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FUNCTIONS OF (ϕ,ψ)–BOUNDED VARIATION AND ITS DOUBLE WALSH–FOURIER COEFFICIENTS

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Abstract. In this paper, we have estimated the order of magnitude of double Walsh-Fourier coefficients of functions of (ϕ,ψ)-bounded variation in the sense of Vitali and Hardy.

1. Introduction

In 1949, N. J. Fine [3] proved using the second mean value theorem that if f is of bounded variation on $[0,1]$ and if $\hat{f}(m)$ denotes its (one dimensional) Walsh-Fourier coefficient, then $\hat{f}(m) = O\left(\frac{1}{m}\right)$, for all $m \neq 0$. In 2002 F. Móricz [5] estimated the order of magnitude of double Fourier coefficients with the help of Riemann-Stieltjes integral of functions of two variables and in 2004 V. Fülöp and F. Móricz [4] estimated the order of magnitude of multiple Fourier coefficients of functions of bounded variation in the sense of Vitali and Hardy in a straightforward way without using Riemann-Stieltjes integral. Also, the order of magnitude of double Fourier coefficients of different classes of functions of generalized bounded variation were estimated in [1, 2, 7, 8]. In this paper, we have estimated the order of magnitude of double Walsh-Fourier coefficients of functions of (ϕ,ψ)-bounded variation in the sense of Vitali and Hardy. Our results with $\phi(x) = \psi(x) = x$ gives Walsh analogues of the results of F. Móricz [5] and V. Fülöp and F. Móricz [4, for $n = 2$], except possibly for the exact constant in their case.

2. Notation and definitions

We consider the Walsh orthonormal system $\{w_m(x) : m \in \mathbb{N}_0\}$ defined on the unit interval $\mathbb{I} = [0,1)$ in the Paley enumeration, where $\mathbb{N}_0 = \{0,1,2,\ldots\}$. To go into some details, let

$$r_0(x) = \begin{cases} 1, & \text{if } x \in \left[0,\frac{1}{2}\right), \\ -1, & \text{if } x \in \left[\frac{1}{2},1\right); \end{cases}$$

and extend $r_0(x)$ for the half-axis $[0,\infty)$ with period 1.

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The Rademacher orthonormal system $\{r_k(x) : k \in \mathbb{N}_0\}$ is defined as

$$r_k(x) = r_0(2^k x), \quad k = 1,2,\dots ; x \in \mathbb{I}.$$

If

$$m = \sum_{k=0}^{\infty} m_k 2^k, \quad \text{each} \quad m_k = 0 \text{ or } 1,$$

is the binary decomposition of $m \in \mathbb{N}_0$, then

$$w_m(x) = \prod_{k=0}^{\infty} r_k^{m_k}(x), \quad x \in \mathbb{I},$$

is called the m^{th} Walsh function in the Paley enumeration.

In particular, we have

$$w_0(x) = 1 \text{ and } w_{2^m}(x) = r_m(x), \; m \in \mathbb{N}_0.$$

Any $x \in \mathbb{I}$ can be written as

$$x = \sum_{k=0}^{\infty} x_k \, 2^{-(k+1)}, \quad \text{each} \quad x_k = 0 \text{ or } 1.$$

For any $x \in \mathbb{I} \backslash Q$, there is only one expression of this form, where Q is a class of dyadic rationals in $\mathbb{I}$. When $x \in Q$ there are two expressions of this form, one which terminates in 0's and one which terminates in 1's.

For any $x, y \in \mathbb{I}$ their dyadic sum is defined as

$$x \dot{+} y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

Observe that, for each $m \in \mathbb{N}_0$, we have

$$w_m(x \dot{+} y) = w_m(x) \, w_m(y), \;\; x, y \in \mathbb{I}, \;\; x \dot{+} y \notin Q.$$

For a real-valued function $f \in L^1(\overline{\mathbb{I}}^2)$, where $\overline{\mathbb{I}} = [0,1]$ and f is 1-periodic in each variable, its double Walsh-Fourier series is defined as

$$f(x,y) \sim \sum_{m \in \mathbb{N}_0} \sum_{n \in \mathbb{N}_0} \hat{f}(m,n) \, w_m(x) \, w_n(y),$$

where the double Walsh-Fourier coefficients $\hat{f}(m,n)$ are defined by

$$\hat{f}(m,n) = \int\int_{\overline{\mathbb{I}}^2} f(x,y) \, w_m(x) \, w_n(y) \, dx \, dy.$$

Let ϕ and ψ be strictly increasing convex functions on $[0,\infty)$ with $\phi(0) = \psi(0) = 0$. A function ϕ is said to be a Δ_2 function if there is a constant $d \geqslant 2$ such that $\phi(2x) \leqslant d\phi(x)$ for all $x \geqslant 0$.

