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A NOTE ON A FAMILY OF LOG–INTEGRALS

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Abstract. A family of log-integrals with three parameters is analyzed. In particular, some difficult integrals are evaluated exactly using the derivatives of the Gamma function.

1. Motivation

The motivation for this note comes from a paper by Srivastava and Choi from 2000 [6]. In this paper, the authors show how higher-order derivatives of the Gamma function can be obtained in closed form. Let $\Gamma(z)$ be the familiar Gamma function given by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \qquad (\Re(z) > 0).$$

The digamma function $\psi(z)$ is defined for all $z \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}$ by

$$\psi(z) = (\ln \Gamma(z))' = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+z} \right),$$

with γ being the Euler-Mascheroni constant

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) = 0,5772156649\ldots.$$

Srivastava and Choi [6] show the following recursions for the values $\Gamma^{(n)}(1)$ and $\Gamma^{(n)}(1/2)$ for $n \geqslant 0$ (Eqs. (2.2) and (2.3) in [6]):

$$\Gamma^{(n+1)}(1) = -\gamma \Gamma^{(n)}(1) + n! \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(n-k)!} \zeta(k+1) \Gamma^{(n-k)}(1) \tag{1}$$

and

$$\Gamma^{(n+1)}(1/2) = -\delta \Gamma^{(n)}(1/2) + n! \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(n-k)!} (2^{k+1}-1) \zeta(k+1) \Gamma^{(n-k)}(1/2) \tag{2}$$

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with $\delta = \gamma + 2\ln(2)$ and where

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \qquad (\Re(s) > 1)$$

is the Riemann zeta function. These recursions allow to compute all higher-order derivatives and the first values are

$$\Gamma^{(1)}(1) = -\gamma,$$

$$\Gamma^{(2)}(1) = \gamma^2 + \zeta(2),$$

$$\Gamma^{(3)}(1) = -\gamma^3 - 3\gamma\zeta(2) - 2\zeta(3),$$

$$\Gamma^{(4)}(1) = \gamma^4 + 6\gamma^2\zeta(2) + 8\gamma\zeta(3) + \frac{27}{2}\zeta(4),$$

$$\Gamma^{(1)}(1/2) = -\delta\sqrt{\pi},$$

$$\Gamma^{(2)}(1/2) = \sqrt{\pi}(\delta^2 + 3\zeta(2)),$$

$$\Gamma^{(3)}(1/2) = \sqrt{\pi}(-\delta^3 - 9\delta\zeta(2) - 14\zeta(3)),$$

$$\Gamma^{(4)}(1/2) = \sqrt{\pi}\Big(\delta^4 + 18\delta^2\zeta(2) + 56\delta\zeta(3) + \frac{315}{2}\zeta(4)\Big),$$

and so on. The authors also show that these values are useful in the evaluation of integrals. As an example they analyze the family of integrals $I(m), m \geqslant 0$, given by

$$I(m) = \int_0^\infty \frac{\ln^m(x)}{(1+x)\sqrt{x}}\,dx. \tag{3}$$

In this note, we study the family of logarithmic integrals given by

$$J(m,n,p) = \int_0^\infty \frac{\ln^m(x)}{(1+x^n)^p}\,dx \tag{4}$$

with the three free parameters $m \geqslant 0$ and $1 < np$. Obviously, some members of the family are easily evaluated. Namely, $J(0,2,1) = \pi/2$, $J(0,1,3/2) = 2$ and maybe a few others but the general case seems to be difficult. The family of integrals does not appear in the compendium [5]. Other logarithmic integrals, of which some are related to $J(m,n,p)$ are discussed by Boros and Moll in their treatise of integrals [1, Chapter 12].

