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LOCATION OF ZEROS OF LACUNARY–TYPE POLYNOMIALS IN ANNULAR REGIONS

IRFAN AHMAD WANI*, MOHAMMAD IBRAHIM MIR AND ISHFAQ NAZIR

Abstract. In this paper, we place certain conditions on the real and imaginary parts of the coefficients of a lacunary-type polynomial and find the annular regions containing all the zeros of a lacunary-type polynomial.

1. Introduction

Various experimental observations and investigations when translated into mathematical language lead to mathematical models. The solution of these models could lead to problems of solving algebraic polynomial equations of a certain degree. The study of zeros of these algebraic complex polynomials is an old theme in the analytic theory of polynomials, and has spawned a vast amount of research over the past millennium including its applications both within and outside of mathematics. In addition to having numerous applications, this study has been the inspiration for much theoretical research (including being the initial motivation for modern algebra). Algebraic and analytic methods for finding zeros of a polynomial, in general, can be quite complicated, so it is desirable to put some restrictions on polynomials. This motivated the study of identifying suitable regions in the complex plane containing the zeros of a polynomial when their coefficients are restricted with special conditions. A classical result on the location of zeros of a polynomial by restricted coefficients known as Eneström-Kakeya theorem (see section 8.3 of [20]) is stated as

THEOREM 1. *If* $P(z)=\sum_{j=0}^{n} a_j z^j$, *is a polynomial of degree* n *and if the coefficients satisfy,* $0<a_0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$, *then* $P(z)$ *has all its zeros in* $|z| \leqslant 1$.

In literature (see [2]–[20]) there exist several generalizations of Eneström-Kakeya theorem. There is always a need for better and better results in this subject because of its application in many areas including signal processing, communication theory, cryptography, control theory, combinatorics, and mathematical biology. In this paper, by using standard techniques we establish an annular region in which zeros of a complex lacunary-type polynomial lie by putting certain restrictions on the real and imaginary parts of the complex coefficients of a given lacunary-type polynomial. In 1996, Aziz and Zargar [1] proved the following result for the regions containing the zeros of the lacunary-type polynomials. In fact they proved

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* Corresponding author.

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THEOREM 2. *If* $P(z) = a_n z^n + a_p z^p + \ldots + a_1 z + a_0$, $0 \leqslant p \leqslant n-1$ *is a polynomial of degree* n *and* $M = \max|\frac{a_j}{a_n}|$, $j = 0, 1, \ldots, p$, *then all the zeros of* $P(z)$ *lie in* $|z| < K$, *where* K *is a unique positive root of the trinomial equation*

$$x^{n-p} - x^{n-p-1} - M = 0.$$

In 1978, Datt and Govil [4], obtained a ring-shaped region containing all the zeros of $P(z)$. In fact they proved,

THEOREM 3. *If* $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$, $0 \leqslant p \leqslant n-1$ *is a polynomial of degree* n *and* $M = \max|\frac{a_j}{a_n}|$, $j = 0, 1, \ldots, n-1$, *then* $P(z)$ *has all its zeros in the ring shaped region*

$$\frac{|a_0|}{2(1+A)^{n-1}\{An+1\}} \leqslant |z| < 1 + \alpha_0 A,$$

where α_0 *is a unique positive root of the equation*

$$x = 1 - \frac{1}{(1+Ax)^n}$$

in the interval $(0, 1)$.

Aziz and Zargar [1], generalized Theorem 3 to the class of lucanary-type polynomials. Moreover, they proved,

THEOREM 4. *If* $P(z) = a_n z^n + a_p z^p + \ldots + a_1 z + a_0$, $0 \leqslant p \leqslant n-1$ *is a polynomial of degree* n *and* $M = \max|\frac{a_j}{a_n}|$, $j = 0, 1, \ldots, p$, *then all the zeros of* $P(z)$ *lie in the ring shaped region*

$$\frac{|a_0|}{2(1+A)^{n-1}\{1+(p+1)A\}} \leqslant |z| < 1 + \alpha_0 A,$$

where α_0 *is a unique positive root of the equation*

$$x = 1 - \frac{1}{(1+Ax)^{p+1}}$$

in the interval $(0, 1)$.

Recently A. Kumar, Z. Manzoor and B. A. Zagar [12] obtained an annular region containing the zeros of the lacunary-type polynomials with real coefficients, where these coefficients are subjected to Eneström-Kakeya theorem. In fact they proved

