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GENERALIZED STATISTICAL RELATIVE UNIFORM $\widetilde{\phi}$–CONVERGENCE OF TRIPLE SEQUENCES OF FUNCTIONS

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Abstract. In this paper we have introduced the notion of $\widetilde{\phi}$-convergence in μ-density and μ-statistical relative uniform convergence of triple sequences of functions defined on a compact subset D of real numbers, where μ is finitely additive measure. We introduce the concept μ_3-statistical relative uniform $\widetilde{\phi}$-convergence which inherits the basic properties of uniform $\widetilde{\phi}$-convergence.

1. Introduction and background

In 1951, Steinhaus [28] and Fast [11] proposed the idea of statistical convergence. Later, Connor [4] and Fridy [12] demonstrated that convergent sequences are statistically convergent, but the reverse of this does not hold, in general. Following that, several researchers afterwards examined the topic of statistical convergence from other aspects (see, for example, [15, 16, 17, 19, 24, 26, 27]. Connor has extended the concept of statistical convergence in [5], where the asymptotic density is replaced by finitely additive set function. In this paper, μ denotes a finitely additive set function taking values in $[0,1]$ defined on a field Γ of subsets of N such that if $|A| < \infty$, then $\mu(A) = 0$; if $A \subset B$ and $\mu(B) = 0$, then $\mu(A) = 0$; $\mu(\mathbb{N}) = 1$. Such a set function satisfying the above criteria will be called measure. Connor has provided us with the following information in [5, 6].

(a) x is μ-density convergent to L if there is an $A \in \Gamma$ such that $(x-L)\chi_A$ is a null sequence and $\mu(A) = 1$ where χ_A is the characteristic function of A.

(b) x is μ-statistically convergent to L and write $st_\mu - \lim x = L$, provided

$$\mu(\{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}) = 0$$

for every $\varepsilon > 0$.

If $T = (t_{nk})$ is a nonnegative regular summability method, then T is used to generate a measure as follows:

For each $n \in \mathbb{N}$, set $\mu_n(A) = \sum_{k=1}^{\infty} t_{nk}\chi_A(k)$ for each $A \subseteq \mathbb{N}$. Let

$$\Gamma = \left\{A \subseteq \mathbb{N} : \lim_n \mu_n(A) = 0 \text{ or } \lim_n \mu_n(A) = 1\right\}.$$

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Define, $\mu_T : \Gamma \longrightarrow [0,1]$ by

$$\mu_T(A) = \lim_{n \longrightarrow \infty} \mu_n(A) = \lim_{n \longrightarrow \infty} \sum_{k=1}^{\infty} t_{nk} \chi_A(k),$$

where μ_T and Γ satisfy the requirements of the preceding definitions. If T is a Cesáro matrix of order one, the μ_T-statistical convergence is equivalent to statistical convergence.

Connor [5] has established that (a) implies (b), but not necessarily conversely. These two definitions are equivalent Connor [5, 6] if μ has additive property for null sets: if given a collection of null sets $\{A_j\}_{j\in\mathbb{N}} \subseteq \Gamma$, there exists a collection $\{B_i\}_{i\in\mathbb{N}} \subseteq \Gamma$ with the properties $|A_i \Delta B_i| < \infty$ for each $i \in \mathbb{N}$, $B = \bigcup_{i=1}^{\infty} B_i \in \Gamma$, and $\mu(B) = 0$. Different classes of μ-statistical convergence of sequences have been introduced and their properties have been studied by Duman and Orhan [9] and Gürdal [14].

The notion of convergence of sequence of function is also considered in measure theory. Wilczynski [29] studied the statistical convergence of sequences of functions in 2000. Some classification may also be found in [20]. Moore [21] was the first who introduced the notion of relative uniform convergence of sequences of functions. Thereafter, Chittenden [3] studied the notion (which is equivalent to Moore's definition as follows:

A sequence of functions (g_n), defined on $J = [a,b]$ converges relatively uniformly to a limit function g if there is a function $\gamma(x)$, called a scale function such that for every $\varepsilon > 0$, there exists an integer $m = m(\varepsilon)$ such that

$$|g_n(x) - g(x)| < \varepsilon |\gamma(x)|, \text{ uniformly in } x \text{ on } J, \forall n \geqslant m.$$

Nowadays many authors prefer to use the notion of relative uniform convergence of sequences of functions (see e.g. frequently quoted works [7, 8]).

