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EXPLICIT EXPRESSIONS FOR SOME LINEAR EULER–TYPE SUMS CONTAINING HARMONIC AND SKEW–HARMONIC NUMBERS

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Abstract. Closed-form expressions for the three general linear Euler-type sums containing the harmonic numbers H_n and the skew-harmonic numbers $\overline{H}_n$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(2n+1)^{2q+1}},\ \sum_{n=1}^{\infty}\frac{\overline{H}_n}{(2n+1)^q},\quad \text{and}\quad \sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{(2n+1)^{2q+1}},$$

where q is a positive integer greater than or equal to zero or two as needed to ensure convergence are given. Closed-form expressions for several other closely related generalised logarithmic integrals and sums are also presented.

1. Introduction

Suppose $n \in \mathbb{Z}_{\geqslant 0}$. The nth harmonic number H_n is defined by $\sum_{k=1}^n \frac{1}{k}$ such that $H_0 \equiv 0$. The nth skew-harmonic number $\overline{H}_n$ is defined by $\sum_{k=1}^n \frac{(-1)^{k+1}}{k}$ such that $\overline{H}_n \equiv 0$. Note the notation we adopt here for the skew-harmonic numbers is that introduced by Flajolet and Salvy [10]. Other notations used for the skew-harmonic numbers include H_n^- by Boyadzhiev [5, 6], L_n by Xu, Zhang, and Zhu [31], and H_n' by Campbell and Sofo [7]. Historically, the study of infinite series containing harmonic numbers was initiated in response to a letter dated December 6, 1742, sent by the German mathematician Christian Goldbach (1690–1764) to the famous Swiss mathematician Leonard Euler [15, p. 741]. Today infinite series containing harmonic numbers, and to a lesser extent the skew-harmonic numbers, continue to occupy the attention of mathematicians with this area of study now vast [23, 27, 28].

For $q \in \mathbb{Z}_{>1}$ Euler gave without proof a general closed-form expression for the first and perhaps most famous of the so-called linear Euler sums [9]. Here

$$\sum_{n=1}^{\infty}\frac{H_n}{n^q} = \frac{1}{2}(q+2)\zeta(q+1) - \frac{1}{2}\sum_{n=1}^{q-2}\zeta(n+1)\zeta(q-n). \tag{1}$$

Note the empty sum that arises when $q=2$ is understood to be nil. Here ζ denotes the Riemann zeta function which is defined by

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \tag{2}$$

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for $s>1$. After Euler, interest in sums of the type given by (1) lapsed and it would be the best part of two centuries before interest in them was finally renewed. Given how fundamental the linear Euler sum of (1) is, it has, unsurprisingly, been repeatedly rediscovered independently a number of times since the time of Euler [16, 30, 20, 11]. In the case of the linear Euler sum containing skew-harmonic numbers (we intend to call this a *skew-Euler sum*), its general closed-form expression came much later, appearing to have been first given by Sitaramachandrarao [21, Thm 3.5]. For $q \in \mathbb{Z}_{>1}$ one has

$$\sum_{n=1}^{\infty} \frac{\overline{H}_n}{n^q} = (2-2^{1-q})\log(2)\zeta(q) + \left(1-\frac{1}{2^q}-\frac{q}{2}\right)\zeta(q+1) + \frac{1}{2}\sum_{k=1}^{q-2}(1-2^{1-q+k})(1-2^{-k})\zeta(q-k)\zeta(k+1). \tag{3}$$

Again the empty sum that arises when $q=2$ is understood to be nil. Alternative derivations to the one given by Sitaramachandrarao can be found in [10], [17, pp. 196–200, Eq. (4.11)]. Similar closed-form expressions can be found for the corresponding alternating cases (H_n and $\overline{H}_n$ in expressions (1) and (3) replaced with $(-1)^n H_n$ and $(-1)^n \overline{H}_n$ respectively) for the case when one has an even index appearing in the denominator of the summand [21, Thm 3.7], [10, Thm 7.1 (ii)], [17, pp. 200–203, Eqs (4.17) and (4.28)].

Closely related to the linear Euler and linear skew-Euler sums are linear Euler-type sums where the n appearing in the denominator of the summand is replaced by the linear factor $2n+1$. In the case of the linear Euler-type sum containing the harmonic numbers, for $q \in \mathbb{Z}_{>1}$ one has [2], [17, pp. 212–213, Eq. (4.73)]:

$$\sum_{n=1}^{\infty} \frac{H_n}{(2n+1)^q} = (2^{1-q}-2)\log(2)\zeta(q) + q\left(1-\frac{1}{2^{q+1}}\right)\zeta(q+1) - \frac{1}{2}\sum_{k=1}^{q-2}(2^{k+1}-1)(2^{-k}-2^{-q})\zeta(q-k)\zeta(k+1). \tag{4}$$

Again the empty sum that arises when $q=2$ is understood to be nil. Beyond this linear Euler-type sum an expression for the corresponding alternating case of (4) containing an odd index (q replaced with $2q+1$) has been given [17, p. 216, Eq. (4.91)]. Nothing, as far as the author is aware, for those cases where the harmonic number term is replaced with a skew-harmonic number term has however been given.

In this paper we give closed-form expressions for the following linear Euler-type sums:

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(2n+1)^{2q+1}}, \sum_{n=1}^{\infty} \frac{\overline{H}_n}{(2n+1)^q}, \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n}{(2n+1)^{2q+1}}. \tag{5}$$

In the first and third sum $q \in \mathbb{Z}_{\geqslant 0}$ while in the second $q \in \mathbb{Z}_{>1}$. For the first sum we give a new proof that leads to an alternative form of an existing result already known in the literature [17, p. 216, Eq. (4.91)] while the results we give for the second and third sums are believed to be completely new. Several other closely related generalised logarithmic integrals and sums in closed-form to the three sums given in (5) are also presented.

2. Some preliminaries

In this section we give a number of preliminary results that are going to be needed when it comes to presenting the main results in the paper.

The digamma function is defined by

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{6}$$

where $\Gamma(x)$ is the classical gamma function defined by the Eulerian integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt.$$

These are respectively entries 5.2.2 and 5.2.1 in [19]. Closely connected to the gamma function is the beta function defined by

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt, \quad x,y > 0, \tag{7}$$

which is related to the gamma function by the identity

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{8}$$

These are respectively entries 8.380.1 and 8.384.1 in [12]. Alternative integral representations for the beta function can be given. One such one is found on substituting $t = \sin^2 \theta$ into (7). Doing so yields

$$\mathrm{B}(x,y) = 2 \int_0^{\frac{\pi}{2}} \sin^{2x-1} \theta \cos^{2y-1} \theta \, d\theta. \tag{9}$$

This is entry 8.380.2 in [12]. Euler's reflexion formula for the gamma function is given by

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \quad z \neq 0, \pm 1, \pm 2, \ldots \tag{10}$$

while Legendre's duplication formula for the gamma function is

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right), \quad 2z \neq 0, -1, -2, \ldots \tag{11}$$

These are respectively entries 5.5.1 and 5.5.5 in [19].

