26) and (27), respectively.

Using (18), (20) and (26), we get

$$\left|a_3-\mu a_2^2\right|+(\sigma_2-\mu)\left|a_2\right|^2 \tag{36}$$

$$=\frac{\eta B_1}{2([3,p,q]-1)\Psi_2}\left[\begin{array}{c}\left[u_2-\vartheta u_1^2\right]\\ +\frac{1}{2}\left(\frac{([2,p,q]-1)\Psi_1^2\left[\eta B_1^2-([2,p,q]-1)(B_2+B_1)\right]}{([3,p,q]-1)\Psi_2\eta B_1^2}-\mu\right)\frac{([3,p,q]-1)\Psi_2\eta B_1}{([2,p,q]-1)^2\Psi_1^2}u_1^2\end{array}\right]$$

$$=\frac{\eta B_1}{2([3,p,q]-1)\Psi_2}\left[\begin{array}{c}\left[u_2-\vartheta u_1^2\right]\\ +\frac{1}{2}\left(1+\frac{B_2}{B_1}+\frac{\eta B_1}{([2,p,q]-1)}\left(1-\mu\frac{([3,p,q]-1)\Psi_2}{([2,p,q]-1)\Psi_1^2}\right)\right)u_1^2\end{array}\right]$$

$$=\frac{\eta B_1}{2([3,p,q]-1)\Psi_2}\left[\left[u_2-\vartheta u_1^2\right]+(1-\vartheta)u_1^2\right].$$

Now, since $\frac{1}{2}\leqslant\vartheta<1$, therefore using Inequality (9) of Lemma 2 in (36), we get

$$\left|a_3-\mu a_2^2\right|+(\sigma_2-\mu)\left|a_2\right|^2\leqslant\frac{\eta B_1}{([3,p,q]-1)\Psi_2}. \tag{37}$$

Using (26) in (37), we get

$$\left|a_3-\mu a_2^2\right|+\left(\frac{([2,p,q]-1)\Psi_1^2\left[\eta B_1^2-([2,p,q]-1)(B_2+B_1)\right]}{([3,p,q]-1)\Psi_2\eta B_1^2}-\mu\right)|a_2|^2$$
$$\leqslant \frac{\eta B_1}{([3,p,q]-1)\Psi_2}.$$

Finally, on simplifying the above inequality, we obtain the Assertion (30).

To show that the bounds are sharp, we define the functions $\mathscr{X}_{\varphi_n}, (n=2,3,4,\dots)$ by

$$1+\frac{1}{\eta}\left(\frac{zD_{pq}\left(\mathscr{T}_{\zeta}^{\lambda,p,q}\mathscr{X}_{\varphi_n}(z)\right)}{\mathscr{T}_{\zeta}^{\lambda,p,q}\mathscr{X}_{\varphi_n}(z)}-1\right)=\varphi\left(z^{n-1}\right),\quad \mathscr{X}_{\varphi_n}(0)=0=\mathscr{X}'_{\varphi_n}(0)-1,$$

and the functions $\mathscr{K}_{\xi}$ and $\mathscr{L}_{\xi}$ $(0\leqslant\xi\leqslant 1)$ by

$$1+\frac{1}{\eta}\left(\frac{zD_{pq}\left(\mathscr{T}_{\zeta}^{\lambda,p,q}\mathscr{K}_{\xi}(z)\right)}{\mathscr{T}_{\zeta}^{\lambda,p,q}\mathscr{K}_{\xi}(z)}-1\right)=\varphi\left(\frac{z(z+\xi)}{1+\xi z}\right),\quad \mathscr{K}_{\xi}(0)=0=\mathscr{K}'_{\xi}(0)-1,$$

and

$$1+\frac{1}{\eta}\left(\frac{zD_{pq}\left(\mathscr{T}_{\zeta}^{\lambda,p,q}\mathscr{L}_{\xi}(z)\right)}{\mathscr{T}_{\zeta}^{\lambda,p,q}\mathscr{L}_{\xi}(z)}-1\right)=\varphi\left(-\frac{1+\xi z}{z(z+\xi)}\right),\quad \mathscr{L}_{\xi}(0)=0=\mathscr{L}'_{\xi}(0)-1.$$