Recently, the concept of statistical convergence for triple sequences was presented by Şahiner et al. [25], as follows: A function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) is called a real triple sequence. A triple sequence (x_{mno}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $|x_{mno} - L| < \varepsilon$ whenever $m,n,o \geqslant N$.

Statistical convergence of any real sequence is identified relatively to absolute value. While we have known that the absolute value of real numbers is special of an Orlicz function [23], that is, a function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ in such a way that it is even, non-decreasing on $\mathbb{R}^+$, continuous on $\mathbb{R}$, and satisfying

$$\widetilde{\phi}(x) = 0 \text{ if and only if } x = 0 \text{ and } \widetilde{\phi}(x) \to \infty \text{ as } x \to \infty.$$

Further, an Orlicz function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ is said to satisfy the $\triangle_2$ condition, if there exists an positive real number M such that $\widetilde{\phi}(2x) \leqslant M.\widetilde{\phi}(x)$ for every $x \in \mathbb{R}^+$. Rao and Ren discuss several key applications of Orlicz functions in numerous domains such as economics, stochastic problems, and so on in [23]. The reader may also consult the new monograph [2] and the work [10] on different methods for systematically generalizing

Orlicz sequence spaces and investigating numerous structural characteristics of such spaces. For the normed sequence spaces and related topics, the reader can refer to the textbooks [1] and [22]. Now, we can give some examples of Orlicz functions, by Example 1:

EXAMPLE 1. (I) For a fixed $r \in \mathbb{N}$, the function $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ defined as $\widetilde{\phi}(x)=|x|^r$ is an Orlicz function.

(II) The function $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ defined as $\widetilde{\phi}(x)=x^2$ is an Orlicz function satisfying the $\triangle_2$ condition.

(III) The function $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ defined as $\widetilde{\phi}(x)=e^{|x|}-|x|-1$ is an Orlicz function not satisfying the $\triangle_2$ condition.

(IV) The function $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ defined as $\widetilde{\phi}(x)=x^3$ is not an Orlicz function.

Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. A triple sequence $x=(x_{mno})$ is said to be statistically $\widetilde{\phi}$-convergent to L if for each $\varepsilon>0$,

$$\lim_{p,q,r\to\infty}\frac{1}{pqr}\left|\left\{m\leqslant p,n\leqslant q,o\leqslant r:\widetilde{\phi}\left(x_{mno}-L\right)\geqslant\varepsilon\right\}\right|=0.$$

Inspired by these literature, in this paper, we investigate the idea of μ_3-statistical relative uniform $\widetilde{\phi}$-convergence. Moreover, we observe that μ_3-statistical uniform $\widetilde{\phi}$-convergence inherits the essential features of uniform $\widetilde{\phi}$-convergence. Related results are contained in [13]

2. Main result

Let $D=[a,1]\subseteq\mathbb{R}$, where $0<a<1$ and (f_{mno}) be a triple sequence of real functions on D. Now, some definitions used in this paper are given.

DEFINITION 1. Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. (f_{mno}) $\widetilde{\phi}$-converges μ_3-density pointwise to f iff for a given $\varepsilon>0$ and for all $x\in D$, there exists $K_x\in\Gamma$, $\mu\left(K_x\right)=1$ and there exists $N=N\left(\varepsilon,x\right)\in K_x$ such that

$$\widetilde{\phi}\left(f_{mno}\left(x\right)-f\left(x\right)\right)<\varepsilon$$

whenever $m\geqslant N,\ n\geqslant N,\ o\geqslant N$ and $(m,n,o)\in K_x$.

In this case we write $f_{mno}\longrightarrow f\ \left(\mu_3^{\widetilde{\phi}}\text{- density}\right)$ on D.