The functional relation for the digamma function is

$$\psi(x+1) = \psi(x) + \frac{1}{x}. \tag{12}$$

This is entry 5.5.2 in [19]. A series representation for the digamma function is

$$\psi(x+1) = -\gamma + \sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+x} \right), \quad x \neq -1, -2, -3, \ldots \tag{13}$$

Here γ is the Euler–Mascheroni constant. This is entry 5.7.6 in [19]. The Maclaurin series expansion for the digamma function is

$$\psi(x+1) = -\gamma - \sum_{n=1}^{\infty} (-1)^n \zeta(n+1) x^n, \qquad |x| < 1. \tag{14}$$

This is entry 5.7.4 in [19]. Some needed values for the digamma function are $\psi(1) = -\gamma$ and

$$\psi\left(\frac{1}{4}\right) = -\gamma - 3\log(2) - \frac{\pi}{2}, \quad \psi\left(\frac{3}{4}\right) = -\gamma - 3\log(2) + \frac{\pi}{2}, \tag{15}$$

the first of these values being entry 6.3.2 in [1], the latter two values being entries 44.7.4 and 44.7.6 in [18].

The polygamma function of order $m \in \mathbb{Z}_{\geqslant 1}$ is defined by

$$\psi^{(m)}(x) = \frac{d^m}{dx^m}\{\psi(x)\} = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(x+n)^{m+1}},$$

and is valid for all $x \in \mathbb{R}$, $x \neq 0, -1, -2, \ldots$. This is entry 6.4.1 in [1]. Note the polygamma function of order zero is given by

$$\psi^{(0)}(x) := \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

and is, by definition, just the digamma function given in (6).

The Maclaurin series expansion for the secant function is

$$\sec(x) = \sum_{n=0}^{\infty} \frac{|E_n| x^n}{n!}, \qquad |x| < \frac{\pi}{2}, \tag{16}$$

where E_n denote the Euler numbers. This is entry 1.411.9 in [12]. The first few of these numbers are:

$$E_0 = 1, E_2 = -1, E_4 = 5, E_6 = -61, E_8 = 1385,$$

while all odd-indexed Euler numbers are equal to zero. The Maclaurin series expansion for the tangent function is

$$\tan(x) = \sum_{n=1}^{\infty} \frac{2^{n+1}(2^{n+1}-1)|B_{n+1}|}{(n+1)!} x^n, \qquad |x| < \frac{\pi}{2}, \tag{17}$$

where B_n denote the Bernoulli numbers. This is entry 1.411.5 in [12]. The first few of these numbers are:

$$B_0 = 1, B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}.$$

In fact, $B_{2n+1} = 0$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

The Dirichlet beta function is defined by

$$\beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}, \tag{18}$$

and is valid for $s>0$. This is entry 3.6.4 in [18]. A special value when the argument is even is $\beta(2)=G$, which is Catalan's constant. When the argument is odd one has:

$$\beta(1)=\frac{\pi}{4},\beta(3)=\frac{\pi^3}{32},\beta(5)=\frac{5\pi^5}{1536},\beta(7)=\frac{61\pi^7}{184\,320}, \text{ and so on.}$$

Indeed, for all odd arguments the Dirichlet beta function can be expressed in terms of a multiple of π as follows:

$$\beta(2n+1)=\frac{|E_{2n}|}{2(2n)!}\left(\frac{\pi}{2}\right)^{2n+1}, \quad n\in\mathbb{Z}_{\geqslant 0}. \tag{19}$$

This is entry 3.13.3 in [18]. A closely related function to the Dirichlet beta function is the Dirichlet lambda function. It is defined by

$$\lambda(s)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^s}, \tag{20}$$

and is valid for $s>1$. This is entry 3.6.2 in [18]. It is related to the Riemann zeta function by

$$\lambda(s)=\left(1-\frac{1}{2^s}\right)\zeta(s). \tag{21}$$

This is entry 3.0.1 in [18].

The series representation of the Dirichlet beta function given in (18) can be formed in terms of a difference between two polygamma functions. For $s\in\mathbb{Z}_{\geqslant 0}$ we have

$$\beta(s+1)=\frac{1}{2^{s+1}}\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)^{s+1}}=\frac{(-1)^{s+1}}{4^{s+1}s!}\left[\psi^{(s)}\left(\frac{1}{4}\right)-\psi^{(s)}\left(\frac{3}{4}\right)\right]. \tag{22}$$

Replacing s with $2q$ and $2q+1$ in (22), where $q\in\mathbb{Z}_{\geqslant 0}$, one has [13]

$$\psi^{(2q)}\left(\frac{3}{4}\right)-\psi^{(2q)}\left(\frac{1}{4}\right)=(-1)^q 4^q E_{2q}\pi^{2q+1}, \tag{23}$$

and

$$\psi^{(2q+1)}\left(\frac{3}{4}\right)-\psi^{(2q+1)}\left(\frac{1}{4}\right)=-4^{2q+2}(2q+1)!\beta(2q+2), \tag{24}$$

respectively. In (23) the Dirichlet beta function has been expressed in terms of the Euler numbers using (19). A similar thing can be done for the Dirichlet lambda function. Here

$$\lambda(s+1)=\frac{1}{2^{s+1}}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{2}\right)^{s+1}}=\frac{(-1)^{s+1}}{4^{s+1}s!}\left[\psi^{(s)}\left(\frac{3}{4}\right)+\psi^{(s)}\left(\frac{1}{4}\right)\right]. \tag{25}$$

Replacing s with $2q$ in (25), where $q = \mathbb{Z}_{\geqslant 0}$, and writing the Dirichlet lambda function in terms of the Riemann zeta function using (21) one obtains [13]

$$\psi^{(2q)}\left(\frac{3}{4}\right) + \psi^{(2q)}\left(\frac{1}{4}\right) = -(2q)!2^{2q+1}\left(2^{2q+1}-1\right)\zeta(2q+1). \tag{26}$$

In the next lemma an important result concerning a power series expansion involving a difference between two digamma functions with differing arguments is given.

LEMMA 1. *For* $|t| < 1$

$$\psi\left(\frac{t}{2}+1\right) - \psi(t+1) = \sum_{k=1}^{\infty}(-1)^k\left(1-\frac{1}{2^k}\right)\zeta(k+1)t^k. \tag{27}$$

Proof. From (13) we see that

$$\psi\left(\frac{t}{2}+1\right) - \psi(t+1) = \sum_{n=1}^{\infty}\left(\frac{1}{n+t} - \frac{1}{n+t/2}\right). \tag{28}$$

Also for $|t| < 1$, as

$$\frac{1}{n+t} = \sum_{k=0}^{\infty}\frac{(-1)^k t^k}{n^{k+1}} = \frac{1}{n} + \sum_{k=1}^{\infty}\frac{(-1)^k t^k}{n^{k+1}},$$

and

$$\frac{1}{n+t/2} = \sum_{k=0}^{\infty}\frac{(-1)^k}{n^{k+1}}\frac{t^k}{2^k} = \frac{1}{n} + \sum_{k=0}^{\infty}\frac{(-1)^k}{n^{k+1}}\frac{t^k}{2^k},$$

and are nothing more than infinite geometric series expansions, the term appearing in the brackets on the right of (28) can be rewritten as

$$\psi\left(\frac{t}{2}+1\right) - \psi(t+1) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{n^{k+1}}\left(1-\frac{1}{2^k}\right)t^k\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}a_{nk}(t),$$

or

$$\psi\left(\frac{t}{2}+1\right) - \psi(t+1) = \sum_{k=1}^{\infty}(-1)^k\left(1-\frac{1}{2^k}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^{k+1}}\right)t^k,$$

after the order of the double summation has been interchanged and is permissible since for $|t| < 1$ and each $n \in \mathbb{Z}_{\geqslant 1}$

$$\sum_{k=1}^{\infty}|a_{nk}(t)| < \frac{t}{n^2 - tn} = M_n(t) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty}M_n(t) < \infty.$$

Recognising the right most term within brackets as the Riemann zeta function given in (2) completes the proof. □

An alternative expression for the digamma function $\psi\left(\frac{1-t}{2}\right)$ follows in the next lemma.

