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ON THE GENERALISATION OF HENSTOCK–KURZWEIL FOURIER TRANSFORM

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Abstract. In this paper, a generalised integral called the Laplace integral is defined on unbounded intervals, and some of its properties, including necessary and sufficient condition for differentiating under the integral sign, are discussed. It is also shown that this integral is more general than the Henstock-Kurzweil integral. Finally, the Fourier transform is defined using the Laplace integral, and its well-known properties are established.

1. Introduction

If $f\colon \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable, its Fourier transform is defined by $\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iyx}$, and its theory is also well established. Now the obvious question from the viewpoint of "generalised integrals" is "*can we replace the Lebesgue integral with a generalised integral in the definition of Fourier transform?*" It is Erik Talvila who first gave an affirmative answer to the above question in [18]. He used Henstock-Kurzweil integral to define the Fourier transform and proved its important properties. Furthermore, he pointed out that some beautiful results, e.g., Riemann-Lebesgue lemma, are not satisfied by the Henstock-Kurzweil Fourier transform. However, in [10], it is proved that the Riemann-Lebesgue lemma is satisfied in an appropriate subspace of the space of all Henstock-Kurzweil integrable functions on $\mathbb{R}$. Further results concerning Henstock-Kurzweil Fourier transform can be found in [2, 3, 9, 11, 19, 20].

Let us discuss some ambiguity we found in the proof of Lemma 25 of [18]. In the paragraph between equation (20) and equation (21) of [18], it is asserted that $F(x) = \int_{-\infty}^{x} f$, is ACG_* on $\mathbb{R}$; and then this fact is used to prove that $H_a \in ACG_*(\mathbb{R})$ (see the last line of [18, p. 1224]). At the end of the proof of Lemma 25 (a), it is proved that $H_{-\infty} \in ACG_*(\overline{\mathbb{R}})$ and then Corollary 6 of [17] is used to prove the Lemma 25 (a). So, what we have seen so far is that the key idea working implicitly in the proof of Lemma 25 (a) is

"f is Henstock integrable on $\mathbb{R}$ if and only if there is a $F \in ACG_*(\mathbb{R})$ such that $F' = f$ a.e. on $\mathbb{R}$."

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However, according to our best knowledge, there is no Denjoy type definition for the method of integration on infinite intervals (see [4, p. 297], [6], [7], [16]). So, we think the proof of Lemma 25 of [18] might needs to be rechecked.

The ambiguity we discussed in the previous paragraph has encouraged us to nurture the problem of defining Fourier transform using generalised integrals. In [8], a new generalised integral on bounded intervals called the Laplace integral is defined by the authors of this paper, which has continuous primitives and is more general than the Henstock-Kurzweil integral. In this paper, the concept of the Laplace integral is extended on $\mathbb{R}$, which is free of any variational condition, and then applied to define the Fourier transform. Moreover, the essential properties of the Fourier transform are studied in this general setting.

NOTATIONS. Let I be an interval bounded or unbounded. We use following notations throughout this paper.

$L^1(I) = \{f \mid f \text{ is Lebesgue integrable on } I\}$,

$\mathscr{HK}(I) = \{f \mid f \text{ is Henstock-Kurzweil integrable on } I\}$,

$\mathscr{BV}(I) = \{f \mid f \text{ is of bounded variation on } I\}$,

$\mathscr{BV}(\pm\infty) = \{f \mid f \text{ is of bounded variation on } \mathbb{R}\setminus(-a,a) \text{ for some } a \in \mathbb{R}\}$,

$V_I[f] = $ Total variation of f on I,

$V_I[f(\cdot,y)] = $ Total variation of f with respect to x on I,

$\|f\|_1 = $ The L^1 norm of f.

2. Preliminaries

DEFINITION 2.1. ([8]) Let f be Laplace integrable (see Definition 3.2 of [8]) on a neighbourhood of x. If $\exists\,\delta > 0$ such that the following limits

$$\lim_{s\to\infty} s\int_0^{\delta} e^{-st} f(x+t)\,dt \quad \text{and} \quad \lim_{s\to\infty} s\int_0^{\delta} e^{-st} f(x-t)\,dt$$

exist and are equal, then the common value is denoted by $LD_0 f(x)$. And we say f is Laplace continuous at x if $LD_0 f(x) = f(x)$.

DEFINITION 2.2. Let f be Laplace integrable on a neighbourhood of x. If $\exists\,\delta > 0$ such that the following limits

$$\lim_{s\to\infty} s^2\int_0^{\delta} e^{-st}[f(x+t)-f(x)]\,dt \quad \text{and} \quad \lim_{s\to\infty}(-s^2)\int_0^{\delta} e^{-st}[f(x-t)-f(x)]\,dt$$

exist and are equal, then we say f is Laplace differentiable at x and the common value is denoted by $LD_1 f(x)$.

If f is a function of two variables, say x and y, then we define the Laplace derivative of f with respect to x as the Laplace derivative of $f_y(x) = f(x,y)$ (y is assumed to be constant) and we denote it by $LD_{1x}f$. Definition of $LD_{1y}f$ is similar. For further properties on Laplace derivative and Laplace continuity see [12, 13, 14, 15].

3. The Laplace integral on unbounded intervals

In Definition 3.4 of [5], Denjoy-Perron type integrals on unbounded intervals are defined and then it is proved that it is equivalent to Henstock-Kurzweil integral on unbounded intervals (Theorem 3.2 of [5]). Similarly, we shall define the Laplace integral on unbounded intervals and establish its properties. Due to similarity, most of the results will be given without proof.

DEFINITION 3.1. Let $I = [a,\infty]$ and let $f : I \to \mathbb{R}$. Then we say f is Laplace integrable on $[a,\infty)$ or on $[a,\infty]$ if

(a) f is Laplace integrable on $[a,c]$ for $c \geqslant a$ and

(b) $\lim_{c\to\infty} \int_a^c f$ exists.