2. Main result and consequences

THEOREM 1. *For integers $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $p \in \mathbb{Q}_+$ with $1 < np$, we have*

$$J(m,n,p) = \frac{1}{n^{m+1}\Gamma(p)} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Gamma^{(k)}\left(\frac{1}{n}\right) \Gamma^{(m-k)}\left(p - \frac{1}{n}\right). \tag{5}$$

Proof. For $n > 0$, consider the function $f(x) = (1+x^n)^{-p}$. Then, from the theory of Mellin transforms (see [3, Chapter 8]) we have

$$g(s) = \int_0^\infty \frac{x^{s-1}}{(1+x^n)^p}\, dx = \frac{1}{n\Gamma(p)} \cdot \Gamma\left(\frac{s}{n}\right)\Gamma\left(p - \frac{s}{n}\right) \qquad 0 < \Re(s) < np.$$

Therefore

$$\begin{aligned} \frac{d^m}{ds^m} g(s) &= \int_0^\infty \frac{\ln^m(x)\, x^{s-1}}{(1+x^n)^p}\, dx \\ &= \frac{1}{n\Gamma(p)} \frac{d^m}{ds^m} \left(\Gamma\left(\frac{s}{n}\right) \cdot \Gamma\left(p - \frac{s}{n}\right)\right) \\ &= \frac{1}{n\Gamma(p)} \sum_{k=0}^{m} \binom{m}{k} \Gamma^{(k)}\left(\frac{s}{n}\right) \Gamma^{(m-k)}\left(p - \frac{s}{n}\right), \end{aligned}$$

where in the last step the Leibniz rule for derivatives was applied. The statement follows by evaluating the derivatives at $s = 1$. □

We proceed with some special cases of Theorem 1.

COROLLARY 1. *For all $m \geqslant 0$ and $n \geqslant 2$, we have*

$$J(m,n,1) = \frac{1}{n^{m+1}} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Gamma^{(k)}\left(\frac{1}{n}\right) \Gamma^{(m-k)}\left(1 - \frac{1}{n}\right). \tag{6}$$

In particular, we have the relation

$$J(m,2,1) = 2^{-(m+1)} I(m), \tag{7}$$

where $I(m)$ is the integral in (3).

Proof. The first part is obvious. The second part follows from the fact that (see [6])

$$I(m) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Gamma^{(k)}\left(\frac{1}{2}\right) \Gamma^{(m-k)}\left(\frac{1}{2}\right). \quad \square$$

It is notable to observe that due to identity (7) we have that for all m odd

$$J(m,2,1) = 0. \tag{8}$$

This is true because $I(m) = 0$ when m is odd as is shown in [6]. Three other explicit evaluations are

$$J(2,2,1) = \frac{\pi^3}{8}, \qquad J(4,2,1) = \frac{5\pi^5}{32}, \qquad J(6,2,1) = \frac{61\pi^7}{128}.$$

COROLLARY 2. *For all $m \geqslant 0$ and $n \geqslant 1$, we have*

$$J(m,n,3/2) = \frac{2}{n^{m+1}\sqrt{\pi}} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Gamma^{(k)}\left(\frac{1}{n}\right) \Gamma^{(m-k)}\left(\frac{3}{2} - \frac{1}{n}\right). \tag{9}$$

In particular,

$$J(0,n,3/2) = \frac{n-2}{n^2\sqrt{\pi}} \Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{n-2}{2n}\right) \qquad (n \neq 2). \tag{10}$$

Proof. Formula (9) is also an immediate consequence of Theorem 1 keeping in mind that $\Gamma(3/2) = \sqrt{\pi}/2$. $\square$

Applying the special evaluations of $\Gamma^{(n)}(1)$ and $\Gamma^{(n)}(1/2)$ we have the following examples:

$$J(1,1,3/2) = 4\ln(2), \qquad J(1,2,3/2) = -\ln(2),$$

$$J(2,1,3/2) = \frac{4}{3}\pi^2 + 8\ln^2(2), \qquad J(2,2,3/2) = \frac{1}{6}\pi^2 + \ln^2(2),$$

$$J(3,1,3/2) = 24\zeta(3) + 16\ln^3(2) + 8\pi^2\ln(2),$$

$$J(3,2,3/2) = -\frac{1}{2}\Big(3\zeta(3) + 2\ln^3(2) + \pi^2\ln(2)\Big),$$

$$J(4,1,3/2) = \frac{24}{5}\pi^4 + 32\pi^2\ln^2(2) + 32\ln^4(2) + 192\ln(2)\zeta(3),$$

$$J(4,2,3/2) = \frac{3}{20}\pi^4 + \pi^2\ln^2(2) + \ln^4(2) + 6\ln(2)\zeta(3).$$

The following relation between $J(m,2,3/2)$ and $J(m,1,3/2)$ holds.