LEMMA 2. *For $x \in \mathbb{R}, x \neq 1,3,5,\ldots$ the following equality holds:*

$$\psi\left(\frac{1-x}{2}\right) = -\left[\psi\left(\frac{x}{2}+1\right) - \psi(x+1)\right] + \psi(x+1) - 2\log(2) - \pi\tan\left(\frac{\pi x}{2}\right). \quad (29)$$

Proof. From Legendre's duplication formula for the gamma function given in (11) one has

$$\Gamma\left(\frac{1-x}{2}\right) = \frac{\sqrt{\pi}2^x\Gamma(1-x)}{\Gamma\left(1-\frac{x}{2}\right)}.$$

And from Euler's reflexion formula of (10) we have

$$\Gamma(1-x) = \frac{\pi}{\sin(\pi x)\Gamma(x)} \quad \text{and} \quad \Gamma\left(1-\frac{x}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi x}{2}\right)\Gamma\left(\frac{x}{2}\right)}.$$

Thus

$$\Gamma\left(\frac{1-x}{2}\right) = \frac{\sqrt{\pi}2^{t-1}\Gamma\left(\frac{x}{2}\right)}{\Gamma(x)\cos\left(\frac{\pi x}{2}\right)},$$

where the double angle formula for the sine function has been used. Taking the logarithmic derivative with respect to x yields

$$\psi\left(\frac{1-x}{2}\right) = 2\psi(x) - \psi\left(\frac{x}{2}\right) - 2\log(2) - \pi\tan\left(\frac{\pi x}{2}\right).$$

Applying the functional relation of (12) to the two digamma function terms appearing to the right of the equality then completes the proof. □

We next give, as lemmas, a number of important logarithmic integrals we are going to have a need for.

LEMMA 3. *If $q \in \mathbb{Z}_{>0}$ then*

$$\int_0^1 \frac{\log^q(x)}{1-x^2}\,dx = (-1)^q q!\left(1-\frac{1}{2^{q+1}}\right)\zeta(q+1). \quad (30)$$

Proof. A proof can be found in [4, Proposition 12.3.1]. □

LEMMA 4. *If $q \in \mathbb{Z}_{\geqslant 0}$ then*

$$\int_0^1 \frac{\log^q(x)}{1+x^2}\,dx = (-1)^q q!\beta(q+1). \quad (31)$$

This is entry 4.271.5 in [12].

LEMMA 5. *If $n \in \mathbb{Z}_{\geqslant 0}$ then the following equality holds:*

$$\int_0^\infty \frac{\log^n(x)\log(1+x^2)}{1+x^2}\,dx = \left(\frac{\pi}{2}\right)^{n+1} n! \sum_{k=1}^n \frac{(-1)^k |E_{n-k}|(2^{k+1}-1)\zeta(k+1)}{\pi^k (n-k)!}$$
$$+ \left(\frac{\pi}{2}\right)^{n+2} n! \sum_{k=1}^n \frac{2^{k+2}(2^{k+1}-1)|B_{k+1}|\cdot|E_{n-k}|}{(n-k)!(k+1)!} + 2\left(\frac{\pi}{2}\right)^{n+1} |E_n|\log(2). \tag{32}$$

The two empty sums that arise when $n = 0$ are understood to be nil.

Proof. Denoting the integral to be prove by I_n, after enforcing a substitution of $x \mapsto \tan x$ one has

$$I_n = -2\int_0^{\frac{\pi}{2}} \log^n(\tan x)\log(\cos x)\,dx.$$

The resulting integral will now be found using an exponential generating function approach [29]. Consider the exponential generating function given by

$$G(t) = \sum_{n=0}^\infty \frac{I_n t^n}{n!}.$$

One therefore has

$$G(t) = -2\int_0^{\frac{\pi}{2}} \log(\cos x) \sum_{n=0}^\infty \frac{(t\log(\tan x))^n}{n!}\,dx = -2\int_0^{\frac{\pi}{2}} \sin^t x \cos^{-t} \log(\cos x)\,dx.$$

Now consider the function defined by

$$J(a) = 2\int_0^{\frac{\pi}{2}} \sin^t x \cos^{a-t} x\,dx, \quad a \geqslant 0,$$

where we observe that $G(t) = -J'(0)$. Now $J(a)$ can be readily written in terms of a beta function as follows

$$J(a) = \mathrm{B}\left(\frac{t+1}{2}, \frac{a+1-t}{2}\right) = \frac{\Gamma\left(\frac{1+t}{2}\right)\Gamma\left(\frac{a+1-t}{2}\right)}{\Gamma\left(1+\frac{a}{2}\right)}.$$

Here we have made use of the integral representation for the beta function given in (9) together with the result given in identity (8). Taking the logarithmic derivative with respect to a yields

$$\frac{J'(a)}{J(a)} = \frac{1}{2}\psi\left(\frac{a+1-t}{2}\right) - \frac{1}{2}\psi\left(\frac{a}{2}+1\right).$$

On setting $a = 0$ we find

$$G(t) = -J'(0) = -\frac{\pi}{2}\sec\left(\frac{\pi t}{2}\right)\left[\gamma + \psi\left(\frac{1-t}{2}\right)\right],$$

where we note the results of $J(0)=\Gamma\left(\frac{1+t}{2}\right)\Gamma\left(\frac{1-t}{2}\right)=\pi\sec\left(\frac{\pi t}{2}\right)$ and $\psi(1)=-\gamma$ have been used. From result (29) given in Lemma 2, the expression for $G(t)$ can be rewritten as

$$G(t)=\frac{\pi}{2}\sec\left(\frac{\pi t}{2}\right)\left\{(-\gamma-\psi(t+1))+\left[\psi\left(\frac{t}{2}+1\right)-\psi(t+1)\right]\right.$$
$$\left.+2\log(2)+\pi\tan\left(\frac{\pi t}{2}\right)\right\}. \tag{33}$$

