Journal of Classical Analysis
Volume 21, Number 1 (2023), 29–34
doi:10.7153/jca-2023-21-03

# NECESSARY AND SUFFICIENT TAUBERIAN CONDITIONS UNDER WHICH STATISTICALLY LOGARITHMIC CONVERGENCE FOLLOWS FROM STATISTICALLY LOGARITHMIC SUMMABILITY

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*Abstract.* Let $(s_n)$ be a sequence of complex numbers. The sequence $(\tau_n)$ of $n$-th logarithmic means of $(s_n)$ is defined by $\tau_n = \frac{1}{\ell_n}\sum_{k=1}^{n}\frac{s_k}{k}$ where $\ell_n = \sum_{k=1}^{n}\frac{1}{k} \sim \log n$. It is well known that if a bounded sequence $(s_n)$ is statistically logarithmic convergent to $s$, then it is statistically logarithmic summable to the same number. However, the converse of this implication is not true in general. In this paper, we obtain conditions, so called Tauberian conditions, under which the converse implication holds.

## 1. Introduction

The concept of statistical convergence was introduced by Fast [2]. A sequence $(s_n)$ is said to be statistically convergent to $s$ if for every $\varepsilon > 0$,

$$\lim_{m\to\infty}\frac{1}{m}\left|\{n \leqslant m : |s_n - s| \geqslant \varepsilon\}\right| = 0. \tag{1}$$

In this case, we write

$$st - \lim_{n\to\infty} s_n = s. \tag{2}$$

Here $|S|$ means the number of elements of the set $S$.

The concept of statistically logarithmic convergence, which is a generalization of the concept of statistical convergence, was introduced by Alghamdi et al. [1]. A sequence $(s_n)$ is said to be statistically logarithmic convergent to $s$ if for every $\varepsilon > 0$,

$$\lim_{m\to\infty}\frac{1}{\ell_m}\left|\left\{n \leqslant m : \frac{1}{n}|s_n - s| \geqslant \varepsilon\right\}\right| = 0, \tag{3}$$

where $\ell_m = \sum_{k=1}^{m}\frac{1}{k} \sim \log m$. In this case, we write

$$st_l - \lim_{n\to\infty} s_n = s. \tag{4}$$

For more results on the statistical convergence and statistically logarithmic convergence, we refer to [6] and [7], respectively.

*Mathematics subject classification* (2020): 40E05, 40A05.

*Keywords and phrases*: Statistically logarithmic convergence, statistically logarithmic summability, Tauberian conditions and theorems.

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Paper JCA-21-03

The $n$-th logarithmic mean of the sequence $(s_n)$ is defined by

$$\tau_n = \frac{1}{l_n}\sum_{k=1}^{n}\frac{s_k}{k}. \tag{5}$$

We say that a sequence $(s_n)$ is statistically logarithmic summable to $s$ if

$$st_l - \lim_{n\to\infty} \tau_n = s. \tag{6}$$

Alghamdi et al. [1] proved that statistical convergence implies statistically logarithmic convergence, but converse is not true in general. They also proved that if a sequence $(s_n)$ is bounded, then

$$st_l - \lim_{n\to\infty} s_n = s \quad \text{implies} \quad st_l - \lim_{n\to\infty} \tau_n = s. \tag{7}$$

But the converse of the implication (7) is not true in general. In this paper, our aim is to obtain conditions under which the converse of the implication (7) is satisfied.

## 2. Lemmas

We need the following lemmas for the proof of main results. The first lemma gives the representation for the difference $s_n - \tau_n$.

LEMMA 1. ([5]) *For $\lambda > 1$ and sufficiently large $n$,*

$$s_n - \tau_n = \frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]} - \ell_n}(\tau_{[n^\lambda]} - \tau_n) - \frac{1}{\ell_{[n^\lambda]} - \ell_n}\sum_{k=n+1}^{[n^\lambda]}\frac{s_k - s_n}{k}. \tag{8}$$

The following lemma shows that the statistical logarithmic limit relation is linear. A similar relation is given for the statistical convergence by Moricz [4].