COROLLARY 3. *For all $m \geqslant 0$,*

$$J(m,2,3/2) = (-1)^m 2^{-(m+1)} J(m,1,3/2). \tag{11}$$

Proof. Calculate

$$\begin{aligned}
2^m\sqrt{\pi}J(m,2,3/2) &= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Gamma^{(k)}\left(\frac{1}{2}\right) \Gamma^{(m-k)}(1) \\
&= \sum_{k=0}^{m} \binom{m}{m-k} (-1)^{k} \Gamma^{(m-k)}\left(\frac{1}{2}\right) \Gamma^{(k)}(1) \\
&= (-1)^m \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Gamma^{(m-k)}\left(\frac{1}{2}\right) \Gamma^{(k)}(1) \\
&= (-1)^m \frac{\sqrt{\pi}}{2} J(m,1,3/2). \quad \square
\end{aligned}$$

We also have the evaluation

$$J(1,3,3/2) = \frac{2}{9\sqrt{\pi}}\Gamma\Big(\frac{1}{3}\Big)\Gamma\Big(\frac{1}{6}\Big)\Big(-1+\frac{\sqrt{3}\pi}{18}+\frac{1}{3}\ln(2)\Big),$$

where we have used

$$\psi\Big(\frac{1}{3}\Big) = -\gamma - \frac{1}{6}\sqrt{3}\pi - \frac{3}{2}\ln(3)$$

and

$$\psi\Big(\frac{1}{6}\Big) = -\gamma - \frac{1}{2}\sqrt{3}\pi - \frac{3}{2}\ln(3) - 2\ln(2).$$

COROLLARY 4.

$$J(m,n,2) = \frac{1}{n^{m+1}}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\Gamma^{(k)}\Big(\frac{1}{n}\Big)\Gamma^{(m-k)}\Big(2-\frac{1}{n}\Big). \tag{12}$$

In particular, for all $n \geqslant 1$

$$J(0,n,2) = \frac{1}{n}\Gamma\Big(\frac{1}{n}\Big)\Gamma\Big(\frac{2n-1}{n}\Big). \tag{13}$$

Also, if m *is odd then*

$$J(m,1,2) = 0. \tag{14}$$

Proof. The first two parts are obvious, so we focus on the last statement. Let m be odd. Then, we can calculate

$$\begin{aligned}
J(m,1,2) &= \sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\Gamma^{(k)}(1)\Gamma^{(m-k)}(1) \\
&= \Big(\sum_{k=0}^{(m-1)/2} + \sum_{k=(m-1)/2+1}^{m}\Big)\binom{m}{k}(-1)^{m-k}\Gamma^{(k)}(1)\Gamma^{(m-k)}(1) \\
&= \sum_{k=0}^{(m-1)/2}\binom{m}{k}(-1)^{m-k}\Gamma^{(k)}(1)\Gamma^{(m-k)}(1) \\
&\quad + \sum_{k=0}^{(m-1)/2}\binom{m}{\frac{m+1}{2}+k}(-1)^{m-(m+1)/2-k}\Gamma^{((m+1)/2+k)}(1)\Gamma^{(m-(m+1)/2-k)}(1) \\
&= \sum_{k=0}^{(m-1)/2}\binom{m}{k}(-1)^{m-k}\Gamma^{(k)}(1)\Gamma^{(m-k)}(1) \\
&\quad + \sum_{k=0}^{(m-1)/2}\binom{m}{\frac{m+1}{2}+k}(-1)^{(m-1)/2-k}\Gamma^{((m+1)/2+k)}(1)\Gamma^{((m-1)/2-k)}(1).
\end{aligned}$$

But,

$$\binom{m}{\frac{m+1}{2}+k} = \binom{m}{\frac{m-1}{2}-k}$$

and changing the order of summation in the second sum we arrive at

$$J(m,1,2)=(1+(-1)^m)\sum_{k=0}^{(m-1)/2}\binom{m}{k}(-1)^{m-k}\Gamma^{(k)}(1)\Gamma^{(m-k)}(1),$$

which finishes the proof. □

Once more, applying the special evaluations of $\Gamma^{(n)}(1)$ we state the following results

$$J(2,1,2)=\frac{\pi^2}{3},\qquad J(4,1,2)=27\zeta(4)+6\zeta^2(2)=\frac{7\pi^4}{15}.$$

and

$$J(6,1,2)=2(\Gamma^{(6)}(1)-6\Gamma^{(1)}(1)\Gamma^{(5)}(1)+15\Gamma^{(2)}(1)\Gamma^{(4)}(1))-20(\Gamma^{(3)}(1))^2=\frac{31\pi^6}{21}.$$