In this case we write $\int_a^\infty f = \lim_{c\to\infty} \int_a^c f$. The set of all Laplace integrable functions on $[a,\infty)$ will be denoted by $\mathscr{L}\mathscr{P}[a,\infty)$ or by $\mathscr{L}\mathscr{P}[a,\infty]$.

Integrability on $(-\infty,b]$ or on $[-\infty,b]$ can be defined analogously. We shall say that f is integrable on $\mathbb{R}$ or on $\overline{\mathbb{R}}$ ($= [-\infty,\infty]$) if there is some $a \in \mathbb{R}$ such that f is integrable on both $(-\infty,a]$ and $[a,\infty)$, and we write $\int_{-\infty}^\infty f = \int_{-\infty}^a f + \int_a^\infty f$. From now on, we assume all integrals are Laplace integral unless otherwise stated.

From Section 3 of [8] and Definition 3.4, Theorem 3.2 of [5], it is evident that Denjoy-Perron integral or Henstock-Kurzweil integral on unbounded intervals is a particular case of Laplace integral; however, the following example will ensure that the set $\mathscr{L}\mathscr{P}(\mathbb{R}) \setminus \mathscr{H}\mathscr{K}(\mathbb{R})$ is non-empty.

EXAMPLE 3.1. Let $f : [0,1] \to \mathbb{R}$ be the function defined in Equation (4.12) of [8] and let $\phi = LD_1 f$ on $[0,1]$ (see Theorem 4.2 of [8]). Now define

$$F(x) = \begin{cases} f(x) & \text{if } x \in [0,1] \\ 0 & \text{if } x \in \mathbb{R} \setminus [0,1]. \end{cases}$$

Then F is continuous and Laplace differentiable on $\mathbb{R}$. If we denote $LD_1 F$ by Φ, then we get

$$\Phi(x) = \begin{cases} \phi(x) & \text{if } x \in [0,1] \\ 0 & \text{if } x \in \mathbb{R} \setminus [0,1]. \end{cases}$$

By Theorem 4.3 of [8] and Definition 3.1, we get that $\Phi \in \mathscr{L}\mathscr{P}(\mathbb{R}) \setminus \mathscr{H}\mathscr{K}(\mathbb{R})$.

THEOREM 3.1. (Cauchy criterion) *Let $I = [a,\infty]$ and let $f : I \to \mathbb{R}$ be such that $f \in \mathscr{L}\mathscr{P}[a,c]$ for all $c \geqslant a$. Then $f \in \mathscr{L}\mathscr{P}(I)$ if and only if for any $\varepsilon > 0$ there is a $K(\varepsilon) \geqslant a$ such that $q > p \geqslant K(\varepsilon)$ implies $|\int_p^q f| \leqslant \varepsilon$.*

THEOREM 3.2. *Let $I = [a,\infty]$. Then*

(a) *if $f, g \in \mathscr{L}\mathscr{P}(I)$ and $c \in \mathbb{R}$, then $cf + g \in \mathscr{L}\mathscr{P}(I)$ and $\int_I (cf+g) = c\int_I f + \int_I g$.*

(b) *if* $f, g \in \mathscr{LP}(I)$ *and* $f \leqslant g$, *then* $\int_I f \leqslant \int_I g$.

(c) *if* $c \in I$ *and* $f \in \mathscr{LP}(I)$, *then* $\int_a^\infty f = \int_a^c f + \int_c^\infty f$.

THEOREM 3.3. (Fundamental theorem of calculus) *Let* $I = [a, \infty]$ *and let* $f : I \to \mathbb{R}$. *If* $f \in \mathscr{LP}(I)$ *and* $F(x) = \int_a^x f$, *then* $LD_1 F = f$ *a.e. on* I.

Proof. Let $I_0 = [a, a+1)$ and $I_n = [a+n, a+n+1)$, $n \in \mathbb{N}$. Then $I = \bigcup_{n=0}^\infty I_n$. Now apply Theorem 5.5 of [8] on each I_n. □

THEOREM 3.4. (Du Bois-Reymond's) *Let* $I = [a, \infty]$, *let* $f : I \to \mathbb{R}$ *and let* $g : I \to \mathbb{R}$. *If*

(a) $f \in \mathscr{LP}[a, c]$ *for all* $c \geqslant a$ *and* $F(x) = \int_a^x f$ *is bounded on* $[a, \infty]$,

(b) $g \in L^1(I) \cap \mathscr{BV}(I)$ *and* $G(x) = \int_a^x g$ *for all* $x \in I$,

(c) $\lim_{x \to \infty} F(x)G(x)$ *exists,*

then $fG \in \mathscr{LP}(I)$.

Proof. As F is bounded, $Fg \in L^1(I)$ which implies $\lim_{x\to\infty} \int_a^x Fg$ exists and equal to $\int_a^\infty Fg$. Let $x \in [a, \infty)$, then by Theorem 6.1 of [8], we have

$$\int_a^x fG = F(x)G(x) - \int_a^x Fg. \tag{3.1}$$

Now applying the last assumption on (3.1), we get $fG \in \mathscr{LP}(I)$. □

COROLLARY 3.1. (Integration by parts) *Let* $I = [a, \infty]$, *let* $f \in \mathscr{LP}(I)$, *let* $g \in L^1(I) \cap \mathscr{BV}(I)$ *and let* $G(x) = \int_a^x g$, $x \in I$. *Then* $fG \in \mathscr{LP}(I)$ *and*

$$\int_a^\infty fG = \lim_{x \to \infty} [F(x)G(x)] - \int_a^\infty Fg,$$

where $F(x) = \int_a^x f$.