LEMMA 2. *If $st_l - \lim_{n\to\infty} s_n = s$ and $st_l - \lim_{n\to\infty} t_n = t$, then*

$$st_l - \lim_{n\to\infty}(s_n + t_n) = s + t \tag{9}$$

*and if $c$ is a constant, then*

$$st_l - \lim_{n\to\infty}(cs_n) = cs. \tag{10}$$

*Proof.* The proof depends on the following lines:

$$\left\{n \leqslant m : \frac{1}{n}|(s_n + t_n) - (s + t)| \geqslant \varepsilon\right\} \subseteq \left\{n \leqslant m : \frac{1}{n}|s_n - s| \geqslant \frac{\varepsilon}{2}\right\}$$
$$\cup \left\{n \leqslant m : \frac{1}{n}|t_n - t| \geqslant \frac{\varepsilon}{2}\right\} \tag{11}$$

and for a constant $c \neq 0$,

$$\frac{1}{n}|cs_n - cs| \geqslant \varepsilon \Leftrightarrow \frac{1}{n}|s_n - s| \geqslant \frac{\varepsilon}{|c|}. \quad \square \tag{12}$$

The following lemma plays a crucial role in the proof of main results.

LEMMA 3. *If a sequence $(s_n)$ is statistically logarithmic summable to $s$, then*

$$st_l - \lim_{n\to\infty} \tau_{[n^\lambda]} = s \tag{13}$$

*for all $\lambda > 1$.*

*Proof.* It is clear that for each $\varepsilon > 0$,

$$\left\{n \leqslant m : \frac{1}{[n^\lambda]}|\tau_{[n^\lambda]} - s| \geqslant \varepsilon\right\} \subseteq \left\{n \leqslant [m^\lambda] : \frac{1}{n}|\tau_n - s| \geqslant \varepsilon\right\}. \tag{14}$$

It follows from (14) that

$$\frac{1}{\ell_m}\left|\left\{n \leqslant m : \frac{1}{[n^\lambda]}|\tau_{[n^\lambda]} - s| \geqslant \varepsilon\right\}\right| \leqslant \frac{\lambda}{\ell_{[m^\lambda]}}\left|\left\{n \leqslant [m^\lambda] : \frac{1}{n}|\tau_n - s| \geqslant \varepsilon\right\}\right|.$$

Hence, (13) follows from the statistically logarithmic summability of $(s_n)$ to $s$. $\square$

The next lemma show that if a sequence $(s_n)$ is statistically logarithmic summable to $s$, then the sequence of its moving logarithmic means is statistically logarithmic summable to the same number.

LEMMA 4. *If a sequence $(s_n)$ is statistically logarithmic summable to $s$, then*

$$st_l - \lim_{n\to\infty} \frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k}{k} = s \tag{15}$$

*for all $\lambda > 1$.*

*Proof.* If $\lambda > 1$ and $n$ is large enough in the sense that $\ell_{[n^\lambda]} > \ell_n$, then

$$\frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k}{k} = \tau_n + \frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]} - \ell_n}(\tau_{[n^\lambda]} - \tau_n) \tag{16}$$

Since for large enough $n$,

$$\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]} - \ell_n} \leqslant \frac{2\lambda}{\lambda - 1}, \tag{17}$$

(15) follows from (16), (17) and Lemma 2 and Lemma 3. $\square$

## 3. Main results

For a sequence of complex numbers, we prove the following two-sided Tauberian theorem. Namely, we give necessary and sufficient Tauberian condition under which statistically logarithmic convergence of a sequence follows from its statistically logarithmic summability.

THEOREM 1. *If a sequence $(s_n)$ of complex numbers is statisticaly logarithmic summable to $s$, then it is statistically logarithmic convergent to $s$ if and only if*

$$\inf_{\lambda>1}\limsup_{m\to\infty}\frac{1}{\ell_m}\left|\left\{n\leqslant m:\left|\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\frac{s_k-s_n}{kn}\right|\geqslant\varepsilon\right\}\right|=0 \tag{18}$$

*for each $\varepsilon>0$.*

*Proof. Necessity.* Assume that both (4) and (6) are satisfied. Applying Lemma 2 and Lemma 4 yields

$$st_l-\lim_{n\to\infty}\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\frac{s_k-s_n}{k}=0 \tag{19}$$

for $\lambda>1$.

*Sufficiency.* Assume that conditions (6) and (18) are satisfied. In order to prove that (4), it is sufficient to prove that

$$st_l-\lim_{n\to\infty}(s_n-\tau_n)=0. \tag{20}$$

It follows from Lemma 1 that for any $\varepsilon>0$,

$$\left\{n\leqslant m:\frac{1}{n}|s_n-\tau_n|\geqslant\varepsilon\right\}\subseteq\left\{n\leqslant m:\frac{l_{[n^\lambda]}}{l_{[n^\lambda]}-l_n}\cdot\frac{1}{n}\left|\tau_{[n^\lambda]}-\tau_n\right|\geqslant\varepsilon/2\right\}$$
$$\cup\left\{n\leqslant m:\left|\frac{1}{l_{[n^\lambda]}-l_n}\sum_{k=n+1}^{[n^\lambda]}\frac{s_k-s_n}{kn}\right|\geqslant\varepsilon/2\right\} \tag{21}$$