Clearly, the functions $\mathscr{X}_{\varphi_n}$, $\mathscr{K}_{\xi}$ and $\mathscr{L}_{\xi}$ belongs to the class $\mathscr{S}^*_{\lambda,p,q}(\eta,\zeta,\varphi)$. The result is sharp for $\mu<\sigma_1$ or $\mu>\sigma_2$ if and only if f is $\mathscr{X}_{\varphi_2}$ or one of its rotations. When $\sigma_1<\mu<\sigma_2$, then the equality holds if and only if f is $\mathscr{X}_{\varphi_3}$ or one of its rotations.

Further, the result is sharp for $\mu=\sigma_1$, if and only if f is $\mathscr{K}_{\xi}$ or one of its rotations. If $\mu=\sigma_2$, then the equality holds if and only if f is $\mathscr{L}_{\xi}$ or one of its rotations. □

Similarly, we can estimate the coefficient bounds for the coefficients of z and z^2 of the functions belonging to the class $\mathscr{C}_{\lambda,p,q}(\eta,\zeta,\varphi)$:

THEOREM 4. *Let* $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\dots$ *with* $B_1>0$ *and* $B_2\geqslant 0$. *Let*

$$\chi_1=\frac{[2,p,q]^2([2,p,q]-1)\Psi_1^2\left[\eta B_1^2+([2,p,q]-1)(B_2-B_1)\right]}{[3,p,q]([3,p,q]-1)\Psi_2\eta B_1^2}, \tag{38}$$

$$\chi_2=\frac{[2,p,q]^2([2,p,q]-1)\Psi_1^2\left[\eta B_1^2+([2,p,q]-1)(B_2+B_1)\right]}{[3,p,q]([3,p,q]-1)\Psi_2\eta B_1^2}, \tag{39}$$

$$\chi_3=\frac{[2,p,q]^2([2,p,q]-1)\Psi_1^2\left[\eta B_1^2+([2,p,q]-1)B_2\right]}{[3,p,q]([3,p,q]-1)\Psi_2\eta B_1^2}. \tag{40}$$

If the function f *given by* (1) *belongs to the class* $\mathscr{C}_{\lambda,p,q}(\eta,\zeta,\varphi)$ *with* $\eta>0$, *then*

$$\left|a_3-\mu a_2^2\right| \leqslant \begin{cases} \dfrac{\frac{\eta B_2}{[3,p,q]([3,p,q]-1)\Psi_2}+\frac{\eta^2 B_1^2}{([2,p,q]-1)[3,p,q]([3,p,q]-1)\Psi_2}}{\left(1-\frac{[3,p,q]([3,p,q]-1)\Psi_2}{[2,p,q]^2([2,p,q]-1)\Psi_1^2}\mu\right)} & \text{if} \quad \mu \leqslant \chi_1; \\ \dfrac{\eta B_1}{[3,p,q]([3,p,q]-1)\Psi_2} & \text{if} \ \chi_1 \leqslant \mu \leqslant \chi_2; \\ \dfrac{-\frac{\eta B_2}{[3,p,q]([3,p,q]-1)\Psi_2}-\frac{\eta^2 B_1^2}{([2,p,q]-1)[3,p,q]([3,p,q]-1)\Psi_2}}{\left(1-\frac{[3,p,q]([3,p,q]-1)\Psi_2}{[2,p,q]^2([2,p,q]-1)\Psi_1^2}\mu\right)} & \text{if} \quad \mu \geqslant \chi_2. \end{cases} \tag{41}$$