DEFINITION 2. Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. (f_{mno}) $\widetilde{\phi}$-converges μ_3-density uniform to f iff for a given $\varepsilon>0$, and there exists $K\in\Gamma$, $\mu\left(K\right)=1$ and there exists $N=N\left(\varepsilon\right)\in K$ such that

$$\widetilde{\phi}\left(f_{mno}\left(x\right)-f\left(x\right)\right)<\varepsilon$$

for all $m\geqslant N,\ n\geqslant N,\ o\geqslant N$ and $(m,n,o)\in K$ and for every $x\in D$.

In this case we write $f_{mno}\rightrightarrows f\left(\mu_3^{\widetilde{\phi}}\text{-density}\right)$ on D.

DEFINITION 3. Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. (f_{mno}) $\widetilde{\phi}$-converges μ_3-density relatively uniform to f iff there exists a function $\gamma(x)$ called scale function such that $|\gamma(x)|>0$ for a given $\varepsilon>0$ there exists $K\in\Gamma$, $\mu(K)=1$ and there exists $N=N(\varepsilon)\in K$ such that for all $m\geqslant N$, $n\geqslant N$, $o\geqslant N$ and $(m,n,o)\in K$ for each $x\in D$,

$$\widetilde{\phi}\left(\frac{f_{mno}(x)-f(x)}{\gamma(x)}\right)<\varepsilon.$$

In this case we will write $f_{mno}\rightrightarrows f(D;\gamma)$ $\left(\mu_3^{\widetilde{\phi}}\text{- density}\right)$.

DEFINITION 4. Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. (f_{mno}) $\widetilde{\phi}$-converges μ_3-statistically pointwise to f iff every $\varepsilon>0$ and for each $x\in D$,

$$\mu\left(\left\{(m,n,o)\in\mathbb{N}^3:\widetilde{\phi}\left(f_{mno}(x)-f(x)\right)\geqslant\varepsilon\right\}\right)=0.$$

In this case we write $f_{mno}\longrightarrow f\left(\mu_3^{\widetilde{\phi}}\text{-stat}\right)$ on D.

DEFINITION 5. Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. The triple sequence (f_{mno}) of bounded functions on D $\widetilde{\phi}$-convergence μ_3-statistically uniform to f iff $st_{\mu_3}-\lim\|f_{mno}-f\|_B=0$, where the norm $\|.\|_B$ is the usual supremum norm on $B(D)$, the space of bounded functions on D.

In this case we write $f_{mno}\rightrightarrows f\left(\mu_3^{\widetilde{\phi}}\text{- stat}\right)$ on D.

DEFINITION 6. Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. (f_{mno}) $\widetilde{\phi}$-converges relatively uniform to f if there exists a function $\gamma(x)$ such that for every $\varepsilon>0$, there exists an integer N_ε such that for every $n>N_\varepsilon$ the inequality $\widetilde{\phi}\left(f_{mno}(x)-f(x)\right)<\varepsilon\widetilde{\phi}\left(\gamma(x)\right)$ holds uniformly in x. The sequence (f_{mno}) is said to $\widetilde{\phi}$-converge uniformly relative to the scale function $\gamma(x)$ or more simply, relatively uniformly.

It is observed that uniform $\widetilde{\phi}$-convergence is the special case of relatively uniform $\widetilde{\phi}$-convergence in which scale function is a nonzero constant.

Now, in this section, we introduced the following definition and established the article's outcomes.

DEFINITION 7. Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. (f_{mno}) $\widetilde{\phi}$-converges μ_3-statistically relatively uniform to f if and only if there exists a function $\gamma(x)$ such that $\widetilde{\phi}(\gamma(x))>0$ called scale function $\gamma(x)$ such that for every $\varepsilon>0$,

$$\mu\left\{(m,n,o)\in\mathbb{N}^3:\sup_{x\in D}\widetilde{\phi}\left(\frac{f_{mno}(x)-f(x)}{\gamma(x)}\right)\geqslant\varepsilon\right\}=0.$$

In this case we will write $f_{mno}\rightrightarrows f(D;\gamma)$ $\left(\mu_3^{\widetilde{\phi}}\text{-}st\right)$.