THEOREM 5. *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leqslant \mu \leqslant n$, $a_0 \neq 0$ *is a polynomial of degree* n, *where for some* $t > 0$ *and some* $\mu \leqslant k \leqslant n$,

$$t^{\mu}|a_{\mu}| \geqslant \ldots \geqslant t^{k-1}|a_{k-1}| \geqslant t^k|a_k| \leqslant t^{k+1}|a_{k+1}| \leqslant \ldots \leqslant t^{n-1}|a_{n-1}| \leqslant t^n|a_n|$$

and $|\arg a_j - \beta| \leqslant \alpha \leqslant \frac{\pi}{2}$ *for* $\mu \leqslant j \leqslant n$ *and some* α *and* β*, then all the zeros of* $P(z)$ *lie in the region defined by the following inequalities*

$$\min(r_2,t) \leqslant |z| \leqslant \max(r_1,t^{-1}),$$

where

$$r_1 = \frac{2M^2}{t^2|ta_n - a_{n-1}|(|a_n|-M) + (t^4|ta_n-a_{n-1}|^2(|a_n|-M)^2 + 4|a_n|t^2M^3)^{\frac{1}{2}}},$$

$$r_2 = \frac{1}{2M_1^2}\left[(t^2|a_0|-M_1)t^2|a_0| + \left(\{t^2|a_0|-M_1)t^2|a_0|\}^2 + 4M_1^3|a_0|t^3\right)^{\frac{1}{2}}\right],$$

$$\begin{aligned} M = {} & t^{n-k}(1+t^2)(|a_0|t^k - |a_k|\cos\alpha) + t^{n-\mu}|a_\mu|(t^2+\cos\alpha+\sin\alpha) \\ & + t^2|a_n|(\cos\alpha+\sin\alpha) + (1-t^2)\cos\alpha\sum_{j=\mu+1}^{k-1}|a_j|t^{n-j} \\ & + (t^2-1)\cos\alpha\sum_{j=k+1}^{n-1}|a_j|t^{n-j} + (1+t^2)\sin\alpha\sum_{j=\mu+1}^{n-1}|a_j|t^{n-j}, \end{aligned}$$

$$\begin{aligned} M_1 = {} & |a_0|t + (1+\cos\alpha+\sin\alpha)(a_\mu|t^{\mu+1} + |a_n|t^{n+1}) - 2\cos\alpha|a_k|t^{k+1} \\ & + 2\sin\alpha\sum_{j=\mu+1}^{n-1}|a_j|t^{n-j}. \end{aligned}$$

For more literature in this direction (see [2], [3], [13], [16], [19]).

2. Main results

In this paper, we obtain an annular regions in which zeros of a complex lacunary-type polynomial lie by imposing certain restrictions on the real and imaginary parts of the complex coefficients of a given lacunary-type polynomial. In fact, we prove the following result:

THEOREM 6. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$*,* $1 \leqslant \mu \leqslant n$*,* $a_0 \neq 0$ *be a polynomial of degree n with complex coefficients. Let* $Re(a_j) = \alpha_j$*,* $Im(a_j) = \beta_j$ *for* $j = 0,1,2,\ldots,n$ *and for some* $t \geqslant 0$*,* $\mu \leqslant k \leqslant n$*,* $\mu \leqslant l \leqslant n$*,*

$$t^\mu\alpha_\mu \geqslant \ldots \geqslant t^{k-1}\alpha_{k-1} \geqslant t^k\alpha_k \leqslant t^{k+1}\alpha_{k+1} \leqslant \ldots \leqslant t^{n-1}\alpha_{n-1} \leqslant t^n\alpha_n$$

and

$$t^\mu\beta_\mu \geqslant \ldots \geqslant t^{l-1}\beta_{l-1} \geqslant t^l\beta_l \leqslant t^{l+1}\beta_{l+1} \leqslant \ldots \leqslant t^{n-1}\beta_{n-1} \leqslant t^n\beta_n,$$

then all the zeros of $P(z)$ *lie in* $r_1 \leqslant |z| \leqslant r_2$, *where* $r_1 = \min\left\{\frac{t^2|a_0|}{M_1}, t\right\}$ *and* $r_2 = \max\left\{\frac{M_2}{|a_n|}, t^{-1}\right\}$, *where*

$$M_1 = |a_0|t + (\alpha_\mu + \beta_\mu + |a_\mu|)t^{\mu+1} - 2(t^{k+1}\alpha_k + t^{l+1}\beta_l) + t^{n+1}(\alpha_n + \beta_n + |a_n|),$$

$$\begin{aligned} M_2 = {} & |a_0|(t^2+1)t^{n-1} + |a_\mu|t^{n-\mu+1} + (\alpha_\mu + \beta_\mu)t^{n-\mu-1} + t(\alpha_n + \beta_n) - (t^2+1) \\ & (t^{n-k-1}\alpha_k + t^{n-l-1}\beta_l) + (1-t^2)\left[\sum_{j=\mu+1}^{k-1} t^{n-j-1}\alpha_j + \sum_{j=\mu+1}^{l-1} t^{n-j-1}\beta_j\right] \\ & + (t^2-1)\left[\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=l+1}^{n-1} t^{n-j-1}\beta_j\right]. \end{aligned}$$

REMARK 1. For $t = 1$, Theorem 6 reduces to Corollary 1 and for $\beta = 0$, Theorem 6 reduces to Corollary 2.