If we now express each of the product terms appearing in (33) as a product between their Maclaurin series expansions, on finding the Cauchy product we find:
For the first term

$$\frac{\pi}{2}\sec\left(\frac{\pi t}{2}\right)\cdot(-\gamma-\psi(t+1))=\frac{\pi}{2t}\sum_{n=1}^{\infty}\frac{|E_{n-1}|\pi^{n-1}t^n}{2^{n-1}(n-1)!}\cdot\sum_{n=1}^{\infty}(-1)^n\zeta(n+1)t^n$$
$$=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{(-1)^k\zeta(k+1)|E_{n-k}|\pi^{n-k+1}n!}{2^{n-k+1}(n-k)!}\right)\frac{t^n}{n!}, \tag{34}$$

where the series expansions of (14) and (16) with x replaced with $\frac{\pi t}{2}$ have been used.
For the second term

$$\frac{\pi}{2}\sec\left(\frac{\pi t}{2}\right)\cdot\left[\psi\left(\frac{t}{2}+1\right)-\psi(t+1)\right]=\frac{\pi}{2t}\sum_{n=1}^{\infty}\frac{|E_{n-1}|\pi^{n-1}t^n}{2^{n-1}(n-1)!} \tag{35}$$
$$\times\sum_{n=1}^{\infty}(-1)^n\left(1-\frac{1}{2^n}\right)\zeta(n+1)t^n$$
$$=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{|E_{n-k}|\pi^{n-k+1}(-1)^k(1-2^{-k})\zeta(k+1)n!}{2^{n-k+1}(n-k)!}\right)\frac{t^n}{n!}, \tag{36}$$

where the series expansions of (16) with x replaced with $\frac{\pi t}{2}$ and (27) have been used.
For the third term

$$\pi\sec\left(\frac{\pi t}{2}\right)\cdot\log(2)=\pi\log(2)+\sum_{n=1}^{\infty}\left(\frac{\log(2)|E_n|\pi^{n+1}}{2^n}\right)\frac{t^n}{n!}, \tag{37}$$

where the series expansion of (16) with x replaced with $\frac{\pi t}{2}$ has been used.
For the fourth term

$$\frac{\pi^2}{2}\sec\left(\frac{\pi t}{2}\right)\cdot\tan\left(\frac{\pi t}{2}\right)=\frac{\pi^2}{2t}\sum_{n=1}^{\infty}\frac{|E_{n-1}|\pi^{n-1}t^n}{2^{n-1}(n-1)!}\cdot\sum_{n=1}^{\infty}\frac{2(2^{n+1}-1)|B_{n+1}|\pi^n}{(n+1)!}$$
$$=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{(2^{k+1}-1)|E_{n-k}|\cdot|B_{k+1}|\pi^{n+2}n!}{2^{n-k}(n-k)!(k+1)!}\right)\frac{t^n}{n!}, \tag{38}$$

where the series expansions of (16) and (17) have been used, and in both cases x has been replaced with $\frac{\pi t}{2}$.

In ensuring all the sums start at $n=1$ we see that

$$G(t)=\sum_{n=0}^{\infty}\frac{I_n t^n}{n!}=I_0+\sum_{n=1}^{\infty}\frac{I_n t^n}{n!}=\pi\log(2)+\sum_{n=1}^{\infty}\frac{I_n t^n}{n!}, \tag{39}$$

where

$$I_0=-2\int_0^{\frac{\pi}{2}}\log(\cos x)\,dx=-2\int_0^{\frac{\pi}{2}}\log(\sin x)\,dx=\pi\log(2),$$

this last result being Euler's famous log–sine integral [26, p. 229]. Substituting the results given in (34), (36), (37), and (38) into (39), on equating equal coefficients for $t^n/n!$ delivers the desired result and completes the proof. □

COROLLARY 1. *If $q\in\mathbb{Z}_{\geqslant 0}$ then the following equality holds:*

$$\int_0^{\infty}\frac{\log^{2q}(x)\log(1+x^2)}{1+x^2}\,dx=\left(\frac{\pi}{2}\right)^{2q+1}(2q)!\sum_{k=1}^{q}\frac{\zeta(2k+1)|E_{2q-2k}|\left(2^{2k+1}-1\right)}{\pi^{2k}(2q-2k)!}$$
$$+2\left(\frac{\pi}{2}\right)^{2q+1}|E_{2q}|\log(2). \tag{40}$$

The empty sum that arises when $q=0$ is understood to be nil.

Proof. Replacing n with $2q$ in (32), for $|E_{2q-k}|$ to be non-zero one requires $(2q-k)$ to be even. Setting $2q-k=2m$ where $m\in\mathbb{Z}_{\geqslant 0}$ we see $k=2(q-m)$ must be even. But if k is even then $k+1$ is odd, so $|B_{k+1}|=0$ and ensures the second term in (32) containing $|B_{k+1}|$ is zero for all $1\leqslant k\leqslant n$. Reindexing the remaining sum by $k\mapsto 2k$ yields the desired result and completes the proof. □

REMARK 1. When the index n is odd in (32) all values for the integrals will consist of a simple multiple of π as the term containing $\log(2)$ is zero due to all odd-index Euler numbers being equal to zero.

We shall also have a need for a number of ordinary generating functions. The first of these is the ordinary generating function for the harmonic numbers. It is given by [17, p. 58, Eq. (2.4)]

$$\sum_{n=1}^{\infty}H_n x^n=-\frac{\log(1-x)}{1-x},\quad |x|<1. \tag{41}$$

The second of these is the ordinary generating function for the skew-harmonic numbers. It is given by [17, p. 74, Eq. (2.28)]

$$\sum_{n=1}^{\infty}\overline{H}_n x^n=\frac{\log(1+x)}{1-x},\quad |x|<1, \tag{42}$$

A final ordinary generating function involving the harmonic numbers is given in the following lemma.

LEMMA 6. *For* $|x| < 1$

$$\sum_{n=1}^{\infty} H_n (x^2 - 1) x^{4n} = \frac{\log(1 - x^4)}{1 + x^2}. \tag{43}$$

Proof. Write the log term appearing on the right-hand side of (43) as

$$\frac{\log(1 - x^4)}{1 + x^2} = \frac{(1 - x^2)\log(1 - x^4)}{1 - x^4}.$$

Replacing x with x^4 in the ordinary generating function for the harmonic numbers given by (41), on substituting the resulting expression into the right-hand side of the above expression, the desired result then follows and completes the proof. □

We conclude this section by giving a very general result for a sum containing the harmonic numbers.

LEMMA 7. *Let x be a real number $x \neq -1, -2, -3, \ldots$ and $q \in \mathbb{Z}_{>1}$. Then the following equality holds:*

$$\sum_{n=1}^{\infty} \frac{H_n}{(n + x)^q} = \frac{(-1)^q}{(q - 1)!} \Bigg[(\psi(x) + \gamma)\, \psi^{(q-1)}(x) - \frac{1}{2} \psi^{(q)}(x) + \sum_{k=1}^{q-2} \binom{q-2}{k} \psi^{(k)}(x) \psi^{(q-k-1)}(x) \Bigg] \tag{44}$$

Proof. A proof can be found in [22]. □

3. Main results

In this section the three general closed-formed expressions for the linear Euler-type sums containing the harmonic and the skew-harmonic numbers found in (5) are given and form the main results of our paper.