LEMMA 1. *Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. (f_{mno}) $\widetilde{\phi}$-converges μ-density uniformly on D implies $f_{mno}\rightrightarrows f(D;\gamma)$ $\left(\mu_3^{\widetilde{\phi}}\text{-density}\right)$ on D, which implies $f_{mno}\rightrightarrows f(D;\gamma)$ $\left(\mu_3^{\widetilde{\phi}}\text{-}st\right)$.*

In general, the converse of Lemma 1 does not necessarily hold, as was seen by the following example:

EXAMPLE 2. Define $f_{mno}:[0,1]\longrightarrow\mathbb{R}$ by

$$f_{mno}(x)=\begin{cases}0, & x=0,\\ \frac{2m^2n^2o^2x}{1+m^3n^3o^3x^2}, & x\neq 0.\end{cases}$$

Define

$$\gamma(x)=\begin{cases}1, & x=0,\\ \frac{1}{x}, & x\in(0,1].\end{cases}$$

Then, $f_{mno}\longrightarrow f=\theta(D;\gamma)\left(\mu_3^{\widetilde{\phi}}\text{-density}\right)$. Hence, $f_{mno}\longrightarrow f=\theta(D;\gamma)\left(\mu_3^{\widetilde{\phi}}\text{-}st\right)$. But (f_{mno}) is not μ-statistical uniformly $\widetilde{\phi}$-convergent to $f=\theta$ in $[0,1]$, where θ is the zero function.

THEOREM 1. *Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function and the triple sequence of functions (f_{mno}) be each continuous on D, a compact subset of $\mathbb{R}$ and let μ be a measure with additive property for null sets. If $f_{mno}\rightrightarrows f(D;\gamma)\left(\mu_3^{\widetilde{\phi}}\text{-}st\right)$ on D and $\gamma(x)$ is continuous, then f is continuous on D.*

Proof. By hypothesis D is a compact subset of $\mathbb{R}$ and for each $(m,n,o)\in\mathbb{N}^3$, f_{mno} is a continuous function. Thus it is clear that for each $(m,n,o)\in\mathbb{N}^3$, f_{mno} is $\widetilde{\phi}$-bounded on D. Hence, there exists $M>0$ where $M=\sup_{x\in D}\left\{\widetilde{\phi}(f_{mno}(x))\right\}$ such that $\widetilde{\phi}(f_{mno}(x))\leqslant M$.

Also, $\gamma(x)$ continuous implies it is $\widetilde{\phi}$-bounded. So there exists $G>0$ such that $\widetilde{\phi}(\gamma(x))\leqslant G$. Let, $L=\max(M,G)$ and $f_{mno}\rightrightarrows f(D;\gamma)$ $\left(\mu_3^{\widetilde{\phi}}\text{-}st\right)$. Then for every $\varepsilon>0$,

$$\mu\left\{(m,n,o)\in\mathbb{N}^3:\sup_{x\in D}\widetilde{\phi}\left(\frac{f_{mno}(x)-f(x)}{\gamma(x)}\right)\geqslant\frac{\varepsilon}{3L}\right\}=0.$$

Let, $x_0\in D$. Since, f_{mno} is continuous for each $n\in\mathbb{N}$ at $x_0\in D$, there exists a $\delta>0$ such that $\widetilde{\phi}(x-x_0)<\delta$ implies $\widetilde{\phi}(f_{mno}(x)-f_{mno}(x_0))<\frac{\varepsilon}{3}$, for each $x\in D$.