COROLLARY 1. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leqslant \mu \leqslant n$, $a_0 \neq 0$ *be a polynomial of degree* n *with complex coefficients. Let* $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$ *and for some* $t \geqslant 0$, $\mu \leqslant k \leqslant n$, $\mu \leqslant l \leqslant n$,

$$\alpha_\mu \geqslant \ldots \geqslant \alpha_{k-1} \geqslant \alpha_k \leqslant \alpha_{k+1} \leqslant \ldots \leqslant \alpha_{n-1} \leqslant \alpha_n$$

and

$$\beta_\mu \geqslant \ldots \geqslant \beta_{l-1} \geqslant \beta_l \leqslant \beta_{l+1} \leqslant \ldots \leqslant \beta_{n-1} \leqslant \beta_n,$$

then all the zeros of $P(z)$ *lie in*

$$\begin{aligned} & \frac{|a_0|}{|a_0| + (\alpha_\mu + \beta_\mu + |a_\mu|) - 2(\alpha_k + \beta_l) + (\alpha_n + \beta_n + |a_n|)} \\ & \leqslant |z| \leqslant \frac{2|a_0| + (\alpha_\mu + \beta_\mu + |a_\mu|) - 2(\alpha_k + \beta_l) + (\alpha_n + \beta_n)}{|a_n|}. \end{aligned}$$

COROLLARY 2. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leqslant \mu \leqslant n$, $a_0 \neq 0$ *be a polynomial of degree* n *with real coefficients, for some* $t \geqslant 0$, $\mu \leqslant k \leqslant n$,

$$t^\mu a_\mu \geqslant \ldots \geqslant t^{k-1}a_{k-1} \geqslant t^k a_k \leqslant t^{k+1}a_{k+1} \leqslant \ldots \leqslant t^{n-1}a_{n-1} \leqslant t^n a_n$$

then all the zeros of $P(z)$ *lie in* $r_3 \leqslant |z| \leqslant r_4$, *where* $r_3 = \min\left\{\frac{t^2|a_0|}{M_3}, t\right\}$ *and* $r_4 = \max\left\{\frac{M_4}{|a_n|}, t^{-1}\right\}$, *where*

$$M_3 = |a_0|t + (a_\mu + |a_\mu|)t^{\mu+1} - 2t^{k+1}a_k + t^{n+1}(a_n + |a_n|),$$

$$\begin{aligned} M_4 = {} & |a_0|(t^2+1)t^{n-1} + |a_\mu|t^{n-\mu+1} + a_\mu t^{n-\mu-1} + t a_n - (t^2+1)t^{n-k-1}a_k \\ & + (1-t^2)\sum_{j=\mu+1}^{k-1} t^{n-j-1}a_j + (t^2-1)\sum_{j=k+1}^{n-1} t^{n-j-1}a_j. \end{aligned}$$

Next, we prove the following result by rearranging coefficients in Theorem 6.

THEOREM 7. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leqslant \mu \leqslant n$, $a_0 \neq 0$ *be a polynomial of degree* n *with complex coefficients. Let* $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$ *and for some* $t \geqslant 0$, $\mu \leqslant k \leqslant n$, $\mu \leqslant l \leqslant n$,

$$t^{\mu}\alpha_{\mu} \geqslant \ldots \geqslant t^{k-1}\alpha_{k-1} \geqslant t^{k}\alpha_{k} \leqslant t^{k+1}\alpha_{k+1} \leqslant \ldots \leqslant t^{n-1}\alpha_{n-1} \leqslant t^{n}\alpha_{n}$$

and

$$t^{\mu}\beta_{\mu} \leqslant \ldots \leqslant t^{l-1}\beta_{l-1} \leqslant t^{l}\beta_{l} \geqslant t^{l+1}\beta_{l+1} \geqslant \ldots \geqslant t^{n-1}\beta_{n-1} \geqslant t^{n}\beta_{n},$$

then all the zeros of $P(z)$ *lie in* $s_1 \leqslant |z| \leqslant s_2$, *where* $s_1 = \min\left\{\frac{t^2|a_0|}{N_1}, t\right\}$ *and* $s_2 = \max\left\{\frac{N_2}{|a_n|}, t^{-1}\right\}$, *where*

$$N_1 = |a_0|t + (\alpha_{\mu} - \beta_{\mu} + |a_{\mu}|)t^{\mu+1} - 2(t^{k+1}\alpha_k - t^{l+1}\beta_l) + t^{n+1}(\alpha_n - \beta_n + |a_n|),$$

$$\begin{aligned} N_2 = {} & |a_0|(t^2+1)t^{n-1} + |a_{\mu}|t^{n-\mu+1} + (\alpha_{\mu} - \beta_{\mu})t^{n-\mu-1} + t(\alpha_n - \beta_n) - (t^2+1) \\ & (t^{n-k-1}\alpha_k - t^{n-l-1}\beta_l) + (1-t^2)\left[\sum_{j=\mu+1}^{k-1} t^{n-j-1}\alpha_j - \sum_{j=\mu+1}^{l-1} t^{n-j-1}\beta_j\right] \\ & + (t^2-1)\left[\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j - \sum_{j=l+1}^{n-1} t^{n-j-1}\beta_j\right]. \end{aligned}$$

REMARK 2. For $t = 1$, Theorem 7 reduces to Corollary 3.