LEMMA 3.1. *Let* $[a, b] \subseteq \mathbb{R}$, *let* $f \in \mathscr{LP}(\mathbb{R})$ *and let* $g \in \mathscr{BV}(\mathbb{R})$. *If* $G(x) = \int_a^x g$, *then*

$$\left| \int_a^b fG \right| \leqslant \left| \int_a^b f \right| \inf_{x \in [a,b]} |G(x)| + \|f\|_{[a,b]} V_{[a,b]}[G] \leqslant \left[\inf_{x \in [a,b]} |G(x)| + V_{[a,b]}[G] \right] \|f\|_{[a,b]}.$$

Proof is similar to that of Lemma 24 of [18]. The norm, $\|\cdot\|_{[a,b]}$, is the norm defined in [18], and it is equivalent to the Alexiewicz's norm (see Theorem 5.2).

THEOREM 3.5. *Let* $I=[a,\infty]$, *let* $f\in\mathscr{LP}(I)$, *let* (g_n) *be a sequence in* $L^1(I)\cap\mathscr{BV}(I)$ *and let* $G_n(x)=\int_a^x g_n$. *If* (g_n) *is of uniform bounded variation, then*

$$\lim_{n\to\infty}\int fG_n=\int f(\lim_{n\to\infty}G_n).$$

For the proof see Theorem 7.2 of [8].

LEMMA 3.2. *Let* $I=[a,\infty]$, *where* $a\in\mathbb{R}$. *Then* $\mathscr{LP}(I)\cap\mathscr{BV}(I)=\mathscr{HK}(I)\cap\mathscr{BV}(I)$.

Proof. It is enough to prove that $f\in\mathscr{LP}(I)\cap\mathscr{BV}(I)$ implies $f\in\mathscr{HK}(I)$. Note that $f\in\mathscr{LP}(I)\cap\mathscr{BV}(I)$ implies $f\in\mathscr{LP}([a,b])\cap\mathscr{BV}([a,b])$, where $a<b<\infty$. As bounded Laplace integrable functions on finite intervals are Lebesgue integrable (see Corollary 5.2 of [8]), $f\in\mathscr{HK}([a,b])$. Now, as

$$\lim_{b\to\infty}(\mathscr{HK})\int_a^b f=\lim_{b\to\infty}\int_a^b f=\int_a^\infty f,$$

Hake's theorem (Theorem 12.8 of [4, p. 291]) implies $f\in\mathscr{HK}(I)$. □

4. A necessary and sufficient condition of Laplace differentiation under the integral sign

One may notice that almost the whole paper [18] depends on Lemma 25 of [18]. Furthermore, Theorem 4 of [17] has a crucial role in proving that lemma. In this section, we also establish results similar to Theorem 4 of [17]. However, before that, we need to define the concept of a $\mathscr{LP}$-primitive.

DEFINITION 4.1. ($\mathscr{LP}$-primitive) Let $I=[a,b]\subseteq\overline{\mathbb{R}}$ and let $F:I\to\mathbb{R}$ be continuous. We say F is a $\mathscr{LP}$-primitive on I if LD_1F exists a.e. on I, $LD_1F\in\mathscr{LP}(I)$ and $\int_\alpha^\beta LD_1F=F(\beta)-F(\alpha)$ for $\alpha,\beta\in I$.

THEOREM 4.1. *Let* $I=[\alpha,\beta]\times[a,b]\subseteq\overline{\mathbb{R}}\times\overline{\mathbb{R}}$ *and let* $f:I\to\mathbb{R}$. *Suppose* $f(.,y)$ *is a* $\mathscr{LP}$*-primitive on* $[\alpha,\beta]$ *for a.e.* $y\in(a,b)$. *Then* $F(x)=\int_a^b f(x,y)\,dy$ *is a* $\mathscr{LP}$*-primitive and* $LD_1F(x)=\int_a^b LD_{1x}f(x,y)\,dy$ *for almost every* $x\in(\alpha,\beta)$ *if and only if*

$$\int_s^t\int_a^b LD_{1x}f(x,y)\,dydx=\int_a^b\int_s^t LD_{1x}f(x,y)\,dxdy \qquad \text{for all } s,t\in[\alpha,\beta]. \tag{4.1}$$

Proof. Let F be a $\mathscr{LP}$-primitive and $LD_1F(x)=\int_a^b LD_{1x}f(x,y)\,dy$ for almost every $x\in(\alpha,\beta)$. Let $s,t\in[\alpha,\beta]$. Then

$$\begin{aligned}\int_s^t\int_a^b LD_{1x}f(x,y)\,dydx&=F(t)-F(s)\\&=\int_a^b[f(t,y)-f(s,y)]\,dy=\int_a^b\int_s^t LD_{1x}f(x,y)\,dxdy.\end{aligned}$$

Conversely, let us assume (4.1) holds. Let $x_0 \in (\alpha,\beta)$ be fixed. Then for $x \in (\alpha,\beta)$, we get

$$\begin{aligned}\int_{x_0}^{x}\int_a^b LD_{1t}f(t,y)\,dydt &= \int_a^b\int_{x_0}^{x} LD_{1t}f(t,y)\,dtdy\\ &= \int_a^b [f(x,y)-f(x_0,y)]\,dy = F(x)-F(x_0).\end{aligned}$$

Hence by Theorem 3.3, we get F is a $\mathscr{L}\mathscr{P}$-primitive and $LD_1F(x) = \int_a^b LD_{1x}f(x,y)\,dy$ for almost every $x \in (\alpha,\beta)$. □

COROLLARY 4.1. *Let $I = [\alpha,\beta]\times[a,b] \subseteq \overline{\mathbb{R}}\times\overline{\mathbb{R}}$ and let $g\colon I \to \mathbb{R}$. Suppose $g(.,y) \in \mathscr{L}\mathscr{P}[\alpha,\beta]$ a.e. $y \in (a,b)$. Define $G(x) = \int_a^b\int_\alpha^x g(t,y)\,dtdy$. Then G is a $\mathscr{L}\mathscr{P}$-primitive and $LD_1G(x) = \int_a^b g(x,y)\,dy$ for a.e. $x \in (\alpha,\beta)$ if and only if*

$$\int_s^t\int_a^b g(x,y)\,dydx = \int_a^b\int_s^t g(x,y)\,dxdy \qquad \text{for } [s,t] \subseteq [\alpha,\beta].$$

5. Convolution

Let $f\colon \mathbb{R}\to\mathbb{R}$ and $g\colon \mathbb{R}\to\mathbb{R}$. Then the convolution $f*g$ of f and g is defined by

$$f*g(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy \qquad \text{for all } x \in \mathbb{R}, \tag{5.1}$$

provided the integral exists. We all know the basic properties of convolution while the integral is Lebesgue integral or Henstock-Kurzweil integral (see Section 3 of [18]). Here, we discuss similar properties when the integral is Laplace integral. However, before that, we need to prove the following Lemmas.