For $I=[a,b]$ and $J=[c,d]$, define

$$f(I\times J)=f(I,d)-f(I,c)=f(b,d)-f(a,d)-f(b,c)+f(a,c).$$

A measurable function f defined on a rectangle $R^2=[a,b]\times[c,d]$ is said to be of (ϕ,ψ)-bounded variation in the sense of Vitali (that is, $f\in(\phi,\psi)BV(R^2)$) if

$$V_{(\phi,\psi)}(f,R^2)=\sup_{\mathscr{I},\mathscr{J}}\left(\sum_k \psi\left(\sum_j \phi(|f(I_j\times J_k)|)\right)\right)<\infty,$$

where $\mathscr{I}$ and $\mathscr{J}$ are finite collections of non-overlapping subintervals $\{I_j\}$ and $\{J_k\}$ in $[a,b]$ and $[c,d]$, respectively.

Consider a function $f:\overline{\mathbb{I}}^2\to\mathbb{R}$ defined by $f(x,y)=g(x)+h(y)$, where g and h are any two arbitrary not necessarily bounded functions from $\overline{\mathbb{I}}$ into $\mathbb{R}$. Then $V_{(\phi,\psi)}(f,\overline{\mathbb{I}}^2)$ $=0$. Thus, a function f with $V_{(\phi,\psi)}(f,\overline{\mathbb{I}}^2)<\infty$ need not be bounded.

If $f\in(\phi,\psi)BV(R^2)$ is such that the marginal functions $f(a,.)\in\phi BV([c,d])$ and $f(.,c)\in\phi BV([a,b])$ (refer [9] for the definition of $\phi BV([a,b])$) then f is said to be of (ϕ,ψ)-bounded variation in the sense of Hardy (that is, $f\in(\phi,\psi)^*BV(R^2)$).

Note that, for $\phi(x)=\psi(x)=x$ classes $(\phi,\psi)BV(R^2)$ and $(\phi,\psi)^*BV(R^2)$ reduce to classes $BV_V(R^2)$ (the class of functions of bounded variation in the sense of Vitali (refer [6, p. 279] for the definition of $BV_V(R^2)$)) and $BV_H(R^2)$ (the class of functions of bounded variation in the sense of Hardy (refer [6, p. 280] for the definition of $BV_H(R^2)$)), respectively.

3. Main results

We prove the following results.

THEOREM 3.1. *If ϕ and ψ are Δ_2, $f\in(\phi,\psi)BV(\overline{\mathbb{I}}^2)\cap L^1(\overline{\mathbb{I}}^2)$ and $\mathbf{k}=(m,n)\in\mathbb{N}^2$, then*

$$\hat{f}(\mathbf{k})=O\left(\phi^{-1}\left(\frac{2}{m}\psi^{-1}\left(\frac{1}{n}\right)\right)\right). \tag{3.1}$$

Proof. For fixed $\mu,\nu\in\mathbb{N}_0$, let $h_1=\frac{1}{2^{\mu+1}}$ and $h_2=\frac{1}{2^{\nu+1}}$. Put

$$g(x,y)=f([x,x\dotplus h_1]\times[y,y\dotplus h_2])\ \text{for all}\ (x,y)\in\overline{\mathbb{I}}^2.$$