COROLLARY 3. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leqslant \mu \leqslant n$, $a_0 \neq 0$ *be a polynomial of degree* n *with complex coefficients. Let* $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$ *and for some* $t \geqslant 0$, $\mu \leqslant k \leqslant n$, $\mu \leqslant l \leqslant n$,

$$\alpha_{\mu} \geqslant \ldots \geqslant \alpha_{k-1} \geqslant \alpha_k \leqslant \alpha_{k+1} \leqslant \ldots \leqslant \alpha_{n-1} \leqslant \alpha_n$$

and

$$\beta_{\mu} \leqslant \ldots \leqslant \beta_{l-1} \leqslant \beta_l \geqslant \beta_{l+1} \geqslant \ldots \geqslant \beta_{n-1} \geqslant \beta_n,$$

then all the zeros of $P(z)$ *lie in*

$$\begin{aligned} & \frac{|a_0|}{|a_0| + (\alpha_{\mu} - \beta_{\mu} + |a_{\mu}|) - 2(\alpha_k - \beta_l) + (\alpha_n - \beta_n + |a_n|)} \\ & \leqslant |z| \leqslant \frac{2|a_0| + (\alpha_{\mu} - \beta_{\mu} + |a_{\mu}|) - 2(\alpha_k - \beta_l) + (\alpha_n - \beta_n)}{|a_n|}. \end{aligned}$$

Next, we prove the following result by rearranging coefficients in Theorem 6 and Theorem 7.

THEOREM 8. *Let $P(z)=a_0+\sum_{j=\mu}^{n} a_j z^j,\ 1\leqslant \mu\leqslant n,\ a_0\neq 0$ be a polynomial of degree n with complex coefficients. Let $Re(a_j)=\alpha_j,\ Im(a_j)=\beta_j$ for $j=0,1,2,\ldots,n$ and for some $t\geqslant 0,\ \mu\leqslant k\leqslant n,\ \mu\leqslant l\leqslant n$,*

$$t^{\mu}\alpha_{\mu}\leqslant\ldots\leqslant t^{k-1}\alpha_{k-1}\leqslant t^{k}\alpha_{k}\geqslant t^{k+1}\alpha_{k+1}\geqslant\ldots\geqslant t^{n-1}\alpha_{n-1}\geqslant t^{n}\alpha_{n}$$

and

$$t^{\mu}\beta_{\mu}\geqslant\ldots\geqslant t^{l-1}\beta_{l-1}\geqslant t^{l}\beta_{l}\leqslant t^{l+1}\beta_{l+1}\leqslant\ldots\leqslant t^{n-1}\beta_{n-1}\leqslant t^{n}\beta_{n},$$

then all the zeros of $P(z)$ lie in $w_1\leqslant|z|\leqslant w_2$, where $w_1=\min\left\{\frac{t^2|a_0|}{O_1},t\right\}$ and $w_2=\max\left\{\frac{O_2}{|a_n|},t^{-1}\right\}$, where

$$O_1=|a_0|t+(\beta_{\mu}-\alpha_{\mu}+|a_{\mu}|)t^{\mu+1}+2(t^{k+1}\alpha_k-t^{l+1}\beta_l)-t^{n+1}(\alpha_n-\beta_n-|a_n|),$$

$$\begin{aligned}O_2=&|a_0|(t^2+1)t^{n-1}+|a_{\mu}|t^{n-\mu+1}+(\beta_{\mu}-\alpha_{\mu})t^{n-\mu-1}-t(\alpha_n-\beta_n)+(t^2+1)\\&(t^{n-k-1}\alpha_k-t^{n-l-1}\beta_l)+(t^2-1)\left[\sum_{j=\mu+1}^{k-1}t^{n-j-1}\alpha_j-\sum_{j=\mu+1}^{l-1}t^{n-j-1}\beta_j\right]\\&+(1-t^2)\left[\sum_{j=k+1}^{n-1}t^{n-j-1}\alpha_j-\sum_{j=l+1}^{n-1}t^{n-j-1}\beta_j\right].\end{aligned}$$

REMARK 3. For $t=1$, Theorem 8 reduces to Corollary 4 and for $\beta=0$, Theorem 8, it reduces to Corollary 5.