LEMMA 5.1. *Let $f \in \mathscr{L}\mathscr{P}[a,\infty]$. Then $\int_a^\infty f(x)\,dx = \int_{a-y}^\infty f(x+y)\,dx$.*

Proof. Let $\varepsilon > 0$ be given. Let U and V be respectively major and minor functions of f on $[a,b]$ with $V(a) = U(a) = 0$ and $0 \leqslant U(b)-V(b) \leqslant \varepsilon$, where $a < b < \infty$. Then it is quite obvious that $U\circ\phi$ and $V\circ\phi$ are respectively major and minor functions of $f\circ\phi$ on $[a-y,b-y]$ with $V\circ\phi(a-y) = U\circ\phi(a-y) = 0$ and $0 \leqslant U\circ\phi(b-y) - V\circ\phi(b-y) \leqslant \varepsilon$, where $\phi(x) = x+y$ for all $x \in [a-y,b-y]$. Thus if $F(x) = \int_a^x f(t)\,dt$ for $x \in [a,b]$, then $F\circ\phi(x) = \int_{a-y}^x f\circ\phi(t)\,dt$ for $x \in [a-y,b-y]$ which implies

$$\int_a^b f(t)\,dt = F(b) = F\circ\phi(b-y) = \int_{a-y}^{b-y} f(t+y)\,dt.$$

Now letting $b\to\infty$, we get the desired result. □

LEMMA 5.2. *Let $f \in \mathscr{LP}(\mathbb{R})$, let $G\colon \mathbb{R}^2 \to \mathbb{R}$ and let $\mathfrak{I}$ be the collection of all open intervals of $\mathbb{R}$. Moreover, let $\partial_x G(x,y) = g(x,y)$. Now define the iterated integrals*

$$\mathrm{I}_1(A,B) = \int\limits_{y\in B}\int\limits_{x\in A} f(x)G(x,y)\,dxdy,$$

$$\mathrm{I}_2(A,B) = \int\limits_{x\in A}\int\limits_{y\in B} f(x)G(x,y)\,dydx,$$

where $(A,B) \in \mathfrak{I}\times\mathfrak{I}$. If $A \in \mathfrak{I}$ is bounded and if

(a) *$V_O[g(\cdot,y)] \in L^1(B)$, for any interval $O \subseteq A$,*

(b) *$|g(x,y)| \leqslant \eta_J(y)$ and $|G(x,y)| \leqslant \kappa_J(y)$ for a.e. $(x,y) \in J\times B$, where J is any interval in A and η_J, $\kappa_J \in L^1(B)$,*

then $\mathrm{I}_1(A,B)$ exists, and $\mathrm{I}_1(A,B) = \mathrm{I}_2(A,B)$. In addition, if $\mathrm{I}_2(\mathbb{R},B)$ exists, then $\mathrm{I}_1(\mathbb{R},B) = \mathrm{I}_2(\mathbb{R},B)$.

Proof. Let J be any bounded interval and let $F(x) = \int_{-\infty}^{x} f$. Then by the second condition of this Lemma, we get $\int_J\int_{-\infty}^{\infty}|F(x)g(x,y)|\,dydx < \infty$, proving that $Fg \in L^1(J\times\mathbb{R})$ for any bounded interval J. So for $-\infty < a < t < b < \infty$ and $(\alpha,\beta) \subseteq \mathbb{R}$, we get

$$\int_{\alpha}^{\beta}\int_{a}^{t} F(x)g(x,y)\,dxdy = \int_{a}^{t}\int_{\alpha}^{\beta} F(x)g(x,y)\,dydx. \tag{5.2}$$

Let $((a,b),(\alpha,\beta)) \in \mathfrak{I}\times\mathfrak{I}$, where (a,b) is bounded. For $t \in (a,b)$, define

$$H_a(t) = \int_{\alpha}^{\beta}\int_{a}^{t} f(x)G(x,y)\,dxdy.$$

Then applying integration by parts and (5.2), we get

$$H_a(t) = F(t)\int_{\alpha}^{\beta} G(t,y)\,dy - \int_{a}^{t}\int_{\alpha}^{\beta} F(x)g(x,y)\,dydx. \tag{5.3}$$

Applying the second condition on (5.3), it can be proved that $H_a(t)$ is continuous on (a,b). Let $\phi(x) = \int_{\alpha}^{\beta} G(x,y)\,dy$. Then on any bounded interval J, ϕ is absolutely continuous. Furthermore, as $|\partial_x G(x,y)| = |g(x,y)| \leqslant \eta_J(y) \in L^1((\alpha,\beta))$, we get $\phi'(x) = \int_{\alpha}^{\beta} g(x,y)\,dy$. Thus by Corollary 6.1 of [8], we have $LD_1 H_a(t) = f(t)\int_{\alpha}^{\beta} G(t,y)\,dy = f(t)\phi(t)$ for a.e. $t \in (a,b)$. Let $P := \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a,b]$. Then by the first condition, we get

$$\sum_{i=0}^{n-1}\left|\phi'(x_{i+1}) - \phi'(x_i)\right| \leqslant \int_{\alpha}^{\beta}\sum_{i=0}^{n-1}|g(x_{i+1},y) - g(x_i,y)|\,dy \leqslant \int_{\alpha}^{\beta} V_{[a,b]}[g(\cdot,y)]\,dy < \infty,$$

proving that $\phi' \in \mathscr{BV}[a,b]$. Thus $f\phi \in \mathscr{LP}[a,b]$. Moreover, integrating $f\phi$, we can prove that H_a is one of its primitive. Now Theorem 4.1 implies that $\mathrm{I}_1(J,(\alpha,\beta)) = \mathrm{I}_2(J,(\alpha,\beta))$, where $(J,(\alpha,\beta)) \in \mathfrak{I}\times\mathfrak{I}$ and J is bounded.