In addition,

$$J(1,2,2)=-J(0,2,2)=\frac{\pi}{4},\quad \text{and}\quad J(2,2,2)=\frac{\pi^3}{16},$$

where we have used $\psi(3/2)=2-\delta$.

We close this section with a few observations concerning the integrals $J(m,n,1/2)$, $n>2$, which cannot be evaluated using the values for $\Gamma^{(n)}(1/2)$ and $\Gamma^{(n)}(1)$.

COROLLARY 5.

$$J(m,n,1/2)=\frac{1}{n^{m+1}\sqrt{\pi}}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\Gamma^{(k)}\left(\frac{1}{n}\right)\Gamma^{(m-k)}\left(\frac{1}{2}-\frac{1}{n}\right). \tag{15}$$

In particular, if m is odd then

$$J(m,4,1/2)=0. \tag{16}$$

Proof. The first part is obvious. For m odd, we have

$$J(m,4,1/2)=\frac{1}{4^{m+1}\sqrt{\pi}}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\Gamma^{(k)}\left(\frac{1}{4}\right)\Gamma^{(m-k)}\left(\frac{1}{4}\right)$$

and we can apply the same idea as in in the proof of Corollary 4. □

As special values we record

$$J(0,3,1/2)=\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}{3\sqrt{\pi}},$$

$$J(1,3,1/2)=\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}{9\sqrt{\pi}}\left(\frac{\pi}{\sqrt{3}}+2\ln(2)\right),$$

and

$$J(2,4,1/2) = \frac{(\Gamma(\frac{1}{4}))^2}{32\sqrt{\pi}}\left(8C+\pi^2\right),$$

where we have used

$$\psi\left(\frac{1}{4}\right) = -\gamma - \frac{\pi}{2} - 3\ln(2)$$

and

$$\psi'\left(\frac{1}{4}\right) = 8C+\pi^2,$$

where C is Catalan's constant.

3. A different approach to evaluate $J(m,n,p)$

A different approach to evaluate $J(m,n,p)$ in form of infinite series uses the Goyal-Laddha generalized Hurwitz-Lerch zeta function [4, 8]

$$\Phi^*_\mu(z,s,a) = \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} \frac{z^k}{(k+a)^s} \tag{17}$$

$$(\mu \in \mathbb{C}, a \in \mathbb{C}\setminus\mathbb{Z}_0^-, s \in \mathbb{C} \text{ when } |z| < 1 \text{ and } \Re(s-\mu) > 1 \text{ when } z = 1)$$

where $(\mu)_n = \Gamma(\mu+n)/\Gamma(\mu)$ denotes the Pochhammer symbol with $(0)_0 := 1$. Using this function we have the following result.

THEOREM 2. *For integers $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $p \in \mathbb{Q}_+$ with $1 < np$, we have*

$$J(m,n,p) = \frac{m!}{n^{m+1}}\left(\Phi^*_p\left(-1, m+1, p-\frac{1}{n}\right) + (-1)^m \Phi^*_p\left(-1, m+1, \frac{1}{n}\right)\right). \tag{18}$$

Proof. It is known that $\Phi^*_\mu(z,s,a)$ possesses the integral representation [8, Eq. (2.10)]

$$\Phi^*_\mu(z,s,a) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}\,e^{-at}}{(1-ze^{-t})^\mu}dt \tag{19}$$

$$(\Re(a), \Re(s) > 0 \text{ when } |z| \leqslant 1\,(z \neq 1) \text{ and } \Re(s) > 1 \text{ when } z = 1).$$

On the other hand, the substitution $x = e^{-t}$ in (4) results in

$$J(m,n,p) = \frac{1}{n^{m+1}}\int_0^\infty \frac{t^m\,e^{-(p-1/n)t}}{(1+e^{-t})^p}dt + \frac{(-1)^m}{n^{m+1}}\int_0^\infty \frac{t^m\,e^{-t/n}}{(1+e^{-t})^p}dt.$$

This completes the proof. □

Theorem 2 turns out to be very useful to provide closed forms for $J(m,n,p)$ for some particular values of the parameters. For instance we have the following evaluations, which extend (8) and in view of (7) also give a closed form for the integral (3) considered by Srivastava and Choi in [6].