COROLLARY 4. *Let $P(z)=a_0+\sum_{j=\mu}^{n} a_j z^j,\ 1\leqslant \mu\leqslant n,\ a_0\neq 0$ be a polynomial of degree n with complex coefficients. Let $Re(a_j)=\alpha_j,\ Im(a_j)=\beta_j$ for $j=0,1,2,\ldots,n$ and for some $t\geqslant 0,\ \mu\leqslant k\leqslant n,\ \mu\leqslant l\leqslant n$,*

$$\alpha_{\mu}\leqslant\ldots\leqslant\alpha_{k-1}\leqslant\alpha_{k}\geqslant\alpha_{k+1}\geqslant\ldots\geqslant\alpha_{n-1}\geqslant\alpha_{n}$$

and

$$\beta_{\mu}\geqslant\ldots\geqslant\beta_{l-1}\geqslant\beta_{l}\leqslant\beta_{l+1}\leqslant\ldots\leqslant\beta_{n-1}\leqslant\beta_{n},$$

then all the zeros of $P(z)$ lie in

$$\frac{|a_0|}{|a_0|-(\alpha_{\mu}-\beta_{\mu}-|a_{\mu}|)+2(\alpha_k-\beta_l)+(\alpha_n-\beta_n-|a_n|)}\leqslant|z|\leqslant\frac{2|a_0|-(\alpha_{\mu}-\beta_{\mu}-|a_{\mu}|)+2(\alpha_k-\beta_l)-(\alpha_n-\beta_n)}{|a_n|}.$$

COROLLARY 5. *Let $P(z)=a_0+\sum_{j=\mu}^{n} a_j z^j,\ 1\leqslant \mu\leqslant n,\ a_0\neq 0$ be a polynomial of degree n with real coefficients, for some $t\geqslant 0,\ \mu\leqslant k\leqslant n$,*

$$t^{\mu}a_{\mu}\leqslant\ldots\leqslant t^{k-1}a_{k-1}\leqslant t^{k}a_{k}\geqslant t^{k+1}a_{k+1}\geqslant\ldots\geqslant t^{n-1}a_{n-1}\geqslant t^{n}a_{n}$$

then all the zeros of $P(z)$ *lie in* $w_3 \leqslant |z| \leqslant w_4$, *where* $w_3 = \min\left\{\frac{t^2|a_0|}{O_3}, t\right\}$ *and* $w_4 = \max\left\{\frac{O_4}{|a_n|}, t^{-1}\right\}$, *where*

$$O_3 = |a_0|t - (a_\mu + |a_\mu|)t^\mu + 2t^{k+1}a_k - t^{n+1}(a_n - |a_n|),$$

$$O_4 = |a_0|(t^2+1)t^{n-1} + |a_\mu|t^{n-\mu+1} - a_\mu t^{n-\mu-1} - ta_n + (t^2+1)t^{n-k-1}a_k + (t^2-1)\sum_{j=\mu+1}^{k-1} t^{n-j-1}a_j + (1-t^2)\sum_{j=k+1}^{n-1} t^{n-j-1}a_j.$$

3. Proof of main results

Proof of Theorem 6. Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leqslant \mu \leqslant n$, $a_0 \neq 0$ be a polynomial of degree n with complex coefficients such that $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ for $j = 0,1,2,\ldots,n$. Consider the polynomial

$$\begin{aligned} Q(z) &= (t-z)P(z) \\ &= a_0(t-z) + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1} \\ &= a_0 t - a_0 z + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1} \\ &= a_0 t - R_1(z) \end{aligned}$$

where

$$R_1(z) = a_0 z - ta_\mu z^\mu - \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j + a_n z^{n+1}.$$

On $|z| = t$,

$$\begin{aligned} |R_1(z)| &\leqslant |a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{n} |ta_j - a_{j-1}|t^j + |a_n|t^{n+1} \\ &\leqslant |a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k} (\alpha_{j-1} - t\alpha_j)t^j + \sum_{j=k+1}^{n} (t\alpha_j - \alpha_{j-1})t^j \\ &\quad + \sum_{j=\mu+1}^{l} (\beta_{j-1} - t\beta_j)t^j + \sum_{j=l+1}^{n} (t\beta_j - \beta_{j-1})t^j + |a_n|t^{n+1} \\ &= |a_0|t + |a_\mu|t^{\mu+1} + (\alpha_\mu + \beta_\mu)t^{\mu+1} - 2(t^{k+1}\alpha_k + t^{l+1}\beta_l) + (\alpha_n + \beta_n + |a_n|)t^{n+1} \\ &= |a_0|t + (\alpha_\mu + \beta_\mu + |a_\mu|)t^{\mu+1} - 2(t^{k+1}\alpha_k + t^{l+1}\beta_l) + (\alpha_n + \beta_n + |a_n|)t^{n+1} \\ &= M_1. \end{aligned}$$

Applying Schwarz Lemma [15] to $R_1(z)$, we get for $|z| \leqslant t$,

$$|R_1(z)| \leqslant \frac{M_1|z|}{t}$$

which implies for $|z| \leqslant t$,

$$|Q(z)| = |a_0 t - R_1(z)| \geqslant t|a_0| - |R_1(z)| \geqslant t|a_0| - \frac{M_1|z|}{t}.$$

Hence, if $|z| < r_1 = \min\left\{\frac{t^2|a_0|}{M_1}, t\right\}$ then $Q(z) \neq 0$ and so $P(z) \neq 0$.