Further, if $\chi_1<\mu\leqslant\chi_3$, *then*

$$\begin{aligned} &\left|a_3-\mu a_2^2\right|+\frac{[2,p,q]^2\left([2,p,q]-1\right)^2\Psi_1^2}{[3,p,q]\left([3,p,q]-1\right)\Psi_2\eta B_1^2} \\ &\times\left[\begin{array}{c} B_1-B_2 \\ -\frac{\eta B_1^2}{[2,p,q]-1}\left(1-\frac{[3,p,q]([3,p,q]-1)\Psi_2}{[2,p,q]^2([2,p,q]-1)\Psi_1^2}\mu\right) \end{array}\right]\left|a_2\right|^2 \\ &\leqslant \frac{\eta B_1}{[3,p,q]\left([3,p,q]-1\right)\Psi_2}, \end{aligned} \tag{42}$$

and if $\chi_3\leqslant\mu<\chi_2$, *then*

$$\begin{aligned} &\left|a_3-\mu a_2^2\right|+\frac{[2,p,q]^2\left([2,p,q]-1\right)^2\Psi_1^2}{[3,p,q]\left([3,p,q]-1\right)\Psi_2\eta B_1^2} \\ &\times\left[\begin{array}{c} B_2+B_1 \\ +\frac{\eta B_1^2}{[2,p,q]-1}\left(1-\frac{[3,p,q]([3,p,q]-1)\Psi_2}{[2,p,q]^2([2,p,q]-1)\Psi_1^2}\mu\right) \end{array}\right]\left|a_2\right|^2 \\ &\leqslant \frac{\eta B_1}{[3,p,q]\left([3,p,q]-1\right)\Psi_2}. \end{aligned} \tag{43}$$

The result is sharp.

Taking $\lambda=b_n=\eta=1$ in Theorem 3, we get the following corollary which obtained by Srivastava et al. [[25], Theorem 3.1]:

COROLLARY 4. *Let* $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\ldots$ *with* $B_1>0$ *and* $B_2\geqslant0$. *Let*

$$\sigma_1=\frac{([2,p,q]-1)\left[B_1^2+([2,p,q]-1)(B_2-B_1)\right]}{([3,p,q]-1)B_1^2},$$

$$\sigma_2=\frac{([2,p,q]-1)\left[B_1^2+([2,p,q]-1)(B_2+B_1)\right]}{([3,p,q]-1)B_1^2},$$

$$\sigma_3 = \frac{([2,p,q]-1)\left[B_1^2 + ([2,p,q]-1)B_2\right]}{([3,p,q]-1)B_1^2}.$$

If the function f given by (1) belongs to the class $\mathscr{S}^*_{p,q}(\varphi)$, then

$$\left|a_3-\mu a_2^2\right| \leqslant \begin{cases} \frac{B_2}{[3,p,q]-1} + \frac{B_1^2}{([2,p,q]-1)([3,p,q]-1)}\left(1-\frac{[3,p,q]-1}{[2,p,q]-1}\mu\right) & \text{if } \mu \leqslant \sigma_1; \\ \frac{B_1}{[3,p,q]-1} & \text{if } \sigma_1 \leqslant \mu \leqslant \sigma_2; \\ -\frac{B_2}{[3,p,q]-1} - \frac{B_1^2}{([2,p,q]-1)([3,p,q]-1)}\left(1-\frac{[3,p,q]-1}{[2,p,q]-1}\mu\right) & \text{if } \mu \geqslant \sigma_2. \end{cases}$$

Further, if $\sigma_1 < \mu \leqslant \sigma_3$, then

$$\left|a_3-\mu a_2^2\right| + \frac{([2,p,q]-1)^2}{([3,p,q]-1)B_1^2}\left[B_1 - B_2 - \frac{B_1^2}{[2,p,q]-1}\left(1-\frac{[3,p,q]-1}{[2,p,q]-1}\mu\right)\right]|a_2|^2 \leqslant \frac{B_1}{[3,p,q]-1},$$

and if $\sigma_3 \leqslant \mu < \sigma_2$, then

$$\left|a_3-\mu a_2^2\right| + \frac{([2,p,q]-1)^2}{([3,p,q]-1)B_1^2}\left[B_2 + B_1 + \frac{B_1^2}{[2,p,q]-1}\left(1-\frac{[3,p,q]-1}{[2,p,q]-1}\mu\right)\right]|a_2|^2 \leqslant \frac{B_1}{[3,p,q]-1}.$$

The result is sharp.