By Lemma 2, Lemma 3 and (17), we have

$$\limsup_{m\to\infty}\frac{1}{\ell_m}\left|\left\{n\leqslant m:\frac{l_{[n^\lambda]}}{l_{[n^\lambda]}-l_n}\cdot\frac{1}{n}\left|\tau_{[n^\lambda]}-\tau_n\right|\geqslant\varepsilon/2\right\}\right|=0. \tag{22}$$

Given $\delta>0$, by (18) there exists some $\lambda>1$ such that

$$\limsup_{m\to\infty}\frac{1}{\ell_m}\left|\left\{n\leqslant m:\left|\frac{1}{l_{[n^\lambda]}-l_n}\sum_{k=n+1}^{[n^\lambda]}\frac{s_k-s_n}{kn}\right|\geqslant\varepsilon/2\right\}\right|\leqslant\delta. \tag{23}$$

Combining (21)–(23) yields

$$\limsup_{m\to\infty}\frac{1}{\ell_m}\left|\left\{n\leqslant m:\frac{1}{n}|s_n-\tau_n|\geqslant\varepsilon\right\}\right|\leqslant\delta. \tag{24}$$

Since $\delta>0$ is arbitrary, we have

$$\lim_{m\to\infty}\frac{1}{\ell_m}\left|\left\{n\leqslant m:\frac{1}{n}|s_n-\tau_n|\geqslant\varepsilon\right\}\right|=0 \tag{25}$$

for every $\varepsilon>0$. This proves (20). By Lemma 2, (4) follows from (6) and (20). □

A sequence $(s_n)$ of complex numbers is said to be statisticaly logarithmic slowly oscillating [3] if for each $\varepsilon>0$,

$$\inf_{\lambda>1}\limsup_{m\to\infty}\frac{1}{\ell_m}\left|\left\{n\leqslant m:\max_{n<k\leqslant[n^\lambda]}\frac{1}{n}|s_k-s_n|\geqslant\varepsilon\right\}\right|=0. \tag{26}$$

It is clear that condition (18) follows from (26). Thus, we have the following corollary.

COROLLARY 1. *Let $(s_n)$ be a sequence of complex numbers which is statistically logarithmic summable to $s$. If $(s_n)$ is statistically logarithmic slowly oscillating, then it is statistically logarithmic convergent to $s$.*

*Proof.* It follows from the inequality

$$\left|\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\frac{s_k-s_n}{kn}\right|\leqslant\max_{n<k\leqslant[n^\lambda]}\frac{1}{n}|s_k-s_n|,$$

that for any $\varepsilon>0$,

$$\left\{n\leqslant m:\left|\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\frac{s_k-s_n}{kn}\right|\geqslant\varepsilon\right\}\subseteq\left\{n\leqslant m:\max_{n<k\leqslant[n^\lambda]}\frac{1}{n}|s_k-s_n|\geqslant\varepsilon\right\}. \tag{27}$$

Now, it is clear by (27) that (26) implies (18). Thus, the proof follows from Theorem 18. □

COROLLARY 2. *Let $(s_n)$ be a sequence of complex numbers which is statistically logarithmic summable to $s$. If there exists a positive constant $M$ such that for every $n$ large enough,*

$$(n\log n)|s_n-s_{n-1}|\leqslant M, \tag{28}$$

*then it is statistically logarithmic convergent to $s$.*

*Proof.* Assume that (28) holds for $n > n_0$. For given $\varepsilon > 0$, choose $\lambda = 1 + \frac{\varepsilon}{M}$. If $n_0 < n < k \leqslant [n^\lambda]$ or equivalently $\log n_0 < \log n < \log k \leqslant \lambda \log n$, we have

$$|s_k - s_n| \leqslant \sum_{i=n+1}^{k} |s_i - s_{i-1}| \leqslant M \sum_{i=n+1}^{k} \frac{1}{i \log i} \leqslant M \sum_{i=n+1}^{k} \frac{1}{i} \leqslant M(\lambda - 1)\log n,$$

or dividing by $n$, we have

$$\frac{1}{n}|s_k - s_n| \leqslant M(\lambda - 1)\frac{\log n}{n} \leqslant M(\lambda - 1) = \varepsilon.$$

So, the set

$$\left\{ n_0 < n \leqslant m : \max_{n<k\leqslant[n^\lambda]} \frac{1}{n}|s_k - s_n| \geqslant \varepsilon \right\}$$

is empty. Thus, condition (26) is satisfied and the proof follows from Corollary 1. □

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(Received July 7, 2022)

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