Next we show that $P(z) \neq 0$ for $|z| > r_2$. For this, we again consider the polynomial

$$\begin{aligned} Q(z) &= (t-z)P(z) \\ &= a_0(t-z) + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1} \\ &= a_0 t - a_0 z + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1} \\ &= R_2(z) - a_n z^{n+1} \end{aligned}$$

where

$$R_2(z) = a_0 t - a_0 z + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j.$$

Now

$$\left| z^n R_2\left(\frac{1}{z}\right) \right| = \left| a_0 t z^n - a_0 z^{n-1} + ta_\mu z^{n-\mu} + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1}) z^{n-j} \right|.$$

On $|z| = t$,

$$\begin{aligned} \left| z^n R_2\left(\frac{1}{z}\right) \right| &\leqslant |a_0|t^{n+1} + |a_0|t^{n-1} + |a_\mu|t^{n-\mu+1} + \sum_{j=\mu+1}^{n} |ta_j - a_{j-1}|t^{n-j} \\ &\leqslant |a_0|t^{n-1}(t^2+1) + |a_\mu|t^{n-\mu+1} + \sum_{j=\mu+1}^{k} (\alpha_{j-1} - t\alpha_j)t^{n-j} \\ &\quad + \sum_{j=k+1}^{n} (t\alpha_j - \alpha_{j-1})t^{n-j} + \sum_{j=\mu+1}^{l} (\beta_{j-1} - t\beta_j)t^{n-j} + \sum_{j=l+1}^{n} (t\beta_j - \beta_{j-1})t^{n-j} \\ &= |a_0|(t^2+1)t^{n-1} + |a_\mu|t^{n-\mu+1} + (\alpha_\mu + \beta_\mu)t^{n-\mu-1} + t(\alpha_n + \beta_n) \\ &\quad - (t^2+1)(t^{n-k-1}\alpha_k + t^{n-l-1}\beta_l) \end{aligned}$$

$$+(1-t^2)\left[\sum_{j=\mu+1}^{k-1} t^{n-j-1}\alpha_j + \sum_{j=\mu+1}^{l-1} t^{n-j-1}\beta_j\right]$$
$$+(t^2-1)\left[\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=l+1}^{n-1} t^{n-j-1}\beta_j\right]$$
$$= M_2.$$

Thus by the Maximum Modulus Theorem [15] for $|z| \leqslant t$,

$$\left|z^n R_2\left(\frac{1}{z}\right)\right| \leqslant M_2,$$

which implies for $|z| \geqslant \frac{1}{t}$,

$$|R_2(z)| \leqslant M_2|z|^n$$

which follows that for $|z| \geqslant \frac{1}{t}$,

$$|Q(z)| = |R_2(z) - a_n z^{n+1}| \geqslant |a_n||z|^{n+1} - M_2|z|^n = |z|^n(|a_n||z| - M_2).$$

Hence, if $|z| > r_2 = \max\left\{\frac{M_2}{|a_n|}, t^{-1}\right\}$, then $Q(z) \neq 0$ and so $P(z) \neq 0$.

That proves the Theorem 6 completely. □

Proof of Theorem 7. Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leqslant \mu \leqslant n$, $a_0 \neq 0$ be a polynomial of degree n with complex coefficients such that $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ for $j = 0, 1, 2, \ldots, n$. Consider the polynomial

$$Q(z) = (t-z)P(z)$$
$$= a_0(t-z) + t a_\mu z^\mu + \sum_{j=\mu+1}^{n} (t a_j - a_{j-1}) z^j - a_n z^{n+1}$$
$$= a_0 t - a_0 z + t a_\mu z^\mu + \sum_{j=\mu+1}^{n} (t a_j - a_{j-1}) z^j - a_n z^{n+1}$$
$$= a_0 t - R_1(z)$$

where

$$R_1(z) = a_0 z - t a_\mu z^\mu - \sum_{j=\mu+1}^{n} (t a_j - a_{j-1}) z^j + a_n z^{n+1}.$$

On $|z| = t$,

$$|R_1(z)| \leqslant |a_0|t + |a_\mu| t^{\mu+1} + \sum_{j=\mu+1}^{n} |t a_j - a_{j-1}| t^j + |a_n| t^{n+1}$$
$$\leqslant |a_0|t + |a_\mu| t^{\mu+1} + \sum_{j=\mu+1}^{k} (\alpha_{j-1} - t\alpha_j) t^j + \sum_{j=k+1}^{n} (t\alpha_j - \alpha_{j-1}) t^j$$
$$+ \sum_{j=\mu+1}^{l} (\beta_{j-1} - t\beta_j) t^j + \sum_{j=l+1}^{n} (t\beta_j - \beta_{j-1}) t^j + |a_n| t^{n+1}$$

$$
\begin{aligned}
&= |a_0|t + |a_\mu|t^{\mu+1} + (\alpha_\mu - \beta_\mu)t^{\mu+1} - 2(t^{k+1}\alpha_k - t^{l+1}\beta_l) + (\alpha_n - \beta_n + |a_n|)t^{n+1} \\
&= |a_0|t + (\alpha_\mu - \beta_\mu + |a_\mu|)t^{\mu+1} - 2(t^{k+1}\alpha_k - t^{l+1}\beta_l) + t^{n+1}(\alpha_n - \beta_n + |a_n|) \\
&= N_1.
\end{aligned}
$$

Applying Schwarz Lemma [15] to $R_1(z)$, we get for $|z| \leqslant t$,

$$
|R_1(z)| \leqslant \frac{N_1|z|}{t}
$$

which implies for $|z| \leqslant t$,

$$
|Q(z)| = |a_0 t - R_1(z)| \geqslant t|a_0| - |R_1(z)| \geqslant t|a_0| - \frac{N_1|z|}{t}.
$$

Hence, if $|z| < r_1 = \min\left\{\frac{t^2|a_0|}{N_1}, t\right\}$ then $Q(z) \neq 0$ and so $P(z) \neq 0$.