As it is assumed that $I_2(\mathbb{R},B)$ exists, for $a\in\mathbb{R}$, we have

$$\begin{aligned}\int_{[a,\infty]}\int_B f(x)G(x,y)\,dydx &= \lim_{t\to\infty}\int_{[a,t]}\int_B f(x)G(x,y)\,dydx\\ &= \lim_{t\to\infty}\int_B\int_{[a,t]} f(x)G(x,y)\,dxdy.\end{aligned}\tag{5.4}$$

Thus $\lim_{t\to\infty} H_a(t)$ exists. Define

$$H_a^*(t)=\begin{cases}H_a(t) & \text{if } a\leqslant t<\infty,\\ \lim_{x\to\infty} H_a(x) & \text{if } t=\infty.\end{cases}\tag{5.5}$$

Then H_a^* is continuous on $[a,\infty]$ and

$$LD_1H_a^*(t)=LD_1H_a(t)=\int_B f(t)G(t,y)\,dy \qquad \text{for a.e. } t\in\mathbb{R}.$$

Moreover, existence of $I_2(\mathbb{R},B)$ implies that $\int_a^\infty LD_1H_a^*(t)\,dt$ exists. Now as H_a is a $\mathscr{LP}$-primitive on every bounded intervals in $\mathbb{R}$, applying (5.4) and (5.5) we have

$$\int_\alpha^\beta LD_1H_a^*(t)\,dt=H_a^*(\beta)-H_a^*(\alpha) \qquad \text{for all } \alpha,\beta\in[a,\infty]$$

which implies that H_a^* is a $\mathscr{LP}$-primitive. Therefore, Corollary 4.1 implies that

$$I_1((a,\infty),B)=I_2((a,\infty),B).$$

Similarly, we can prove that $I_1((-\infty,a),B)=I_2((-\infty,a),B)$, and this completes the proof. □

THEOREM 5.1. *Let* $f\colon\mathbb{R}\to\mathbb{R}$ *and* $g\colon\mathbb{R}\to\mathbb{R}$. *Then*

(a) $f*g=g*f$, *provided* (5.1) *exists.*

(b) *if* $f\in\mathscr{LP}(\mathbb{R})$, $h\in L^1(\mathbb{R})$ *and* $g''\in L^1(\mathbb{R})$, *then* $(f*g)*h=f*(g*h)$.

(c) *for* $z\in\mathbb{R}$, $\tau_z(f*g)=(\tau_zf)*g=f*(\tau_zg)$, *where* $\tau_zf(x)=f(x-z)$.

(d) *if* $A=\{x+y\mid x\in supp(f),\ y\in supp(g)\}$, *then* $supp(f*g)\subseteq\overline{A}$.

Proof. Proof of (a) is a straight forward consequence of Lemma 5.1 and that of (c) and (d) are same as in the case of Lebesgue Fourier transformation. So we give the proof of (b) only.

Applying (a), we have

$$(f*g)*h(x)=\int\int f(y)g(x-y-z)h(z)\,dydz.$$

Let $G^x(y,z) = g(x-y-z)h(z)$. As g' is bounded and g'', h are Lebesgue integrable on $\mathbb{R}$, $G^x(y,z)$ satisfies all conditions of Lemma 5.2. Thus

$$\begin{aligned}(f*g)*h(x) &= \int\int f(y)g(x-y-z)h(z)\,dzdy \\ &= \int\int f(y)g*h(x-y)\,dy = f*(g*h)(x). \quad \square\end{aligned}$$

THEOREM 5.2. *Let $f \in \mathscr{LP}(\mathbb{R})$, let $g \in L^1(\mathbb{R}) \cap \mathscr{BV}(\mathbb{R})$, let $F(x) = \int_{-\infty}^x f$ and let $G(x) = \int_{-\infty}^x g$. If F and G are Lebesgue integrable and $G(\infty) = \lim_{x\to\infty} G(x) = 0$, then*

$$\int_{-\infty}^t \int_{-\infty}^\infty f(x-y)G(y)\,dydx = \int_{-\infty}^\infty \int_{-\infty}^t f(x-y)G(y)\,dxdy.$$

Moreover, we have

$$\begin{aligned}\|f*G\|_1 &\leqslant \|F\|_1\|g\|_1, \\ \|f*G\|_{\mathscr{A}} &\leqslant \|f\|_{\mathscr{A}}\|G\|_1,\end{aligned} \tag{5.6}$$

where $\|f\|_{\mathscr{A}} = \sup_{x\in\mathbb{R}} \left|\int_{-\infty}^x f\right|$, the Alexiewicz's norm (see [1]) of f.