COROLLARY 6. *We have*

$$J(m,2,1)=\begin{cases}0, & \text{if } m \text{ is odd,}\\ 2^{-(m+1)}(-1)^{m/2}E_m\pi^{m+1}, & \text{if } m \text{ is even,}\end{cases} \tag{20}$$

and

$$I(m)=\begin{cases}0, & \text{if } m \text{ is odd,}\\ (-1)^{m/2}E_m\pi^{m+1}, & \text{if } m \text{ is even,}\end{cases} \tag{21}$$

where E_n are the Euler numbers which are obtained by the Taylor series expansion of $1/\cosh(z), |z|<\pi/2$.

Proof. From the expression

$$J(m,2,1)=\frac{m!}{2^{m+1}}\Big(\Phi_1^*\Big(-1,m+1,\frac{1}{2}\Big)+(-1)^m\Phi_1^*\Big(-1,m+1,\frac{1}{2}\Big)\Big)$$

the first part (for m odd) is deduced immediately. When m is even, then

$$J(m,2,1)=\frac{m!}{2^m}\Phi_1^*\Big(-1,m+1,\frac{1}{2}\Big).$$

As $(1)_k=k!$ we have the relation

$$\Phi_1^*\Big(-1,m+1,\frac{1}{2}\Big)=2^{m+1}\beta(m+1),$$

where $\beta(z)$ is the Dirichlet Beta function defined by

$$\beta(z)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^z},\qquad (\Re(z)>0).$$

From here, we use the known expression valid for $q\in\mathbb{N}_0$

$$\begin{aligned}\beta(2q+1)&=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^{2q+1}}\\ &=\frac{(-1)^q E_{2q}}{(2q)!2^{2q+2}}\pi^{2q+1}. \quad\square\end{aligned}$$

The curious and remarkable identity, valid for all even m, may be interesting on its own

$$\sum_{k=0}^{m}\binom{m}{k}(-1)^k\Gamma^{(k)}\Big(\frac{1}{2}\Big)\Gamma^{(m-k)}\Big(\frac{1}{2}\Big)=(-1)^{m/2}E_m\pi^{m+1}. \tag{22}$$

4. Concluding remarks

This short note was about investigating some logarithmic integrals that admit an evaluation using combinations of higher-order derivatives of the Gamma function. The integrals discussed in the text cannot be evaluated by standard and advanced calculus techniques. See [2] for a recent exploration of methods. It is noteworthy, however, that a bit more is possible. Without much effort, we can add a fourth free parameter $a > 0$ to the family of integrals $J(m,n,p)$ and consider

$$J(m,n,p,a) = \int_0^\infty \frac{\ln^m(x)}{(1+ax^n)^p}\,dx.$$

Then, the main result for $J(m,n,p)$ presented in Theorem 1 can be generalized to $J(m,n,p,a)$. We leave the details to the reader but give the next result as a taste of what is the outcome for $p = 3/2$:

$$\begin{aligned} J(m,n,3/2,a) &= \int_0^\infty \frac{\ln^m(x)}{(1+ax^n)^{3/2}}\,dx \\ &= \frac{2}{n^{m+1}\sqrt{\pi}a^{1/n}} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} \ln^{m-j}(a) \\ &\quad \times \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} \Gamma^{(k)}\Big(\frac{1}{n}\Big)\Gamma^{(j-k)}\Big(\frac{3}{2}-\frac{1}{n}\Big). \end{aligned}$$