Next we show that $P(z) \neq 0$ for $|z| > r_2$. For this, we again consider the polynomial

$$
\begin{aligned}
Q(z) &= (t-z)P(z) \\
&= a_0(t-z) + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1} \\
&= a_0 t - a_0 z + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1} \\
&= R_2(z) - a_n z^{n+1}
\end{aligned}
$$

where

$$
R_2(z) = a_0 t - a_0 z + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j.
$$

Now

$$
\left|z^n R_2\left(\frac{1}{z}\right)\right| = \left|a_0 t z^n - a_0 z^{n-1} + ta_\mu z^{n-\mu} + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^{n-j}\right|.
$$

On $|z| = t$,

$$
\begin{aligned}
\left|z^n R_2\left(\frac{1}{z}\right)\right| &\leqslant |a_0|t^{n+1} + |a_0|t^{n-1} + |a_\mu|t^{n-\mu+1} + \sum_{j=\mu+1}^{n} |ta_j - a_{j-1}|t^{n-j} \\
&\leqslant |a_0|t^{n-1}(t^2+1) + |a_\mu|t^{n-\mu+1} + \sum_{j=\mu+1}^{k} (\alpha_{j-1} - t\alpha_j)t^{n-j} \\
&\quad + \sum_{j=k+1}^{n} (t\alpha_j - \alpha_{j-1})t^{n-j} + \sum_{j=\mu+1}^{l} (\beta_{j-1} - t\beta_j)t^{n-j} + \sum_{j=l+1}^{n} (t\beta_j - \beta_{j-1})t^{n-j}
\end{aligned}
$$

$$
\begin{aligned}
&= |a_0|(t^2+1)t^{n-1} + |a_\mu|t^{n-\mu+1} + (\alpha_\mu - \beta_\mu)t^{n-\mu-1} + t(\alpha_n - \beta_n) \\
&\quad - (t^2+1)(t^{n-k-1}\alpha_k - t^{n-l-1}\beta_l) \\
&\quad + (1-t^2)\left[\sum_{j=\mu+1}^{k-1} t^{n-j-1}\alpha_j - \sum_{j=\mu+1}^{l-1} t^{n-j-1}\beta_j\right] \\
&\quad + (t^2-1)\left[\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j - \sum_{j=l+1}^{n-1} t^{n-j-1}\beta_j\right]. \\
&= N_2.
\end{aligned}
$$

Thus by the Maximum Modulus Theorem [15] for $|z| \leqslant t$,

$$\left|z^n R_2\left(\frac{1}{z}\right)\right| \leqslant N_2,$$

which implies for $|z| \geqslant \frac{1}{t}$,

$$|R_2(z)| \leqslant N_2|z|^n$$

which follows that for $|z| \geqslant \frac{1}{t}$,

$$|Q(z)| = |R_2(z) - a_n z^{n+1}| \geqslant |a_n||z|^{n+1} - N_2|z|^n = |z|^n(|a_n||z| - N_2).$$

Hence, if $|z| > r_2 = \max\left\{\frac{N_2}{|a_n|}, t^{-1}\right\}$, then $Q(z) \neq 0$ and so $P(z) \neq 0$.

That proves the Theorem 2.4 completely. □

Proof of Theorem 8. Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leqslant \mu \leqslant n$, $a_0 \neq 0$ be a polynomial of degree n with complex coefficients such that $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ for $j = 0, 1, 2, \ldots, n$. Consider the polynomial

$$
\begin{aligned}
Q(z) &= (t-z)P(z) \\
&= a_0(t-z) + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1} \\
&= a_0 t - a_0 z + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1} \\
&= a_0 t - R_1(z)
\end{aligned}
$$

where

$$R_1(z) = a_0 z - ta_\mu z^\mu - \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j + a_n z^{n+1}.$$