Proof. Let $h(x) = \int_{-\infty}^\infty f(x-y)g(y)\,dy$ for $x \in \mathbb{R}$. Then using integration by parts, we have

$$h(x) = \int_{-\infty}^\infty F(x-y)g(y)\,dy = F*g(x). \tag{5.7}$$

Now as both F and g are Lebesgue integrable, using Fubini's theorem and integration by parts, we have

$$\begin{aligned}\int_{-\infty}^t h(x)\,dx &= \int_{-\infty}^\infty \left(\int_{-\infty}^t F(x-y)\,dx\right) g(y)\,dy \\ &= \int_{-\infty}^\infty F(t-y)G(y)\,dy = \int_{-\infty}^\infty \int_{-\infty}^t f(x-y)G(y)\,dxdy\end{aligned} \tag{5.8}$$

which implies $\int_{-\infty}^t \int_{-\infty}^\infty f(x-y)G(y)\,dydx = \int_{-\infty}^\infty \int_{-\infty}^t f(x-y)G(y)\,dxdy$. From (5.7) and (5.8), we shall get (5.6). $\square$

6. Fourier transform

Let $f\colon \mathbb{R} \to \mathbb{R}$. Then the Fourier transform of f is defined by

$$\widehat{f}(y) = \int_{-\infty}^\infty f(x)e^{-2\pi iyx}\,dx, \tag{6.1}$$

provided the integral exists for $y \in \mathbb{R}$.

Now, if we denote $G_y(x) = e^{-2\pi iyx}$, then it is easy to verify that G_y' is of bounded variation on finite intervals. Hence if the integral in (6.1) exists, it implies that $f \in \mathscr{LP}_{loc}(\mathbb{R})$, where by $\mathscr{LP}_{loc}(\mathbb{R})$ we mean the set of all locally Laplace integrable functions. Now we establish some basic properties of the Fourier transform in this general setting.

THEOREM 6.1. (Existence theorems) (a) *If* $f \in \mathscr{LP}_{loc}(\mathbb{R})$ *and* f *is Lebesgue integrable in a neighbourhood of infinity then* $\widehat{f}$ *exists.*

(b) *If* $f \in \mathscr{LP}(\mathbb{R}) \cap \mathscr{BV}(\pm\infty)$, *then* $\widehat{f}$ *exists on* $\mathbb{R}$.

Proof.

(a) Let $a \in (0,\infty)$ be such that $f \in L^1(\mathbb{R} \setminus (-a,a))$. Then $\int_{|x|\geqslant a} f(x)e^{-2\pi iyx}\,dx$ exists. Now, as $g_y(x) = (-2\pi iy)e^{-2\pi iyx}$ is of bounded variation on $[-a,a]$,

$$\int_{-a}^{a} f(x)e^{-2\pi iyx}\,dx$$

exists. Thus $\widehat{f}$ exists on $\mathbb{R}$.

(b) It is a straightforward consequence of the previous part, Lemma 3.2, and Theorem 3.1 of [10]. □

The above theorem implies that $\widehat{\Phi}$ (see Example 3.1) exists in our setting; however, $\widehat{\Phi}$ does not exist in the sense of Henstock-Kurzweil Fourier transform since Φ is not locally Henstock integrable.

THEOREM 6.2. *Suppose* $f,g \in \mathscr{LP}(\mathbb{R})$.

(a) *If* $\widehat{f}$ *exists, then* $\widehat{\tau_\zeta f}(y) = e^{-2\pi i\zeta y}\widehat{f}(y)$ *and* $\tau_\eta \widehat{f} = \widehat{h}$, *where* $h(x) = e^{2\pi i\eta x} f(x)$.

(b) *Let* $\widehat{f}$ *exists at* $y \in \mathbb{R}$, *let* $g \in L^1(\mathbb{R}) \cap \mathscr{BV}(\mathbb{R})$, *let* $G(x) = \int_{-\infty}^{x} |g|$ *and let* $F_y(x) = \int_{-\infty}^{x} e^{-2\pi iyt} f(t)\,dt$. *If* F_y *and* G *are Lebesgue integrable and* $G(\infty) = \lim_{x\to\infty} G(x) = 0$, *then*

$$\widehat{f * G}(y) = \widehat{f}(y)\widehat{G}(y).$$

(c) *Let* $x^n f \in \mathscr{LP}(\mathbb{R})$ *and let* $\widehat{x^n f}$ *exists for* $n = 0,1$. *Then* $\dfrac{d\widehat{f}}{dy} = \widehat{h}$ *a.e. on* $\mathbb{R}$, *where* $h(x) = (-2\pi ix)f(x)$.

(d) *Let* $f, f' \in \mathscr{LP}(\mathbb{R})$. *If* $\widehat{f}$ *exists, then* $\widehat{f'}$ *exists and* $\widehat{f'}(y) = (2\pi iy)\widehat{f}(y)$.

(e) *Let* $f \in \mathscr{LP}(\mathbb{R})$. *If*

  (i) f *has compact support, then* $\widehat{f}$ *is continuous on* $\mathbb{R}$.

  (ii) $f \in \mathscr{BV}(\pm\infty)$, *then* $\widehat{f}$ *is continuous on* $\mathbb{R} \setminus \{0\}$.

(f) (Riemann-Lebesgue Lemma) *If* $f \in \mathscr{LP}(\mathbb{R}) \cap \mathscr{BV}(\mathbb{R})$, *then* $\lim_{|t|\to\infty} \widehat{f}(t) = 0$.

Proof. Proof of (a) and (b) follow from Lemma 5.1 and Theorem 5.2, respectively, and proof of (f) follows from Theorem 5.2 of [10] and Lemma 3.2. So we shall prove the rest.