THEOREM 1. *For $q \in \mathbb{Z}_{\geqslant 0}$ the following equality holds*

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(2n + 1)^{2q+1}} = (2q + 1)\beta(2q + 2) - \left(\frac{\pi}{2}\right)^{2q+1} \frac{|E_{2q}|}{(2q)!} \log(2) - \frac{1}{2}\left(\frac{\pi}{2}\right)^{2q+1} \sum_{k=1}^{q} \frac{(2^{2k+1} - 1)|E_{2q-2k}|\zeta(2k + 1)}{\pi^{2k}(2q - 2k)!}. \tag{45}$$

The empty sum that arises when $q = 0$ is understood to be nil.

Proof. For $q \in \mathbb{Z}_{\geqslant 1}$ let

$$L_{q-1} = \int_0^1 \frac{\log^{q-1}(x)\log(1+x^2)}{1+x^2}\,dx.$$

From the ordinary generating function for the harmonic numbers given by (41) with x replaced with $-x^2$, we can write

$$L_{q-1} = -\sum_{n=1}^{\infty}(-1)^n H_n \int_0^1 x^{2n}\log^{q-1}(x)\,dx,$$

where the interchange made between the integration and summation signs is permissible due to Fubini's theorem. Integrating by parts $(q-1)$-times one finds

$$L_{q-1} = (-1)^q (q-1)! \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(2n+1)^q},$$

or after rearranging

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(2n+1)^q} = \frac{(-1)^q}{(q-1)!} L_{q-1}.$$

A reindexing of $q \mapsto 2q+1$ then produces

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(2n+1)^{2q+1}} = \frac{-1}{(2q)!} L_{2q}. \tag{46}$$

Now consider the integral

$$\int_0^{\infty} \frac{\log^{q-1}(x)\log(1+x^2)}{1+x^2}\,dx = \int_0^1 \frac{\log^{q-1}(x)\log(1+x^2)}{1+x^2}\,dx + \int_1^{\infty} \frac{\log^{q-1}(x)\log(1+x^2)}{1+x^2}\,dx.$$

Enforcing a substitution of $x \mapsto \frac{1}{x}$ in the second of the integrals to the right of the equality leads to

$$\int_0^{\infty} \frac{\log^{q-1}(x)\log(1+x^2)}{1+x^2}\,dx = \left(1+(-1)^{q-1}\right)\int_0^1 \frac{\log^{q-1}(x)\log(1+x^2)}{1+x^2}\,dx - 2(-1)^{q-1}\int_0^1 \frac{\log^q(x)}{1+x^2}\,dx, \tag{47}$$

or

$$L_{2q} = \int_0^1 \frac{\log^{2q+1}(x)}{1+x^2}\,dx + \frac{1}{2}\int_0^{\infty} \frac{\log^{q-1}(x)\log(1+x^2)}{1+x^2}\,dx, \tag{48}$$

after rearranging and a reindexing of $q \mapsto 2q+1$ has been made. The first of the integrals appearing to the right of the equality in (48) is given by (30) with q replaced with $2q+1$, the second is given by (40), which upon substituting into (46) yields the desired result and completes the proof. □

REMARK 2. An alternative closed-form expression for the sum appearing in (45) in terms of a limit of a $2q$-order derivative can be found in [17, p. 216, Eq. (4.91)]. Expression (45) has subsequently appeared[1] in a paper [25, p. 20] that was submitted prior to the initial submission of this paper and published somewhat after this time. Here an alternative proof to the one given above was used.

COROLLARY 2. *If $q \in \mathbb{Z}_{\geqslant 1}$ then the following equality holds:*

$$\int_0^\infty \frac{\log^{2q-1}(x)\log(1+x^2)}{1+x^2}\,dx = \left(\frac{\pi}{2}\right)^{2q+1}|E_{2q}|. \tag{49}$$

Proof. If in (47) q is replaced with $2q$ one obtains

$$\int_0^\infty \frac{\log^{2q-1}(x)\log(1+x^2)}{1+x^2}\,dx = 2\int_0^1 \frac{\log^{2q}(x)}{1+x^2}\,dx.$$

The integral appearing to the right of the equality is just (31) with q replaced with $2q$. The result then follows on applying (19) to the Dirichlet beta function term, and completes the proof. □

REMARK 3. The result given in Corollary 2 confirms the observation made in Remark 1. For an alternative derivation of this result, see [24, Thm 2].

COROLLARY 3. *If $q \in \mathbb{Z}_{\geqslant 1}$ then the following equality holds:*

$$\sum_{k=1}^{q} \binom{2q}{2k} 2^{2k}(2^{2k}-1)|B_{2k}|\cdot|E_{2q-2k}| = 2q|E_{2q}|.$$

Proof. Replacing n with $2q-1$ in (32) before setting the result equal to (49) yields

$$\frac{|E_{2q}|}{(2q-1)!} = \sum_{k=1}^{2q-1} \frac{2(-1)^k|E_{2q-k-1}|(2^{2k+1}-1)\zeta(k+1)}{\pi^{k+1}(2q-k-1)!} + \sum_{k=1}^{2q-1} \frac{2^{k+2}(2^{k+2}-1)|B_{k+1}|\cdot|E_{2q-k-1}|}{(2q-k-1)!(k+1)!}.$$

For the Euler number term appearing in each of the sums to be non-zero we require $2q-k-1 = 2m$ where $m \in \mathbb{Z}_{>0}$, that is, $k = 2(q-m)-1$ to be odd. If a reindexing of $k \mapsto 2k-1$ in each sum is made, on recalling the result [19, Entry 25.6.2]

$$\zeta(2k) = \frac{|B_{2k}|2^{2k-1}\pi^{2k}}{(2k)!},$$

and writing the product between the factorials in the denominator in terms of a binomial coefficient, the desired result then follows and completes the proof. □

[1] Or be it with a typo present in the argument for the Dirichlet beta function term.

REMARK 4. More curious is the case when the index appearing in the term in the denominator of the summand of (45) is even. For all but the lowest order case when the index is equal to two and a closed-form expression for the sum in terms of known mathematical constants can be found, one has sums of the type

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(2n+1)^{2q}}.$$

For $q \in \mathbb{Z}_{>1}$ these represent a new variety of mathematical constants that are not known to be related to any of the classical constants of mathematics. We will write these as:

$$\Lambda(s) = \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(2n+1)^s}.$$

When s is a positive odd integer their values are as given in Theorem 1. When $s=2$ it is known that [17, pp. 270–271] (see also [24, p. 69] for an alternative, equivalent expression)

$$\Lambda(2) = \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(2n+1)^2} = \frac{3\pi^3}{32} + \frac{\pi}{8}\log^2(2) - \log(2)G + 2\Im \operatorname{Li}_3(1-i).$$

Here $i = \sqrt{-1}$ is the imaginary unit, $\Im$ denotes the imaginary part, while Li_3 denotes the trilogarithm defined by [14, p. 153, Eq. (6.1)]

$$\operatorname{Li}_3(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3}, \quad |x| \leqslant 1.$$

The first of these new constants is $\Lambda(4)$. It has a numerical value equal to

$$\Lambda(4) = -0.010\,493\,456\,297\,335\ldots$$

REMARK 5. The term $\Im \operatorname{Li}_3(1-i)$ appearing in the constant $\Lambda(2)$ can be related to a new, recently defined constant $\mathscr{G}^*$ which is a variant of a Catalan-like constant [8]. Here $\Im \operatorname{Li}_3(1-i) = -\mathscr{G}^*$.