Since $w_m(h_1)=-1$ for $2^\mu\leqslant m<2^{\mu+1}$ and $w_n(h_2)=-1$ for $2^\nu\leqslant n<2^{\nu+1}$, we have

$$\begin{aligned}\hat{g}(m,n)&=\int\int_{\overline{\mathbb{I}}^2} g(x,y)\,w_m(x)\,w_n(y)\,dx\,dy\\&=\int\int_{\overline{\mathbb{I}}^2} f(x,y)\{w_m(x)w_n(y)-w_m(x)w_n(y\dotplus h_2)\\&\qquad -w_m(x\dotplus h_1)w_n(y)+w_m(x\dotplus h_1)w_n(y\dotplus h_2)\}\\&=\{1-w_n(h_2)-w_m(h_1)+w_m(h_1)w_n(h_2)\}\hat{f}(m,n)\\&=4\hat{f}(m,n)\end{aligned}$$

and

$$
\begin{aligned}
|\hat{f}(m,n)| &\leqslant \frac{1}{4}\int\int_{\overline{\mathbb{I}}^2}\left|f\left(\left[x, x\dotplus\frac{1}{2^{\mu+1}}\right]\times\left[y, y\dotplus\frac{1}{2^{\nu+1}}\right]\right)\right| dx\,dy\\
&= \frac{1}{4}\int\int_{\overline{\mathbb{I}}^2}\left|f\left(\left[x\dotplus\frac{1}{2^{\mu}}, x\dotplus\frac{1}{2^{\mu}}\dotplus\frac{1}{2^{\mu+1}}\right]\times\left[y\dotplus\frac{1}{2^{\nu}}, y\dotplus\frac{1}{2^{\nu}}\dotplus\frac{1}{2^{\nu+1}}\right]\right)\right| dx\,dy\\
&= \frac{1}{4}\int\int_{\overline{\mathbb{I}}^2}\left|f\left(\left[x\dotplus\frac{2}{2^{\mu+1}}, x\dotplus\frac{3}{2^{\mu+1}}\right]\times\left[y\dotplus\frac{2}{2^{\nu+1}}, y\dotplus\frac{3}{2^{\nu+1}}\right]\right)\right| dx\,dy.
\end{aligned}
$$

Similarly, we get

$$
|\hat{f}(m,n)| \leqslant \frac{1}{4}\int\int_{\overline{\mathbb{I}}^2}\left|f\left(\left[x\dotplus\frac{4}{2^{\mu+1}}, x\dotplus\frac{5}{2^{\mu+1}}\right]\times\left[y\dotplus\frac{4}{2^{\nu+1}}, y\dotplus\frac{5}{2^{\nu+1}}\right]\right)\right| dx\,dy
$$

and in general we have

$$
\begin{aligned}
|\hat{f}(m,n)| &\leqslant \frac{1}{4}\int\int_{\overline{\mathbb{I}}^2}|\Delta f_{jk}(x,y)|\,dx\,dy\\
&\leqslant \int\int_{\overline{\mathbb{I}}^2}|\Delta f_{jk}(x,y)|\,dx\,dy,
\end{aligned}
$$

where

$$
\Delta f_{jk}(x,y) = f\left(\left[x\dotplus\frac{2j}{2^{\mu+1}}, x\dotplus\frac{(2j+1)}{2^{\mu+1}}\right]\times\left[y\dotplus\frac{2k}{2^{\nu+1}}, y\dotplus\frac{(2k+1)}{2^{\nu+1}}\right]\right)
$$

for all $j = 1,\ldots,2^{\mu}$ and for all $k = 1,\ldots,2^{\nu}$.

For $c > 0$, using Jensen's inequality for integrals, we have

$$
\phi(c|\hat{f}(m,n)|) \leqslant \int\int_{\overline{\mathbb{I}}^2}\phi(c|\Delta f_{jk}(x,y)|)\,dx\,dy.
$$

Summing both the sides of the above inequality over j from 1 to 2^{μ}, we get

$$
2^{\mu}\phi(c|\hat{f}(m,n)|) \leqslant \int\int_{\overline{\mathbb{I}}^2}\sum_{j=1}^{2^{\mu}}\phi(c|\Delta f_{jk}(x,y)|)\,dx\,dy.
$$

Again, using Jensen's inequality for integrals, we have

$$
\psi(2^{\mu}\phi(c|\hat{f}(m,n)|)) \leqslant \int\int_{\overline{\mathbb{I}}^2}\psi\left(\sum_{j=1}^{2^{\mu}}\phi(c|\Delta f_{jk}(x,y)|)\right)dx\,dy.
$$

Summing both the sides of the above inequality over k from 1 to 2^{ν}, we get

$$
\begin{aligned}
2^{\nu}\psi(2^{\mu}\phi(c|\hat{f}(m,n)|)) &\leqslant \int\int_{\overline{\mathbb{I}}^2}\sum_{k=1}^{2^{\nu}}\psi\left(\sum_{j=1}^{2^{\mu}}\phi(c|\Delta f_{jk}(x,y)|)\right)dx\,dy\\
&\leqslant V_{(\phi,\psi)}(cf,\overline{\mathbb{I}}^2), \qquad (3.2)
\end{aligned}
$$

where $cf \in (\phi,\psi)BV(\overline{\mathbb{I}}^2)$ for $c \in (0,1]$.