(c) Let $A=\mathbb{R}$ and $B=[n,n+1]$, $n\in\mathbb{Z}$. Let $h(x)=(-2\pi ix)f(x)$ and let $G(x,y)=e^{-2\pi ixy}$. Then

(i) $g(x,y)=\partial_x G(x,y)=(-2\pi iy)e^{-2\pi ixy}$,

(ii) $V_{[a,b]}[g(\cdot,y)]=2\pi(b-a)y\in L^1(B)$, for any compact interval $[a,b]\subseteq A$,

(iii) $|g(x,y)|=2\pi y\in L^1(B)$ and $|G(x,y)|=1\in L^1(B)$ for $(x,y)\in A\times B$, and

(iv) $\mathrm{I}_2(A,B)=\int_{x\in A}\int_{y\in B}h(x)G(x,y)\,dydx=[\widehat{f}(n+1)-\widehat{f}(n)]\in\mathbb{R}$

which implies $\mathrm{I}_1(A,B)=\int_{y\in B}\int_{x\in A}h(x)G(x,y)\,dxdy=\mathrm{I}_2(A,B)$ (see Lemma 5.2). Hence, we get

$$\int_s^t\int_{-\infty}^{\infty}h(x)G(x,y)\,dxdy=\int_{-\infty}^{\infty}\int_s^t h(x)G(x,y)\,dxdy, \tag{6.2}$$

where $s,t\in[n,n+1]$. Now, as $\widehat{f}(y)=\int_{-\infty}^{\infty}H(x,y)\,dx$, where $H(x,y)=f(x)G(x,y)$, and $\partial_y H(x,y)=h(x)G(x,y)$, by (6.2) and Theorem 4.1, we have $\dfrac{d\widehat{f}}{dy}=\widehat{h}$ a.e. on $[n,n+1]$ for all $n\in\mathbb{Z}$ and hence a.e. on $\mathbb{R}$.

(d) As f,f' are integrable, $\lim_{|x|\to\infty}f(x)=0$. For $[u,v]\subseteq\mathbb{R}$, we have

$$\int_u^v f'(x)e^{-2\pi ixy}\,dx=\left[e^{-2\pi ixy}f(x)\right]_u^v+(2\pi iy)\int_u^v f(x)e^{-2\pi ixy}\,dx.$$

Now taking $u\to-\infty$ and $v\to\infty$, we arrive at our conclusion.

(e) (i) Let $a\in(0,\infty)$ be such that $f(x)=0$ for $|x|>a$. Let $y_0\in\mathbb{R}$ be arbitrary and let $I_{y_0}=[y_0-1,y_0+1]$. If we denote, $g_y(x)=e^{-2\pi iyx}$, then it is easy to verify that the set $\{g_y'\mid y\in I_{y_0}\}$ is of uniform variation. Hence, by Theorem 7.2 of [8], we get

$$\begin{aligned}\lim_{y\to y_0}\widehat{f}(y)&=\lim_{y\to y_0}\int_{-\infty}^{\infty}f(x)e^{-2\pi iyx}\,dx\\&=\lim_{y\to y_0}\int_{-a}^{a}f(x)e^{-2\pi iyx}\,dx=\widehat{f}(y_0).\end{aligned}$$

(ii) Let $a\in(0,\infty)$ be such that $f\in\mathscr{BV}(\mathbb{R}\setminus(-a,a))$. Note that $f=f_1+f_2+f_3$, where

$$f_1=f\chi_{[-\infty,-a]},\qquad f_2=f\chi_{[-a,a]},\qquad f_3=f\chi_{[a,\infty]}.$$

To prove $\widehat{f}$ is continuous on $\mathbb{R}\setminus\{0\}$ it is enough to show that $\widehat{f_1},\widehat{f_3}$ are continuous on $\mathbb{R}\setminus\{0\}$. Now Lemma 3.2 implies $f_1,f_3\in\mathscr{HK}(\mathbb{R})\cap\mathscr{BV}(\mathbb{R})$ and hence Theorem 4.2 of [10] completes the proof. □

LEMMA 6.1. *Let ψ and ϕ be real-valued function on $\mathbb{R}$ and let $\widehat{\psi}$ exists a.e. on $\mathbb{R}$. If $\phi(y)$, $y\phi(y)$ and $y^2\phi(y)$ are Lebesgue integrable on $\mathbb{R}$ and if $\int_{-\infty}^{\infty}\psi\widehat{\phi}$ exists, then $\int_{-\infty}^{\infty}\psi\widehat{\phi}=\int_{-\infty}^{\infty}\widehat{\psi}\phi$.*

Proof. Let $G(x,y)=\phi(y)e^{-2\pi iyx}$ and $g(x,y)=\partial_x G(x,y)$. Then

(a) $V_{[a,b]}[g(\cdot,y)]=\int_a^b|\partial_x g(x,y)|\,dx=\int_a^b 4\pi^2y^2|\phi(y)|\,dx=4\pi^2(b-a)y^2|\phi(y)|$;

(b) $\int_{-\infty}^{\infty}V_{[a,b]}[g(\cdot,y)]\,dy=4\pi^2(b-a)\int_{-\infty}^{\infty}y^2|\phi(y)|\,dy=O_{[a,b]}<\infty$;

(c) $|g(x,y)|=2\pi|y\phi(y)|=\eta(y)\in L^1(\mathbb{R})$;

(d) $|G(x,y)|=|\phi(y)|=\kappa(y)\in L^1(\mathbb{R})$.

Therefore, by Lemma 5.2, we get $\int_{-\infty}^{\infty}\psi\widehat{\phi}=\int_{-\infty}^{\infty}\widehat{\psi}\phi$. □

Let us denote

$$\widetilde{f}(x)=\int_{-\infty}^{\infty}\widehat{f}(t)e^{2\pi ixt}\,dt \tag{6.3}$$

THEOREM 6.3. (The Inversion Theorem) *Let $f:\mathbb{R}\to\mathbb{R}$ be such that $\widehat{f}$ exists almost everywhere on $\mathbb{R}$. Let $x_0\in\mathbb{R}$ be such that $\widetilde{f}$ exists at x_0. Then $f(x_0)=\widetilde{f}(x_0)$ provided*

$$\sup_{x\in[-\delta,\delta]}\left|s\int_{-\delta}^{x}e^{-s|t|}\left(f(x_0+t)-f(x_0)\right)dt\right|\to 0 \qquad \text{as } s\to\infty \tag{6.4}$$

for some $\delta(>0)$.