LEMMA 8. *For $q \in \mathbb{Z}_{>1}$ the second of the sums appearing in (5) can be represented by an integral as follows:*

$$\sum_{n=1}^{\infty} \frac{\overline{H}_n}{(2n+1)^q} = \frac{(-1)^{q+1}}{(q-1)!} \int_0^1 \frac{\log^{q-1}(x)\log(1+x^2)}{1-x^2}\,dx. \tag{50}$$

Proof. From the ordinary generating function for the skew-harmonics of (42) with x replaced with x^2, using this allows one to rewrite the integral as

$$\int_0^1 \frac{\log^{q-1}(x)\log(1+x^2)}{1-x^2}\,dx = \int_0^1 \log^{q-1}(x) \sum_{n=1}^{\infty} \overline{H}_n x^{2n}\,dx = \sum_{n=1}^{\infty} \overline{H}_n \int_0^1 x^{2n}\log^{q-1}(x)\,dx.$$

The interchange made between the order of the summation and integration is permissible due to its absolute convergence since for fixed $q \in \mathbb{Z}_{>1}$ all terms involved are unsigned (all negative if q is even, all positive if q is odd). Integrating by parts $(q-1)$-times, the result then follows and completes the proof. □

THEOREM 2. *For $q \in \mathbb{Z}_{>1}$ the following equality holds*

$$\sum_{n=1}^{\infty} \frac{\overline{H}_n}{(2n+1)^q} = \left(1 - \frac{1}{2^q}\right) \log(2)\zeta(q) - q\left(1 - \frac{1}{2^{q+1}}\right)\zeta(q+1) + \sum_{k=0}^{q-1} \beta(q-k)\beta(k+1). \tag{51}$$

Proof. Denote integral (50) appearing in Lemma 8 by J_q and define on the interval $x \in [0,1]$ the function

$$R_q(x) = \int_0^x \frac{\log^{q-1}(t)}{1-t^2}\, dt.$$

Setting $t \mapsto xt$ produces

$$R_q(x) = \int_0^1 \frac{x \log^q(xt)}{1-x^2t^2}\, dt.$$

Note that $R_q(0) = 0$ and

$$R_q(1) = \int_0^1 \frac{\log^{q-1}(t)}{1-t^2}\, dt = (-1)^{q-1}(q-1)!\left(1 - \frac{1}{2^q}\right)\zeta(q), \tag{52}$$

which is just the result given in (30) with q replaced with $q-1$. Integrating by parts gives

$$\begin{aligned} J_q &= \left[R_q(x)\log(1+x^2)\right]_0^1 - \int_0^1 \frac{2xR_q(x)}{1+x^2}\, dx \\ &= R_q(1)\log(2) - 2\int_0^1 \int_0^1 \frac{x^2 \log^{q-1}(xt)}{(1+x^2)(1-x^2t^2)}\, dt\, dx. \end{aligned}$$

From the partial fraction decomposition of

$$\frac{x^2}{(1+x^2)(1-x^2t^2)} = \frac{1}{(1+t^2)(1-x^2t^2)} - \frac{1}{(1+t^2)(1+x^2)},$$

the integral for J_q may be rewritten as

$$\begin{aligned} J_q = R_q(1)\log(2) &+ 2\int_0^1 \int_0^1 \frac{\log^{q-1}(xt)}{(1+x^2)(1+t^2)}\, dt\, dx \\ &- 2\int_0^1 \int_0^1 \frac{\log^{q-1}(xt)}{(1+t^2)(1-x^2t^2)}\, dt\, dx. \end{aligned}$$

On expanding the first of the $\log^{q-1}(xt)$ terms using the binomial expansion, namely

$$\log^{q-1}(xt) = (\log(x) + \log(t))^{q-1} = \sum_{k=0}^{q-1} \binom{q-1}{k} \log^{q-k-1}(x)\log^k(t),$$

allows one to rewrite the integral as

$$J_q = R_q(1)\log(2) + 2\sum_{k=0}^{q-1}\binom{q-1}{k}\int_0^1 \frac{\log^{q-k-1}(x)}{1+x^2}\,dx\int_0^1 \frac{\log^k(t)}{1+t^2}\,dt - \int_0^1 \frac{2}{t(1+t^2)}\left(\int_0^1 \frac{t\log^{q-1}(xt)}{1-x^2t^2}\,dx\right)dt. \tag{53}$$

The first two single integrals appearing to the right of the equality in (53) are related to the Dirichlet beta function from their connection with the integral appearing in Lemma 3. Here

$$\int_0^1 \frac{\log^{q-k-1}(x)}{1+x^2}\,dx = (-1)^{q-k-1}(q-k-1)!\beta(q-k),$$

and

$$\int_0^1 \frac{\log^k(t)}{1+t^2}\,dt = (-1)^k k!\beta(k+1).$$

Thus the integral in (53) becomes

$$J_q = R_q(1)\log(2) - 2(-1)^q(q-1)!\sum_{k=0}^{q-1}\beta(q-k)\beta(k+1) - \int_0^1 \frac{2R_q(t)}{t(1+t^2)}\,dt. \tag{54}$$

Now consider the last remaining integral separately. From the partial fraction decomposition of

$$\frac{1}{t(1+t^2)} = \frac{1}{t} - \frac{t}{1+t^2},$$

integrating by parts we have

$$\begin{aligned}\int_0^1 \frac{2R_q(t)}{t(1+t^2)}\,dt &= \left[R_q(t)\left(2\log(t)-\log(1+t^2)\right)\right]_0^1 \\ &\quad - \int_0^1 \left(2\log(t)-\log(1+t^2)\right)\frac{\log^{q-1}(t)}{1-t^2}\,dt \\ &= -R_q(1)\log(2) - 2\int_0^1 \frac{\log^q(t)}{1-t^2}\,dt + \int_0^1 \frac{\log^{q-1}(t)\log(1+t^2)}{1-t^2}\,dt \\ &= -R_q(1)\log(2) - 2R_{q+1}(1) - J_q.\end{aligned}$$

Returning to (54), one has

$$J_q = 2R_q(1)\log(2) + 2R_{q+1}(1) - 2(-1)^q(q-1)!\sum_{k=0}^{q-1}\beta(q-k)\beta(k+1) - J_q,$$

or, after rearranging and simplifying

$$J_q = R_q(1)\log(2) + R_{q+1}(1) - (-1)^q(q-1)!\sum_{k=0}^{q-1}\beta(q-k)\beta(k+1).$$

The values for $R_q(1)$ and $R_{q+1}(1)$ come from (52). Here one has

$$J_q = (-1)^{q+1}(q-1)!\left(1-\frac{1}{2^q}\right)\zeta(q)\log(2)+(-1)^q q!\left(1-\frac{1}{2^{q+1}}\right)\zeta(q+1)$$
$$+(-1)^{q+1}(q-1)!\sum_{k=0}^{q-1}\beta(q-k)\beta(k+1). \tag{55}$$

Substituting this result for the integral into the result given in Lemma 8, the result for the required sum then follows and completes the proof. □

In the following example we record the first five of the sums as given by (51) in Theorem 2.