Since ϕ and ψ are convex and $\phi(0) = \psi(0) = 0$, for $c \in (0,1]$ we have $\phi(cx) \leqslant c\phi(x)$ and $\psi(cx) \leqslant c\psi(x)$, and hence we can choose sufficiently small $c \in (0,1]$ such that $V_{(\phi,\psi)}(cf,\overline{\mathbb{I}}^2) \leqslant \frac{1}{2}$. Thus, in view of inequality (3.2), we get

$$|\hat{f}(m,n)| \leqslant \frac{1}{c}\phi^{-1}\left(\frac{2}{m}\psi^{-1}\left(\frac{1}{n}\right)\right).$$

This completes the proof of theorem. □

COROLLARY 3.2. *If ϕ and ψ are Δ_2 and a measurable function $f \in (\phi,\psi)^*BV(\overline{\mathbb{I}}^2)$, then* (3.1) *holds true.*

Proof. For any $f \in (\phi,\psi)^*BV(\overline{\mathbb{I}}^2)$,

$$\begin{aligned} |f(x,y)| &\leqslant |f([0,x]\times[0,y])| + |f(0,y) - f(0,0)| + |f(x,0) - f(0,0)| + |f(0,0)| \\ &\leqslant \phi^{-1}(\psi^{-1}(V_{(\phi,\psi)}(f,\overline{\mathbb{I}}^2))) + \phi^{-1}(V_\phi(f(0,.),\overline{\mathbb{I}})) + \phi^{-1}(V_\phi(f(.,0),\overline{\mathbb{I}})) \\ &\quad + |f(0,0)| \end{aligned}$$

implies f is bounded on $\overline{\mathbb{I}}^2$. Since $(\phi,\psi)^*BV(\overline{\mathbb{I}}^2) \subset (\phi,\psi)BV(\overline{\mathbb{I}}^2)$, the corollary follows from Theorem 3.1. □

COROLLARY 3.3. *If ϕ and ψ are Δ_2, $f \in (\phi,\psi)^*BV(\overline{\mathbb{I}}^2)$ and $\mathbf{k} = (m,0) \in \mathbb{N}_0^2$ is such that $m \neq 0$, then*

$$\hat{f}(\mathbf{k}) = O\left(\phi^{-1}\left(\frac{1}{m}\right)\right). \tag{3.3}$$

Proof. For fixed $\mu,\nu \in \mathbb{N}_0$, let $h_1 = \frac{1}{2^{\mu+1}}$. Put

$$g(x,y) = f(x,y) - f(x \dotplus h_1, y) \text{ for all } (x,y) \in \overline{\mathbb{I}}^2.$$

Since $w_m(h_1) = -1$ for $2^\mu \leqslant m < 2^{\mu+1}$, we have

$$\begin{aligned} \hat{g}(m,0) &= \int\int_{\overline{\mathbb{I}}^2} g(x,y)\, w_m(x)\, dx\, dy = \int\int_{\overline{\mathbb{I}}^2} f(x,y)\{w_m(x) - w_m(x \dotplus h_1)\}\, dx\, dy \\ &= \{1 - w_m(h_1)\}\hat{f}(m,0) = 2\hat{f}(m,0) \end{aligned}$$

and

$$\begin{aligned} |\hat{f}(m,0)| &\leqslant \frac{1}{2}\int\int_{\overline{\mathbb{I}}^2}\left|f(x,y) - f\left(x \dotplus \frac{1}{2^{\mu+1}}, y\right)\right| dx\, dy \\ &= \frac{1}{2}\int\int_{\overline{\mathbb{I}}^2}\left|f\left(x \dotplus \frac{1}{2^\mu}, y\right) - f\left(x \dotplus \frac{1}{2^\mu} \dotplus \frac{1}{2^{\mu+1}}, y\right)\right| dx\, dy \\ &= \frac{1}{2}\int\int_{\overline{\mathbb{I}}^2}\left|f\left(x \dotplus \frac{2}{2^{\mu+1}}, y\right) - f\left(x \dotplus \frac{3}{2^{\mu+1}}, y\right)\right| dx\, dy. \end{aligned}$$