Taking $\lambda = b_n = p = 1$ in Theorem 3, we get the following corollary which obtained by Seoudy and Aouf [[23], Theorem 3]:

COROLLARY 5. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$ with $B_1 > 0$ and $B_2 \geqslant 0$. Let*

$$\sigma_1 = \frac{\left([2]_q-1\right)\left[\eta B_1^2 + \left([2]_q-1\right)(B_2-B_1)\right]}{\left([3]_q-1\right)\eta B_1^2},$$

$$\sigma_2 = \frac{\left([2]_q-1\right)\left[\eta B_1^2 + \left([2]_q-1\right)(B_2+B_1)\right]}{\left([3]_q-1\right)\eta B_1^2},$$

$$\sigma_3 = \frac{\left([2]_q-1\right)\left[\eta B_1^2 + \left([2]_q-1\right)B_2\right]}{\left([3]_q-1\right)\eta B_1^2}.$$

If the function f given by (1) belongs to the class $\mathscr{S}_{q,\eta}(\varphi)$ with $\eta > 0$, then

$$\left|a_3-\mu a_2^2\right| \leqslant \begin{cases} \frac{\eta B_2}{[3]_q-1} + \frac{\eta^2 B_1^2}{\left([2]_q-1\right)\left([3]_q-1\right)}\left(1-\frac{[3]_q-1}{[2]_q-1}\mu\right) & \text{if } \mu \leqslant \sigma_1; \\ \frac{\eta B_1}{[3]_q-1} & \text{if } \sigma_1 \leqslant \mu \leqslant \sigma_2; \\ -\frac{\eta B_2}{[3]_q-1} - \frac{\eta^2 B_1^2}{\left([2]_q-1\right)\left([3]_q-1\right)}\left(1-\frac{[3]_q-1}{[2]_q-1}\mu\right) & \text{if } \mu \geqslant \sigma_2. \end{cases}$$

Further, if $\sigma_1 < \mu \leqslant \sigma_3$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{\left([2]_q - 1\right)^2}{\left([3]_q - 1\right)\eta B_1^2}\left[B_1 - B_2 - \frac{\eta B_1^2}{[2]_q - 1}\left(1 - \frac{[3]_q - 1}{[2]_q - 1}\mu\right)\right]|a_2|^2 \leqslant \frac{\eta B_1}{[3]_q - 1},$$

and if $\sigma_3 \leqslant \mu < \sigma_2$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{\left([2]_q - 1\right)^2}{\left([3]_q - 1\right)\eta B_1^2}\left[B_2 + B_1 + \frac{\eta B_1^2}{[2]_q - 1}\left(1 - \frac{[3]_q - 1}{[2]_q - 1}\mu\right)\right]|a_2|^2 \leqslant \frac{\eta B_1}{[3]_q - 1}.$$

The result is sharp.

Taking $\lambda = b_n = \eta = 1$ in Theorem 4, we get the following corollary which obtained by Srivastava et al. [[25], Theorem 3.2]:

COROLLARY 6. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$ with $B_1 > 0$ and $B_2 \geqslant 0$. Let*

$$\chi_1 = \frac{[2,p,q]^2\left([2,p,q]-1\right)\left[B_1^2 + \left([2,p,q]-1\right)\left(B_2 - B_1\right)\right]}{[3,p,q]\left([3,p,q]-1\right)B_1^2},$$

$$\chi_2 = \frac{[2,p,q]^2\left([2,p,q]-1\right)\left[B_1^2 + \left([2,p,q]-1\right)\left(B_2 + B_1\right)\right]}{[3,p,q]\left([3,p,q]-1\right)B_1^2},$$

$$\chi_3 = \frac{[2,p,q]^2\left([2,p,q]-1\right)\left[B_1^2 + \left([2,p,q]-1\right)B_2\right]}{[3,p,q]\left([3,p,q]-1\right)B_1^2}.$$

If the function f given by (1) belongs to the class $\mathscr{C}_{p,q}(\varphi)$, then

$$\left|a_3 - \mu a_2^2\right| \leqslant \begin{cases} \frac{B_2}{[3,p,q]([3,p,q]-1)} + \frac{B_1^2}{([2,p,q]-1)[3,p,q]([3,p,q]-1)}\left(1 - \frac{[3,p,q]([3,p,q]-1)}{[2,p,q]^2([2,p,q]-1)}\mu\right) & \text{if } \mu \leqslant \chi_1; \\ \frac{B_1}{[3,p,q]([3,p,q]-1)} & \text{if } \chi_1 \leqslant \mu \leqslant \chi_2; \\ -\frac{B_2}{[3,p,q]([3,p,q]-1)} - \frac{B_1^2}{([2,p,q]-1)[3,p,q]([3,p,q]-1)}\left(1 - \frac{[3,p,q]([3,p,q]-1)}{[2,p,q]^2([2,p,q]-1)}\mu\right) & \text{if } \mu \geqslant \chi_2. \end{cases}$$