EXAMPLE 1. Setting $q = 2,3,4,5$, and 6 in (51) the first five sums are:

$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{(2n+1)^2} = \frac{\pi^2}{8}\log(2)+\frac{\pi}{2}G-\frac{7}{4}\zeta(3);$$
$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{(2n+1)^3} = \frac{7}{8}\zeta(3)\log(2)+G^2-\frac{\pi^4}{64};$$
$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{(2n+1)^4} = \frac{\pi^4}{96}\log(2)+\frac{\pi^3}{16}G+\frac{\pi}{2}\beta(4)-\frac{31}{8}\zeta(5);$$
$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{(2n+1)^5} = \frac{31}{32}\log(2)\zeta(5)-\frac{\pi^6}{384}+2G\beta(4);$$
$$\sum_{n=1}^{\infty}\frac{\overline{H}_n}{(2n+1)^6} = \frac{\pi^6}{960}\log(2)+\frac{5\pi^5}{768}G+\frac{\pi^3}{16}\beta(4)+\frac{\pi}{2}\beta(6)-\frac{381}{64}\zeta(7).$$

We now move to the third of the sums appearing in (5) which contains an alternating skew-harmonic number term.

LEMMA 9. *For $q \in \mathbb{Z}_{\geqslant 1}$ the following equality holds:*

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{(2n+1)^q} = \frac{(-1)^{q+1}}{(q-1)!}\int_0^1\frac{\log^{q-1}(x)\log(1-x^2)}{1+x^2}\,dx. \tag{56}$$

Proof. Consider the integral

$$J_{q-1} = \int_0^1\frac{\log^{q-1}(x)\log(1-x^2)}{1+x^2}\,dx.$$

If in the ordinary generating function for the skew-harmonic numbers of (42) x is replaced with $-x^2$ one obtains

$$\frac{\log(1-x^2)}{1+x^2} = \sum_{n=1}^{\infty}(-1)^n\overline{H}_n x^{2n}, \quad |x|<1.$$

Replacing the term on the left with this term that appears in the integral with the expression on the right, yields

$$J_{q-1}=\int_0^1 \log^{q-1}(x)\sum_{n=1}^{\infty}(-1)^n\overline{H}_n x^{2n}\,dx=\sum_{n=1}^{\infty}(1-)^n\overline{H}_n\int_0^1 x^{2n}\log^{q-1}(x)\,dx.$$

The interchange made here between the integration and the summation signs is permissible based on its absolute convergence according to Fubini's theorem as we have

$$\sum_{n=1}^{\infty}\int_0^1\left|(-1)^n\overline{H}_n x^{2n}\log^{q-1}(x)\right|dx<\int_0^1\frac{\log^{q-1}(x)\log(1+x^2)}{1-x^2}\,dx<\infty,$$

with the integral being the convergent integral appearing in Lemma 8. Integrating by parts $(q-1)$-times yields

$$J_{q-1}=(-1)^{q+1}(q-1)!\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{(2n+1)^q},$$

which upon rearranging produces the desired result and completes the proof. □

LEMMA 10. *For $q\in\mathbb{Z}_{\geqslant 1}$ the following equality holds:*

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{(2n+1)^q}=\sum_{n=1}^{\infty}\frac{H_n}{(4n+3)^q}-\sum_{n=1}^{\infty}\frac{H_n}{(4n+1)^q}+\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{(2n+1)^q} \tag{57}$$

Proof. As $(1-x^4)=(1-x^2)(1+x^2)$ the integral J_{q-1} appearing in the proof of Lemma 9 can be written as

$$J_{q-1}=\int_0^1\frac{\log^{q-1}(x)\log(1-x^4)}{1+x^2}\,dx-\int_0^1\frac{\log^{q-1}\log(1+x^2)}{1+x^2}\,dx. \tag{58}$$

We now consider each integral that has appeared separately. For the first of the integrals appearing in (58), one applying the result given in (43) one obtains

$$\int_0^1\frac{\log^{q-1}(x)\log(1-x^4)}{1+x^2}\,dx=\sum_{n=1}^{\infty}H_n\int_0^1\left(x^{4n+2}-x^{4n}\right)\log^{q-1}(x)\,dx,$$

where the interchange made between the integration and summations signs is permissible due Fubini's theorem. Integrating by parts $(q-1)$-times produces

$$\int_0^1\frac{\log^{q-1}(x)\log(1-x^4)}{1+x^2}\,dx=(-1)^{q-1}(q-1)!\left\{\sum_{n=1}^{\infty}\frac{H_n}{(4n+3)^q}-\sum_{n=1}^{\infty}\frac{H_n}{(4n+1)^q}\right\}.$$

The second of the integrals is L_{q-1} which appears in the proof of Theorem 1. Here

$$L_{q-1}=\int_0^1\frac{\log^{q-1}\log(1+x^2)}{1+x^2}\,dx=(-1)^q(q-1)!\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{(2n+1)^q}.$$

The result then readily follows on combining (58) with (56) and completes the proof. □

EXAMPLE 2. Setting $q = 1,2,3,4$, and 5 in (57), the first five sums are:

$$\sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n}{2n+1} = \frac{\pi}{4}\log(2) - G;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n}{(2n+1)^2} = \frac{\pi^3}{32} + \frac{\pi}{8}\log^2(2) + 2G\log(2) + 2\Im\,\mathrm{Li}_3(1-i);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n}{(2n+1)^3} = \frac{\pi^2}{4}G + \frac{\pi^3}{32}\log(2) - 3\beta(4);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n}{(2n+1)^4} = \frac{7}{4}\zeta(3)G - \frac{5\pi^5}{384} + 3\log(2)\beta(4) + \Lambda(4);$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n}{(2n+1)^5} = \frac{5\pi^5}{1536}\log(2) + \frac{\pi^4}{48}G + \frac{\pi^2}{4}\beta(4) - 5\beta(6).$$

When the index q in (57) is odd, an explicit expression for the sum can be given. This we give in the next theorem.

THEOREM 3. *For $q \in \mathbb{Z}_{\geqslant 1}$ the following equality holds:*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n}{(2n+1)^{2q+1}} &= \frac{\pi}{2^{2q+2}}\left(2^{2q+1}-1\right)\zeta(2q+1) + \frac{1}{2}\left(\frac{\pi}{2}\right)^{2q+1}\frac{|E_{2q}|}{(2q)!}\log(2)\\ &\quad - (2q+1)\beta(2q+2) - \frac{1}{2}\left(\frac{\pi}{2}\right)^{2q+1}\sum_{k=1}^{q}\frac{\left(2^{2k+1}-1\right)|E_{2q-2k}|\zeta(2k+1)}{\pi^{2k}(2q-2k)!}\\ &\quad - \frac{1}{2^{4q+2}(2q)!}\sum_{k=1}^{2q-1}\binom{2q-1}{k}\left\{\psi^{(k)}\left(\frac{3}{4}\right)\psi^{(2q-k)}\left(\frac{3}{4}\right)\right.\\ &\quad \left. - \psi^{(k)}\left(\frac{1}{4}\right)\psi^{(2q-k)}\left(\frac{1}{4}\right)\right\}.\end{aligned}$$