On $|z| = t$,

$$
\begin{aligned}
|R_1(z)| &\leqslant |a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{n} |ta_j - a_{j-1}|t^j + |a_n|t^{n+1} \\
&\leqslant |a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k} (\alpha_{j-1} - t\alpha_j)t^j + \sum_{j=k+1}^{n} (t\alpha_j - \alpha_{j-1})t^j \\
&\quad + \sum_{j=\mu+1}^{l} (\beta_{j-1} - t\beta_j)t^j + \sum_{j=l+1}^{n} (t\beta_j - \beta_{j-1})t^j + |a_n|t^{n+1} \\
&= |a_0|t + |a_\mu|t^{\mu+1} - (\alpha_\mu - \beta_\mu)t^{\mu+1} + 2(t^{k+1}\alpha_k - t^{l+1}\beta_l) - (\alpha_n - \beta_n - |a_n|)t^{n+1} \\
&= |a_0|t + (\beta_\mu - \alpha_\mu + |a_\mu|)t^{\mu+1} + 2(t^{k+1}\alpha_k - t^{l+1}\beta_l) - t^{n+1}(\alpha_n - \beta_n - |a_n|) \\
&= O_1.
\end{aligned}
$$

Applying Schwarz Lemma [15] to $R_1(z)$, we get for $|z| \leqslant t$,

$$
|R_1(z)| \leqslant \frac{O_1|z|}{t}
$$

which implies for $|z| \leqslant t$,

$$
|Q(z)| = |a_0t - R_1(z)| \geqslant t|a_0| - |R_1(z)| \geqslant t|a_0| - \frac{O_1|z|}{t}.
$$

Hence, if $|z| < r_1 = \min\left\{\frac{t^2|a_0|}{O_1}, t\right\}$ then $Q(z) \neq 0$ and so $P(z) \neq 0$.

Next we show that $P(z) \neq 0$ for $|z| > r_2$. For this, we again consider the polynomial

$$
\begin{aligned}
Q(z) &= (t-z)P(z) \\
&= a_0(t-z) + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1} \\
&= a_0t - a_0z + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1} \\
&= R_2(z) - a_n z^{n+1}
\end{aligned}
$$

where

$$
R_2(z) = a_0t - a_0z + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j.
$$

Now

$$
\left|z^n R_2\left(\frac{1}{z}\right)\right| = \left|a_0tz^n - a_0z^{n-1} + ta_\mu z^{n-\mu} + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^{n-j}\right|.
$$

On $|z| = t$,

$$\begin{aligned}
\left|z^n R_2\left(\frac{1}{z}\right)\right| &\leqslant |a_0|t^{n+1} + |a_0|t^{n-1} + |a_\mu|t^{n-\mu+1} + \sum_{j=\mu+1}^{n} |ta_j - a_{j-1}|t^{n-j} \\
&\leqslant |a_0|t^{n-1}(t^2+1) + |a_\mu|t^{n-\mu+1} + \sum_{j=\mu+1}^{k} (\alpha_{j-1} - t\alpha_j)t^{n-j} \\
&\quad + \sum_{j=k+1}^{n} (t\alpha_j - \alpha_{j-1})t^{n-j} + \sum_{j=\mu+1}^{l} (\beta_{j-1} - t\beta_j)t^{n-j} + \sum_{j=l+1}^{n} (t\beta_j - \beta_{j-1})t^{n-j} \\
&= |a_0|(t^2+1)t^{n-1} + |a_\mu|t^{n-\mu+1} + (\beta_\mu - \alpha_\mu)t^{n-\mu-1} - t(\alpha_n - \beta_n) \\
&\quad + (t^2+1)(t^{n-k-1}\alpha_k - t^{n-l-1}\beta_l) \\
&\quad + (t^2-1)\left[\sum_{j=\mu+1}^{k-1} t^{n-j-1}\alpha_j - \sum_{j=\mu+1}^{l-1} t^{n-j-1}\beta_j\right] \\
&\quad + (1-t^2)\left[\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j - \sum_{j=l+1}^{n-1} t^{n-j-1}\beta_j\right] \\
&= O_2.
\end{aligned}$$

Thus by the Maximum Modulus Theorem [15] for $|z| \leqslant t$,

$$\left|z^n R_2\left(\frac{1}{z}\right)\right| \leqslant O_2,$$

which implies for $|z| \geqslant \frac{1}{t}$,

$$|R_2(z)| \leqslant O_2|z|^n$$

which follows that for $|z| \geqslant \frac{1}{t}$,

$$|Q(z)| = |R_2(z) - a_n z^{n+1}| \geqslant |a_n||z|^{n+1} - N_2|z|^n = |z|^n(|a_n||z| - O_2).$$

Hence, if $|z| > r_2 = \max\left\{\frac{O_2}{|a_n|}, t^{-1}\right\}$, then $Q(z) \neq 0$ and so $P(z) \neq 0$.

That proves the Theorem 2.6 completely. □

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Irfan Ahmad Wani
Department of Mathematics
University of Kashmir, South Campus
Anantnag-192101, Jammu and Kashmir, India
e-mail: irfanmushtaq62@gmail.com

Mohammad Ibrahim Mir
Department of Mathematics
University of Kashmir, South Campus
Anantnag-192101, Jammu and Kashmir, India
e-mail: ibrahimmath80@gmail.com

Ishfaq Nazir
Department of Mathematics
University of Kashmir, South Campus
Anantnag-192101, Jammu and Kashmir, India
e-mail: ishfaqnazir02@gmail.com