Further, if $\chi_1 < \mu \leqslant \chi_3$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{[2,p,q]^2\left([2,p,q]-1\right)^2}{[3,p,q]\left([3,p,q]-1\right)B_1^2}\left[\begin{array}{c} B_1 - B_2 \\ -\frac{B_1^2}{[2,p,q]-1}\left(1 - \frac{[3,p,q]([3,p,q]-1)}{[2,p,q]^2([2,p,q]-1)}\mu\right)\end{array}\right]|a_2|^2$$
$$\leqslant \frac{B_1}{[3,p,q]\left([3,p,q]-1\right)},$$

and if $\chi_3 \leqslant \mu < \chi_2$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{[2,p,q]^2\left([2,p,q]-1\right)^2}{[3,p,q]\left([3,p,q]-1\right)B_1^2}\left[\begin{array}{c} B_2 + B_1 \\ +\frac{B_1^2}{([2,p,q]-1)}\left(1-\frac{[3,p,q]([3,p,q]-1)}{[2,p,q]^2([2,p,q]-1)}\mu\right)\end{array}\right]\left|a_2\right|^2$$
$$\leqslant \frac{B_1}{[3,p,q]\left([3,p,q]-1\right)}.$$

The result is sharp.

Taking $\lambda = b_n = p = 1$ in Theorem 4, we obtain the following result which improves the result of Seoudy and Aouf [[23], Theorem 4]:

COROLLARY 7. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$ *with* $B_1 > 0$ *and* $B_2 \geqslant 0$. *Let*

$$\chi_1 = \frac{[2]_q^2\left([2]_q-1\right)\left[\eta B_1^2 + \left([2]_q-1\right)(B_2-B_1)\right]}{[3]_q\left([3]_q-1\right)\eta B_1^2},$$

$$\chi_2 = \frac{[2]_q^2\left([2]_q-1\right)\left[\eta B_1^2 + \left([2]_q-1\right)(B_2+B_1)\right]}{[3]_q\left([3]_q-1\right)\eta B_1^2},$$

$$\chi_3 = \frac{[2]_q^2\left([2]_q-1\right)\left[\eta B_1^2 + \left([2]_q-1\right)B_2\right]}{[3]_q\left([3]_q-1\right)\eta B_1^2}.$$

If the function f *given by* (1) *belongs to the class* $\mathscr{C}_{q,\eta}(\varphi)$ *with* $\eta > 0$, *then*

$$\left|a_3 - \mu a_2^2\right| \leqslant \begin{cases} \frac{\eta B_2}{[3]_q\left([3]_q-1\right)} + \frac{\eta^2 B_1^2}{[3]_q\left([3]_q-1\right)\left([2]_q-1\right)}\left(1-\frac{[3]_q\left([3]_q-1\right)}{[2]_q^2\left([2]_q-1\right)}\mu\right) & \text{if} \quad \mu \leqslant \chi_1; \\ \frac{\eta B_1}{[3]_q\left([3]_q-1\right)} & \text{if} \ \chi_1 \leqslant \mu \leqslant \chi_2; \\ -\frac{\eta B_2}{[3]_q\left([3]_q-1\right)} - \frac{\eta^2 B_1^2}{[3]_q\left([3]_q-1\right)\left([2]_q-1\right)}\left(1-\frac{[3]_q\left([3]_q-1\right)}{[2]_q^2\left([2]_q-1\right)}\mu\right) & \text{if} \quad \mu \geqslant \chi_2. \end{cases}$$