Proof. Setting $x = \frac{3}{4}$ and $x = \frac{1}{4}$ in (44) respectively, before considering their difference, we find

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n}{(4n+3)^q} - \sum_{n=1}^{\infty} \frac{H_n}{(4n+1)^q} &= \frac{(-1)^q}{2^{2q}(q-1)!}\left\{3\log(2)\left[\psi^{(q-1)}\left(\frac{1}{4}\right) - \psi^{(q-1)}\left(\frac{3}{4}\right)\right]\right.\\ &\quad + \frac{\pi}{2}\left[\psi^{(q-1)}\left(\frac{1}{4}\right) + \psi^{(q-1)}\left(\frac{3}{4}\right)\right] + \frac{1}{2}\left[\psi^{(q)}\left(\frac{1}{4}\right) - \psi^{(q)}\left(\frac{3}{4}\right)\right]\\ &\quad \left. + \sum_{k=1}^{q-2}\binom{q-2}{k}\left[\psi^{(k)}\left(\frac{3}{4}\right)\psi^{(q-k-1)}\left(\frac{3}{4}\right) - \psi^{(k)}\left(\frac{1}{4}\right)\psi^{(q-k-1)}\left(\frac{1}{4}\right)\right]\right\}.\end{aligned}$$

Note the two values quoted for the digamma function in (15) have been used here. Enforcing a reindexing of $q \mapsto 2q+1$, on applying each of the polygamma results given by (23), (24), and (26) where applicable, produces

$$\sum_{n=1}^{\infty}\frac{H_n}{(4n+3)^q}-\sum_{n=1}^{\infty}\frac{H_n}{(4n+1)^q}=\frac{3}{2}\left(\frac{\pi}{2}\right)^{2q+1}\frac{|E_{2q}|}{(2q)!}\log(2)-2(2q+1)\beta(2q+2)$$
$$+\frac{\pi}{2^{2q+2}}\left(2^{2q+1}-1\right)\zeta(2q+1)$$
$$-\frac{1}{2^{4q+2}(2q)!}\sum_{k=1}^{2q-1}\binom{2q-1}{k}\left\{\psi^{(k)}\left(\frac{3}{4}\right)\psi^{(2q-k)}\left(\frac{3}{4}\right)-\psi^{(k)}\left(\frac{1}{4}\right)\psi^{(2q-k)}\left(\frac{1}{4}\right)\right\}.$$

On combining this result with (45) into (57), the desired result then follows and completes the proof. □

4. A related result

In this section we give a generalised alternating sum related to the second of the sums appearing in (5). It contains the difference between two harmonic numbers of half-integer orders.

LEMMA 11. *If $n>-1$ then the following identity holds:*

$$\int_0^1 \frac{x^n}{1+x}\,dx=\frac{1}{2}\left(H_{\frac{n}{2}}-H_{\frac{n-1}{2}}\right). \tag{59}$$

Here H_z is the analytic continuation of the nth harmonic numbers.

Proof. A proof can be found in [17, p. 156]. □

THEOREM 4. *For $p\in\mathbb{Z}_{>1}$ the following equality holds:*

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^q}\left(H_{\frac{n}{2}}-H_{\frac{n-1}{2}}\right)=2q\left(1-\frac{1}{2^{q+1}}\right)\zeta(q+1)-2\sum_{k=0}^{q-1}\beta(q-k)\beta(k+1). \tag{60}$$

Proof. Denoting the sum to be found by S_q, write it as

$$S_q=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cdot\frac{1}{(2n+1)^{q-1}}\left(H_{\frac{n}{2}}-H_{\frac{n-1}{2}}\right).$$

Observing that

$$\frac{1}{(2n+1)^{q-1}}=\frac{(-1)^q}{(q-2)!}\int_0^1 x^{2n}\log^{q-2}(x)\,dx,$$

we can write the expression for S_q as

$$S_q = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \frac{(-1)^q}{(q-2)!} \int_0^1 t^{2n} \log^{q-2}(t)\, dt \cdot 2 \int_0^1 \frac{x^2}{1+x}\, dx$$
$$= \frac{2(-1)^q}{(q-2)!} \int_0^1 \int_0^1 \frac{\log^{q-2}(t)}{1+x} \sum_{n=0}^{\infty} \frac{(-1)^n (t\sqrt{x})^{2n}}{2n+1}\, dx\, dt$$
$$= \frac{2(-1)^q}{(q-2)!} \int_0^1 \int_0^1 \frac{\log^{q-2}(t) \arctan(t\sqrt{x})}{(1+x)t\sqrt{x}}\, dx\, dt,$$

where result (59) in Lemma 11 has been employed and the Maclaurin series expansion for the arctangent function recognised. Enforcing a substitution of $x \mapsto x^2$ leads to

$$S_q = \frac{2(-1)^q}{(q-2)!} \int_0^1 \int_0^1 \frac{\log^{q-2}(t) \arctan(xt)}{(1+x^2)t}\, dx\, dt.$$

Integrating the inner x-integral by parts produces

$$S_q = \frac{4(-1)^{q+1}}{(q-1)!} \int_0^1 \int_0^1 \frac{x \log^{q-1}(t)}{(1+x^2)(1+x^2t^2)}\, dx\, dt$$
$$= \frac{2(-1)^{q+1}}{(q-1)!} \int_0^1 \frac{\log^{q-1}(t) \log\left(\frac{2}{1+t^2}\right)}{1-t^2}\, dt$$
$$= \frac{2(-1)^q}{(q-1)!} \int_0^1 \frac{\log^{q-1}(t) \log(1+t^2)}{1-t^2}\, dt - \frac{2(-1)^q \log(2)}{(q-1)!} \int_0^1 \frac{\log^{q-1}(t)}{1-t^2}\, dt.$$

The first of the integrals to the right of the equality is the integral J_q whose value is given by (55). The second is (30) with q replaced with $q-1$. The result then follows and completes the proof. □

EXAMPLE 3. The first five sums that result from setting $q = 2, 3, 4, 5$, and 6 in (60) are:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \left(H_{\frac{n}{2}} - H_{\frac{n-1}{2}}\right) = \frac{7}{2}\zeta(3) - \pi G;$$
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \left(H_{\frac{n}{2}} - H_{\frac{n-1}{2}}\right) = \frac{\pi^4}{32} - 2G^2;$$
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^4} \left(H_{\frac{n}{2}} - H_{\frac{n-1}{2}}\right) = \frac{31}{4}\zeta(5) - \frac{\pi^3}{8} G - \pi\beta(4);$$
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^5} \left(H_{\frac{n}{2}} - H_{\frac{n-1}{2}}\right) = \frac{\pi^6}{192} - 4G\beta(4);$$
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^6} \left(H_{\frac{n}{2}} - H_{\frac{n-1}{2}}\right) = \frac{381}{32}\zeta(7) - \frac{5\pi^5}{384} G - \frac{\pi^3}{8}\beta(4) - \pi\beta(6).$$

REMARK 6. The second of the sums appearing in Example 3 was recently given as a problem in [3].

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