Similarly, we get

$$|\hat{f}(m,0)| \leqslant \frac{1}{2}\int\int_{\overline{\mathbb{I}}^2}\left|f\left(x\dotplus\frac{4}{2^{\mu+1}},y\right)-f\left(x\dotplus\frac{5}{2^{\mu+1}},y\right)\right| dx\,dy$$

and in general we have

$$\begin{aligned}|\hat{f}(m,0)| &\leqslant \frac{1}{2}\int\int_{\overline{\mathbb{I}}^2}|\Delta f_j(x,y)|\,dx\,dy\\ &\leqslant \int\int_{\overline{\mathbb{I}}^2}|\Delta f_j(x,y)|\,dx\,dy,\end{aligned}$$

where

$$\Delta f_j(x,y) = f\left(x\dotplus\frac{2j}{2^{\mu+1}},y\right)-f\left(x\dotplus\frac{(2j+1)}{2^{\mu+1}},y\right)$$

for all $j=1,\ldots,2^{\mu}$.

For $c>0$, using Jensen's inequality for integrals, we have

$$\phi(c|\hat{f}(m,0)|) \leqslant \int\int_{\overline{\mathbb{I}}^2}\phi(c|\Delta f_j(x,y)|)\,dx\,dy.$$

Summing both the sides of the above inequality over j from 1 to 2^{μ}, we get

$$\begin{aligned}2^{\mu}\phi(c|\hat{f}(m,0)|) &\leqslant \int\int_{\overline{\mathbb{I}}^2}\sum_{j=1}^{2^{\mu}}\phi(c|\Delta f_j(x,y)|)\,dx\,dy\\ &\leqslant V_{\phi}(cf(.,y),\overline{\mathbb{I}}).\end{aligned} \tag{3.4}$$

As ϕ is satisfying Δ_2 condition and is increasing implies

$$\phi(a+b)\leqslant\phi(2\max\{a,b\})\leqslant d(\phi(a)+\phi(b)),\ \text{for any } a,b\geqslant 0.$$

Therefore, for any $0<y\leqslant 1$,

$$V_{\phi}(f(.,y),\overline{\mathbb{I}})\leqslant d[\psi^{-1}(V_{\phi}(f,\overline{\mathbb{I}}^2))+V_{\phi}(f(.,0),\overline{\mathbb{I}})].$$

Thus, in view of (3.4), we get

$$2^{\mu}\phi(c|\hat{f}(m,0)|)\leqslant d[\psi^{-1}(V_{\phi}(cf,\overline{\mathbb{I}}^2))+V_{\phi}(cf(.,0),\overline{\mathbb{I}})]. \tag{3.5}$$

Since ϕ is convex and $\phi(0)=0$, for $c\in(0,1]$ we have $\phi(cx)\leqslant c\phi(x)$, and hence we can choose sufficiently small $c\in(0,1]$ such that $V_{\phi}(cf,\overline{\mathbb{I}}^2)\leqslant\psi\left(\frac{1}{4d}\right)$ and $V_{\phi}(cf(.,0),\overline{\mathbb{I}})$ $\leqslant\frac{1}{4d}$. Thus, in view of inequality (3.5), we get

$$|\hat{f}(m,0)|\leqslant\frac{1}{c}\phi^{-1}\left(\frac{1}{m}\right).$$

This completes the proof of corollary. $\square$

Similarly, one gets the analogue of the above corollary, which is stated below.

COROLLARY 3.4. *If* ϕ *and* ψ *are* Δ_2, $f \in (\phi,\psi)^*BV(\overline{\mathbb{I}}^2)$ *and* $\mathbf{k} = (0,n) \in \mathbb{N}_0^2$ *is such that* $n \neq 0$, *then*

$$\hat{f}(\mathbf{k}) = O\left(\phi^{-1}\left(\frac{1}{n}\right)\right).$$

REMARK 3.5. *Our results with* $\phi(x) = \psi(x) = x$ *gives Walsh analogues of the results of F. Móricz* [5] *and V. Fülöp and F. Móricz* [4, for $n = 2$], *except possibly for the exact constant in their case.*

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