Further, if $\chi_1 < \mu \leqslant \chi_3$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{[2]_q^2\left([2]_q-1\right)^2}{[3]_q\left([3]_q-1\right)\eta B_1^2}\left[B_1 - B_2 - \frac{\eta B_1^2}{[2]_q-1}\left(1-\frac{[3]_q\left([3]_q-1\right)}{[2]_q^2\left([2]_q-1\right)}\mu\right)\right]\left|a_2\right|^2$$
$$\leqslant \frac{\eta B_1}{[3]_q\left([3]_q-1\right)},$$

and if $\chi_3 \leqslant \mu < \chi_2$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{[2]_q^2\left([2]_q - 1\right)^2}{[3]_q\left([3]_q - 1\right)\eta B_1^2}$$
$$\times\left[B_2 + B_1 + \frac{\eta B_1^2}{[2]_q - 1}\left(1 - \frac{[3]_q\left([3]_q - 1\right)}{[2]_q^2\left([2]_q - 1\right)}\mu\right)\right]|a_2|^2$$
$$\leqslant \frac{\eta B_1}{[3]_q\left([3]_q - 1\right)}.$$

The result is sharp.

Taking $q \longrightarrow 1^-$ in Corollary 7, we obtain the following result which improves the result of Seoudy and Aouf [[23], Corollary 4]:

COROLLARY 8. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$ *with* $B_1 > 0$ *and* $B_2 \geqslant 0$. *Let*

$$\chi_1 = \frac{2\left[\eta B_1^2 + B_2 - B_1\right]}{3\eta B_1^2}, \qquad \chi_2 = \frac{2\left[\eta B_1^2 + B_2 + B_1\right]}{3\eta B_1^2},$$

and

$$\chi_3 = \frac{2\left[\eta B_1^2 + B_2\right]}{3\eta B_1^2}.$$

If the function f *given by* (1) *belongs to the class* $\mathscr{C}_\eta(\varphi)$ *with* $\eta > 0$, *then*

$$\left|a_3 - \mu a_2^2\right| \leqslant \begin{cases} \frac{\eta B_2}{6} + \frac{\eta^2 B_1^2}{6}\left(1 - \frac{3}{2}\mu\right) & \text{if} \quad \mu \leqslant \chi_1; \\ \frac{\eta B_1}{6} & \text{if} \quad \chi_1 \leqslant \mu \leqslant \chi_2; \\ -\frac{\eta B_2}{6} - \frac{\eta^2 B_1^2}{6}\left(1 - \frac{3}{2}\mu\right) & \text{if} \quad \mu \geqslant \chi_2. \end{cases}$$

Further, if $\chi_1 < \mu \leqslant \chi_3$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{2}{3\eta B_1^2}\left[B_1 - B_2 - \eta B_1^2\left(1 - \frac{3}{2}\mu\right)\right]|a_2|^2 \leqslant \frac{\eta B_1}{6},$$

and if $\chi_3 \leqslant \mu < \chi_2$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{2}{3\eta B_1^2}\left[B_2 + B_1 + \eta B_1^2\left(1 - \frac{3}{2}\mu\right)\right]|a_2|^2 \leqslant \frac{\eta B_1}{6}.$$

The result is sharp.

REFERENCES

[1] T. ACAR, A. ARAL AND S. A. MOHIUDDINE, *On Kantorovich modification of (p,q)-Baskakov operators*, J Inequal Appl., **98**, (2016), 1–14.

[2] F. M. AL-OBOUDI AND M. M. HAIDAN, *Spirallike functions of complex order*, J. Natur. Geom., **19**, 1 (2000), 53–72.

[3] M. ARIF, M. U. HAQ AND J. L. LIU, *A subfamily of univalent functions associated with q-analogue of Noor integral operator*, J. Funct. Spaces, **2018** (ID 3818915), 1–5.

[4] S. D. BERNARDI, *Convex and starlike univalent functions*, Trans. Am. Math. Soc., **135**, (2016), 429–446.

[5] J. D. BUKWELI-KYEMBA, M. N. HOUNKONNOU, *Quantum deformed algebras: coherent states and special functions*, arXiv preprint arXiv:1301.0116, (2013).

[6] T. BULBOACA, *Differential Subordinations and Superordinations, Recent Results*, House of Scientific Book Publ., Cluj-Napoca, 2005.