Thus for all $x\in D$, for which $|x-x_0|<\delta$ we have

$$\begin{aligned}\widetilde{\phi}(f(x)-f(x_0))&\leqslant\widetilde{\phi}(f(x)-f_{mno}(x))+\widetilde{\phi}(f_{mno}(x)-f_{mno}(x_0))\\&\quad+\widetilde{\phi}(f_{mno}(x_0)-f(x_0))\\&<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

Since, $x_0 \in D$ is arbitrary, so f is continuous on D. □

We may get the following conclusion from the aforementioned cases:

COROLLARY 1. *Let $\widetilde{\phi}: \mathbb{R} \to \mathbb{R}$ be an Orlicz function and all functions (f_{mno}) be continuous on a compact subset D of $\mathbb{R}$. If $f_{mno} \rightrightarrows f(D;\gamma)$ $\left(\mu_3^{\widetilde{\phi}}\text{-density}\right)$ on D, then f is continuous on D.*

THEOREM 2. *A necessary and sufficient condition for a real valued continuous function f on D be the μ_3-st relative uniform limit of a sequence of real valued continuous functions (f_{mno}) is that there exists a sequence (D_{pqr}) of subsets of D such that $D = \bigcup_{p,q,r=1,1,1}^{\infty} D_{pqr}$ and the restricted functions $f_{pqr_{|D_{pqr}}}$ converges $\mu_3^{\widetilde{\phi}}$-st relative uniformly to f.*

Proof. Let, (f_{mno}) be $\widetilde{\phi}$-convergent μ_3-statistically relative uniform to f. To prove that $f_{pqr_{|D_{pqr}}}$ $\widetilde{\phi}$-converges μ_3-*st* relative uniformly to f. By hypothesis, there exists a scale function $\gamma(x)$ such that $\widetilde{\phi}(\gamma(x)) > 0$ and for every $\varepsilon > 0$, we have

$$\mu\left\{(m,n,o) \in \mathbb{N}^3 : \sup_{x\in D} \widetilde{\phi}\left(\frac{f_{mno}(x) - f(x)}{\gamma(x)}\right) \geqslant \varepsilon\right\} = 0.$$

By definition of restriction function, we have

$$f_{mno|D_{mno}}(x) = f_{mno}(x), \ \forall x \in D_{mno}.$$

Also, $D_{mno} \subseteq D$ shows that

$$\mu\left\{(m,n,o) \in \mathbb{N}^3 : \sup_{x\in D_{mno}} \widetilde{\phi}\left(\frac{f_{mno}(x) - f(x)}{\gamma(x)}\right) \geqslant \varepsilon\right\}$$
$$\leqslant \mu\left\{(m,n,o) \in \mathbb{N}^3 : \sup_{x\in D} \widetilde{\phi}\left(\frac{f_{mno}(x) - f(x)}{\gamma(x)}\right) \geqslant \varepsilon\right\}$$

and, so we get

$$\mu\left\{(m,n,o) \in \mathbb{N}^3 : \sup_{x\in D_{mno}} \widetilde{\phi}\left(\frac{f_{mno}(x) - f(x)}{\gamma(x)}\right) \geqslant \varepsilon\right\} = 0.$$

Conversely, let each $f_{pqr_{|D_{pqr}}}$ $\widetilde{\phi}$-converges μ_3-*st* relative uniformly to f. We established that (f_{mno}) $\widetilde{\phi}$-converges μ_3-statistical relative uniform to f. Here, $D = \bigcup_{p,q,r=1,1,1}^{\infty} D_{pqr}$. Therefore,

$$\mu\left\{(m,n,o) \in \mathbb{N}^3 : \sup_{x\in D} \widetilde{\phi}\left(\frac{f_{mno}(x) - f(x)}{\gamma(x)}\right) \geqslant \varepsilon\right\}$$
$$\leqslant \sum_{m,n,o=1,1,1,}^{\infty} \mu\left\{(m,n,o) \in \mathbb{N}^3 : \sup_{x\in D_{mno}} \widetilde{\phi}\left(\frac{f_{mno}(x) - f(x)}{\gamma(x)}\right) \geqslant \varepsilon\right\}.$$