Proof. For simplicity, we assume $x_0=0$. As e^{-t^2}, te^{-t^2} and $t^2e^{-t^2}$ belong to $L^1(\mathbb{R})$, by Lemma 6.1, we get

$$\int_{-\infty}^{\infty}e^{-\lambda^2\pi t^2}\widehat{f}(t)\,dt=\int_{-\infty}^{\infty}\widehat{e^{-\lambda^2\pi t^2}}f(t)\,dt.$$

Now as $G_\lambda(t)=\frac{d}{dt}(e^{-\lambda^2\pi t^2})\in L^1(\mathbb{R})\cap\mathscr{BV}(\mathbb{R})$ and the set $\{V_{\mathbb{R}}[G_\lambda]\mid 0\leqslant\lambda\leqslant 1\}$ is uniformly bounded on $\mathbb{R}$, Theorem 3.5 implies

$$\lim_{\lambda\to 0^+}\int_{-\infty}^{\infty}e^{-\lambda^2\pi t^2}\widehat{f}(t)\,dt=\int_{-\infty}^{\infty}\widehat{f}(t)\,dt=\widetilde{f}(0). \tag{6.5}$$

Let $\delta>0$ be such that (6.4) holds. Then

$$\begin{aligned}\int_{-\infty}^{\infty}\widehat{e^{-\lambda^2\pi t^2}}f(t)\,dt&=\int_{-\infty}^{\infty}\lambda^{-1}e^{-\pi t^2/\lambda^2}f(t)\,dt\\&=\int_{|t|<\delta}\lambda^{-1}e^{-\pi t^2/\lambda^2}f(t)\,dt+\int_{|t|\geqslant\delta}\lambda^{-1}e^{-\pi t^2/\lambda^2}f(t)\,dt.\end{aligned}$$

Let $I_1=\int_{|t|<\delta}\lambda^{-1}e^{-\pi t^2/\lambda^2}f(t)\,dt$, $I_2=\int_{|t|\geqslant\delta}\lambda^{-1}e^{-\pi t^2/\lambda^2}f(t)\,dt$ and $J_\delta=\mathbb{R}\setminus(-\delta,\delta)$. Then by simple calculation, we can prove that

$$H_\lambda(t)=\frac{d}{dt}(\lambda^{-1}e^{-\pi t^2/\lambda^2})\in L^1(\mathbb{R})\cap\mathscr{BV}(\mathbb{R})$$

and the set $\{V_{\mathbb{R}}[H_\lambda] \mid 0 \leqslant \lambda \leqslant \varepsilon\}$ is uniformly bounded on J_δ, where $\varepsilon \in [0,1)$ is sufficiently small. Thus again, Theorem 3.5 implies that $I_2 \to 0$ as $\lambda \to 0^+$. Now

$$\begin{aligned} I_1 &= f(0)\int_{-\delta}^{\delta} \lambda^{-1} e^{-\pi t^2/\lambda^2}\,dt + \int_{-\delta}^{\delta} \lambda^{-1} e^{-t/\lambda}(f(t)-f(0))e^{-\pi t^2/\lambda^2+t/\lambda}\,dt \\ &= f(0)\int_{-\delta}^{\delta} \lambda^{-1} e^{-\pi t^2/\lambda^2}\,dt + \int_{-\delta}^{\delta} g_\lambda(t)h_\lambda(t)\,dt, \end{aligned}$$

where $g_\lambda(t) = \lambda^{-1}e^{-t/\lambda}(f(t)-f(0))$ and $h_\lambda(t) = e^{-\pi t^2/\lambda^2+t/\lambda}$. By Lemma 3.1, we get

$$\left|\int_{-\delta}^{\delta} g_\lambda(t)h_\lambda(t)\,dt\right| \leqslant \left[\inf_{[-\delta,\delta]} |h_\lambda(t)| + V_{[-\delta,\delta]}[h_\lambda]\right] \|g_\lambda\|_{[-\delta,\delta]}.$$

For sufficiently small λ, it can be proved that $V_{[-\delta,\delta]}[h_\lambda]$ is independent of λ. Thus by (6.4), we get

$$\begin{aligned} \lim_{\lambda\to 0^+}\int_{-\infty}^{\infty} \widehat{e^{-\lambda^2\pi t^2}} f(t)\,dt &= \lim_{\lambda\to 0^+} I_1 = \lim_{\lambda\to 0^+} f(0)\int_{-\delta}^{\delta} \lambda^{-1}e^{-\pi t^2/\lambda^2}\,dt \\ &= f(0)\pi^{-1/2}\lim_{\lambda\to 0^+}\int_0^{\frac{\pi\delta^2}{\lambda^2}} t^{-1/2}e^{-t}\,dt \\ &= f(0)\pi^{-1/2}\Gamma(1/2) = f(0). \end{aligned} \tag{6.6}$$

Equating (6.5) and (6.6), we get $f(0) = \int_{-\infty}^{\infty} \widehat{f}(t)\,dt$. □

COROLLARY 6.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that $\widehat{f} = 0$ a.e. on $\mathbb{R}$. Then $f = 0$ a.e. on $\mathbb{R}$.*

Proof. Let $I_n = [n, n+1]$ for $n \in \mathbb{Z}$. Then $f \in \mathscr{LP}(I_n)$ for all $n \in \mathbb{Z}$. Now, Corollary 6.2 of [8] implies that f is Laplace continuous a.e. on I_n and hence (6.4) is satisfied a.e. on I_n for all $n \in \mathbb{Z}$. Since $\widehat{f} = 0$ a.e. on $\mathbb{R}$, we obtain $f = 0$ a.e. on I_n for all $n \in \mathbb{Z}$ which completes the proof. □

7. Conclusions

The definition of Laplace integral depends on a generalised derivative called the Laplace derivative. Suppose it is possible to define the total Laplace derivative on $\mathbb{R}^n$ ($n \geqslant 2$) and establish its interrelations with the partial Laplace derivatives. In that case, it may be possible to find excellent applications of the Fourier transform (in our setting) to the generalised PDEs, i.e., to the PDEs using partial Laplace derivatives, which, we hope, will be an interesting problem to deal with.

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