[7] A. CETINKAYA, Y. KAHRAMANER AND Y. POLATOGLU, *Feteke-szegő inequalities for q-starlike and q-convex functions*, Acta Univ. Apulensis, **53**, (2018), 55–64.

[8] R. CHAKRABARTI, R. JAGANNATHAN, *A (p,q)-oscillator realization of two-parameter quantum*, J. Phys. A Math. General, **24**, 13 (1991), L711–L718.

[9] S. M. EL-DEEB AND T. BULBOACĂ, *Differential sandwich-type results for symmetric functions connected with a q-analog integral operator*, Mathematics, **7**, 12 (2019), 1185.

[10] S. M. EL-DEEB AND T. BULBOACĂ, *Fekete-Szegő inequalities for certain class of analytic functions connected with q-analogue of Bessel function*, J. Egypt. Math. Soc., **27**, (2019), 1–11.

[11] S. M. EL-DEEB, T. BULBOACĂ AND B. M. EL-MATARY, *Maclaurin Coefficient Estimates of Bi-Univalent Functions Connected with the q-Derivative*, Mathematics, **8**, 3 (2020), 418.

[12] M. FEKETE AND G. SZEGŐ, *Eine Bemerkung über ungerade schlichte Funktionen*, J. Lond. Math. Soc., **8**, (1933), 85–89.

[13] B. A. FRASIN, *Family of analytic functions of complex order*, Acta Math. Acad. Paedagog. Nyházi. (N. S.), **22**, 2 (2006), 179–191.

[14] F. H. JACKSON, *On q-definite integrals*, Quart. J. Pure Appl. Math., **41**, (1910), 193–203.

[15] F. H. JACKSON, *q-difference equations*, Am. J. Math., **32**, 4 (1910), 305–314.

[16] W. MA AND D. MINDA, *A unified treatment of some special classes of univalent functions*, in Proceedings of the Conference on Complex Analysis, Z. Li, F. Ren, L. Yang, and S. Zhang (Eds), (1992), 157–169.

[17] S. S. MILLER AND P. T. MOCANU, *Differential Subordination: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, 225, Marcel Dekker Inc., New York and Basel, 2000.

[18] S. PORWAL, *An application of a Poisson distribution series on certain analytic functions*, J. Complex Anal., **2014**, (2014), 984135.

[19] J. K. PRAJAPAT, *Subordination and superordination preserving properties for generalized multiplier transformation operator*, Math. Comput. Model., **55**, (2012), 1456–1465.

[20] V. RAVICHANDRAN, Y. POLATOGLU, M. BOLCAL AND A. SEN, *Certain subclasses of starlike and convex functions of complex order*, Hacettepe J. Math. Stat., **34**, (2005), 9–15.

[21] M. S. ROBERTSON, *On the theory of univalent functions*, Ann. Math., **37**, (1936), 374–408.

[22] P. N. SADJANG, *On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas*, arXiv:1309.3934 [math.QA], (2013).

[23] T. M. SEOUDY AND M. K. AOUF, *Coefficient estimates of new classes of q-starlike and q-convex functions of complex order*, J. Math. Inequal, **10**, 1 (2016), 135–145.

[24] H. M. SRIVASTAVA, A. K. MISHRA, AND M. K. DAS, *The fekete-szegő problem for a subclass of close-to-convex functions*, Complex Var. Elliptic Equ., **44**, (2001), 145–163.

[25] H. M. SRIVASTAVA, N. RAZA, E. S. A. ABUJARAD AND M. H. ABUJARAD, *Fekete-Szegő inequality for classes of (p,q)-starlike and (p,q)-convex functions*, RACSAM, **113**, (2019), 3563–3584.

[26] P. WIATROWSKI, *On the coefficients of some family of holomorphic functions*, Zeszyty Nauk. Uniw. Łódz, Nauk. Mat.-Przyr., **39**, (1970), 75–85.

[27] F. YATKIN AND E. KADIOĞLU, *Fekete-Szegö Inequality for (p,q)-Starlike and (p,q)-Convex Functions of Complex Order*, J. Inst. Sci. and Tech., **10**, 2 (2020), 1247–1253.

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