Since $\mu\left\{(m,n,o)\in\mathbb{N}^3:\sup_{x\in D_{mno}}\widetilde{\phi}\left(\frac{f_{mno}(x)-f(x)}{\gamma(x)}\right)\geqslant\varepsilon\right\}=0$, it follows that

$$\mu\left\{(m,n,o)\in\mathbb{N}^3:\sup_{x\in D}\widetilde{\phi}\left(\frac{f_{mno}(x)-f(x)}{\gamma(x)}\right)\geqslant\varepsilon\right\}=0.$$

This completes the proof. □

$C(\mu,ru)$ will be used throughout to represent the space of all continuous μ-statistical relative uniform $\widetilde{\phi}$-convergent sequences of real functions formed on D, where the scale function is continuous. Now we may define a norm for this class of sequences as follows:

Let $(f_{mno})\in C(\mu,ru)$ denote

$$\|(f_{mno})\|=\sup_{m,n,o\geqslant1}\sup_{\substack{\|x\|\leqslant1\\x\in D}}\frac{\widetilde{\phi}(f_{mno}(x))}{\|x\|\,\gamma(x)}. \tag{1}$$

Now we will demonstrate the following outcome.

THEOREM 3. *Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. The class of sequences $C(\mu,ru)$ is a Banach space with respect to the norm defined by (1).*

Proof. First we established that $C(\mu,ru)$ is a normed linear space. Let (f_{mno}), (g_{mno})
$\in C(\mu,ru)$ and

$N1)$ $\|(f_{mno})\|\geqslant0$ and $\|(f_{mno})\|=0$ iff $f_{mno}=\theta$, for each $m,n,o\in\mathbb{N}$

$N2)$ $\|(f_{mno})+(g_{mno})\|=\|(f_{mno}+g_{mno})\|=\sup_{m,n,o\geqslant1}\sup_{\substack{\|x\|\leqslant1\\x\in D}}\frac{\widetilde{\phi}((f_{mno}+g_{mno})(x))}{\|x\|\,\gamma(x)}$

$$\leqslant\sup_{m,n,o\geqslant1}\sup_{\substack{\|x\|\leqslant1\\x\in D}}\frac{\widetilde{\phi}(f_{mno}(x))}{\|x\|\,\gamma(x)}+\sup_{m,n,o\geqslant1}\sup_{\substack{\|x\|\leqslant1\\x\in D}}\frac{\widetilde{\phi}(g_{mno}(x))}{\|x\|\,\gamma(x)}$$
$$=\|(f_{mno})\|+\|(g_{mno})\|.$$

$N3)$ $\|\alpha(f_{mno})\|=\sup_{m,n,o\geqslant1}\sup_{\substack{\|x\|\leqslant1\\x\in D}}\frac{\widetilde{\phi}(\alpha f_{mno}(x))}{\|x\|\,\widetilde{\phi}(\gamma(x))}=\sup_{m,n,o\geqslant1}\sup_{\substack{\|x\|\leqslant1\\x\in D}}\frac{|\alpha|\,\widetilde{\phi}(f_{mno}(x))}{\|x\|\,\widetilde{\phi}(\gamma(x))}$
$=|\alpha|\,\|(f_{mno})\|.$

In order to show that $C(\mu,ru)$ is complete with respect to the above norm, we start with a Cauchy sequence $\left(f^{(mno)}\right)$, where $f^{(mno)}=\left(f_{m_1n_1o_1},f_{m_2n_2o_2},f_{m_3n_3o_3},\ldots\right)$. Then by definition of Cauchy sequence, for a given $\varepsilon>0$, there exists a positive integer

$n_0 \in \mathbb{N}$ such that for all $m \geqslant \vartheta \geqslant n_0,\ n \geqslant \iota \geqslant n_0,\ o \geqslant \kappa \geqslant n_0$

$$\left\| \left(f^{(mno)}\right) - \left(f^{(\vartheta\iota\kappa)}\right) \right\| < \varepsilon,\ \forall m \geqslant \vartheta \geqslant n_0, n \geqslant \iota \geqslant n_0, o \geqslant \kappa \geqslant n_0$$