Another direction we can go is to use properties of Mellin transforms. Here, we think of the following property: If $M(f(x),s) = g(s)$ is the Mellin transform of the (suitably chosen) function $f(x)$, then $M(1/xf(1/x),s) = g(1-s)$. Working with $f(x) = (1+x^n)^{-p}$ we get, valid for $\Re(s) < 1 < np + \Re(s)$,

$$M\Big(\frac{1}{x}f\Big(\frac{1}{x}\Big),s\Big) = \int_0^\infty \frac{x^{s-1}x^{np-1}}{(1+x^n)^p}\,dx = \frac{1}{n\Gamma(p)} \cdot \Gamma\Big(\frac{1-s}{n}\Big)\Gamma\Big(p-\frac{1-s}{n}\Big).$$

This gives

$$\begin{aligned} \frac{d^m}{ds^m} g(1-s) &= \int_0^\infty \frac{\ln^m(x)\,x^{np+s-2}}{(1+x^n)^p}\,dx \\ &= \frac{1}{n\Gamma(p)} \sum_{k=0}^{m} \binom{m}{k} \Gamma^{(k)}\Big(\frac{1-s}{n}\Big)\Gamma^{(m-k)}\Big(p-\frac{1-s}{n}\Big), \end{aligned}$$

The last identity allows to evaluate some logarithmic integrals with an additional factor x^q, for some q, in the numerator. For instance, proceeding as before with $s = -1, n = p = 2$ we get the formula

$$\int_0^\infty \frac{\ln^m(x)\,x}{(1+x^2)^2}\,dx = \frac{1}{2^{m+1}} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \Gamma^{(k)}(1)\Gamma^{(m-k)}(1),$$

from which we easily get the expressions

$$\int_0^\infty \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2},$$

$$\int_0^\infty \frac{\ln(x)\,x}{(1+x^2)^2}\,dx = 0,$$

$$\int_0^\infty \frac{\ln^2(x)\,x}{(1+x^2)^2}\,dx = \frac{1}{4}\zeta(2) = \frac{\pi^2}{24},$$

and in general

$$\int_0^\infty \frac{\ln^m(x)\,x}{(1+x^2)^2}\,dx = 0 \qquad (m\,\text{odd}).$$

Similarly, with $s=-2, n=3$ and $p=2$,

$$\int_0^\infty \frac{\ln^m(x)\,x^2}{(1+x^3)^2}\,dx = \frac{1}{3^{m+1}}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\Gamma^{(k)}(1)\Gamma^{(m-k)}(1),$$

and we can deduce

$$\int_0^\infty \frac{x^2}{(1+x^3)^2}\,dx = \frac{1}{3},$$

$$\int_0^\infty \frac{\ln(x)\,x^2}{(1+x^3)^2}\,dx = 0,$$

$$\int_0^\infty \frac{\ln^2(x)\,x^2}{(1+x^3)^2}\,dx = \frac{2}{27}\zeta(2) = \frac{\pi^2}{81},$$

and in general

$$\int_0^\infty \frac{\ln^m(x)\,x^2}{(1+x^3)^2}\,dx = 0 \qquad (m\,\text{odd}).$$

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REFERENCES

[1] G. BOROS AND V. H. MOLL, *Irresistible Integrals Symbolics, Analysis and Experiments in the Evaluation of Integrals*, Cambridge University Press, USA, 2004.

[2] K. N. BOYADZHIEV, *Special Techniques for Solving Integrals: Examples and Problems*, World Scientific, USA, 2021.

[3] L. DEBNATH AND D. BHATTA, *Integral Transforms and Their Applications*, 3rd edition, Chapman & Hall/CRC, USA, 2014.

[4] S. P. GOYAL AND R. K. LADDHA, *On the generalized zeta function and the generalized Lambert function*, Gaṇita Sandesh 11 (1997), 99-108.

[5] I. GRADSHTEYN AND I. RYZHIK, *Table of Integrals, Series, and Products*, 7th edition, Elsevier Academic Press, Amsterdam, 2007.

[6] H. M. SRIVASTAVA AND J. CHOI, *Evaluation of higher-order derivatives of the Gamma function*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **11** (2000), 9–18.

[7] H. M. SRIVASTAVA AND J. CHOI, *Series associated with the Zeta and related functions*, Springer Science+Business Media, 2001.

[8] H. M. SRIVASTAVA, R. K. SAXENA, T. K. POGÁNY AND R. SAXENA, *Integral and computational representations of the extended Hurwitz-Lerch zeta function*, Integral Transforms Spec. Funct. 22 (2011), no. 7, 487–506.

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