$$\Rightarrow \sup_{i \geqslant 1} \sup_{\substack{\|x\| \leqslant 1 \\ x \in D}} \frac{\widetilde{\phi}\left(\left(f_{m_i n_i o_i} - f_{\vartheta_i \iota_i \kappa_i}\right)(x)\right)}{\|x\| \left|\gamma(x)\right|} < \varepsilon.$$

$$\Rightarrow \frac{\widetilde{\phi}\left(\left(f_{m_i n_i o_i} - f_{\vartheta_i \iota_i \kappa_i}\right)(x)\right)}{\|x\| \widetilde{\phi}\left(\gamma(x)\right)} < \varepsilon, \text{ with } \|x\| \leqslant 1$$

which clearly implies that

$$\widetilde{\phi}\left(\left(f_{m_i n_i o_i} - f_{\vartheta_i \iota_i \kappa_i}\right)(x)\right) < \varepsilon \|x\| \widetilde{\phi}\left(\gamma(x)\right) < \varepsilon. \tag{2}$$

Thus, $\left(f_{m_i n_i o_i}(x)\right)$ is a triple Cauchy sequence of reals, hence it is $\widetilde{\phi}$-convergent. Let $\lim_{m,n,o \longrightarrow \infty} f_{m_i n_i o_i}(x) = f_i(x)$, for all $x \in D$ with $\|x\| \leqslant 1$. Keeping m,n,o fixed and letting $\vartheta, \iota, \kappa \longrightarrow \infty$ in (2) we have

$$\widetilde{\phi}\left(\left(f_{m_i n_i o_i} - f_i\right)(x)\right) < \varepsilon \|x\| \widetilde{\phi}\left(\gamma(x)\right) < \varepsilon, \tag{3}$$

for all $m,n,o \geqslant n_0$. So, we get

$$\sup_{\substack{\|x\| \leqslant 1 \\ x \in D}} \widetilde{\phi}\left(\left(f_{m_i n_i o_i} - f_i\right)(x)\right) < \varepsilon$$

and

$$\lim_{m,n,o \longrightarrow \infty} \sup_{\substack{\|x\| \leqslant 1 \\ x \in D}} \widetilde{\phi}\left(\left(f_{m_i n_i o_i} - f_i\right)(x)\right) = 0.$$

Thus, $f_{m_i n_i o_i} \longrightarrow f_i$ uniformly on $\{x \in D : \|x\| \leqslant 1\}$, implying $\left(f_{m_i n_i o_i}\right)$ $\widetilde{\phi}$-converges relatively uniformly to f_i. Hence, we have

$$f_{m_i n_i o_i} \rightrightarrows f_i \ (D;\gamma) \left(\mu_3^{\widetilde{\phi}}\text{-}st\right).$$

Thus, there exists a scale function $\gamma(x)$ such that $\widetilde{\phi}\left(\gamma(x)\right) > 0$ and for every $\varepsilon > 0$, we have

$$\mu\left\{(m,n,o) \in \mathbb{N}^3 : \sup_{x \in D} \widetilde{\phi}\left(\frac{f_{m_i n_i o_i}(x) - f_i(x)}{\gamma(x)}\right) \geqslant \varepsilon\right\} = 0,$$

where $\|x\| \leqslant 1$. On the other hand, from (3), we get

$$\sup_{i \geqslant 1} \sup_{\substack{\|x\| \leqslant 1 \\ x \in D}} \frac{\widetilde{\phi}\left(\left(f_{m_i n_i o_i} - f_i\right)(x)\right)}{\|x\| \widetilde{\phi}\left(\gamma(x)\right)} < \varepsilon.$$

So, we obtain $\left\| \left(f^{(mno)}\right) - (f) \right\| < \varepsilon,\ \forall m,n,o \geqslant n_0$. Notice that

$$(f) = \left\{(f) - \left(f^{(mno)}\right)\right\} + \left(f^{(mno)}\right) \in C(\mu, ru).$$

Hence, we have $\left(f^{(mno)}\right) \longrightarrow (f)$ in $(C(\mu,ru),\|.\|)$ as desired. $\square$

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